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A NEW INVARIANT DESCRIPTION OF THE NUCLEON MEAN-SQUARE RADIUS

The nucleon mean-square radius is defined from the expansion of form factor $F(t)$ for small momentum transfer $t = q^2 = (p-k)^2$

$$F(t) = F(0) + \frac{1}{6} \langle r_0^2 \rangle t + \ldots \tag{1}$$

or, identically

$$\langle r_0^2 \rangle = \frac{6 \left. \frac{\partial F(t)}{\partial t} \right|_{t=0}}{F(0)} . \tag{2}$$

To make correspond the mean-square radius thus defined to the spatial distribution of a nucleon it is usual to choose a special reference frame, the Breit frame. In this system $\vec{p} = -\vec{k}$ because of that the time component of momentum is zero $q_0 = p_0 - k_0 = 0$ and the four-dimensional Fourier transformation reduces to the three-dimensional one

$$f(r) = \frac{1}{(2\pi)^3} \int F(-\vec{q}^{\,2}) e^{i\vec{q}\vec{r}} d\vec{q} . \tag{3}$$

By means of thus defined spatial distribution for a nucleon $\langle r_0^2 \rangle$ in the Breit system is determined by the formula

$$\langle r_0^2 \rangle = \frac{\left\{ \left(i \frac{\partial}{\partial \vec{q}} \right)^2 F(-\vec{q}^{\,2}) \right\} \Big|_{\vec{q}^{\,2}=0}}{F(0)} = \frac{-6 \left. \frac{\partial F(-\vec{q}^{\,2})}{\partial \vec{q}^{\,2}} \right|_{\vec{q}^{\,2}=0}}{F(0)} = \frac{\int r^2 f(r) d\vec{r}}{\int f(r) d\vec{r}} . \tag{4}$$

However, such a description of the spatial distribution of a nucleon is not satisfactory since in the Breit system the nucleon itself is moving.

In the present note we will show that this difficulty can be overcome if in describing the nucleon spatial distribution one makes use of the introduced in [1] relativistic *configurational representation* which, as will be seen below, provides the invariant description of the nucleon spatial distribution.

To this end we note that on the mass shell* $p_0^2 - \vec{p}^{\,2} = M^2$ the momentum transfer squared $t = q^2 = (p-k)^2$ can be expressed through the 3-vector $\vec{\Delta} = \vec{p}(-)\vec{k}$ which is the difference of two vectors in the Lobachevsky space [1,2]:

$$\vec{\Delta} = \vec{p}(-)\vec{k} = (\overrightarrow{\Lambda_p^{-1} k}) = \vec{p} - \frac{\vec{k}}{M}\left(p_0 - \frac{\vec{p}\vec{k}}{k_0 + M}\right), \qquad (5)$$

$$\Delta_0 = (p(-)k)_0 = \frac{p_0 k_0 - \vec{p}\vec{k}}{M} = \sqrt{M^2 + \vec{\Delta}^2} \qquad (6)$$

in the following way

$$t = (p-k)^2 = 2M^2 - 2pk = 2M^2 - 2M\sqrt{M^2 + \vec{\Delta}^2}\,. \qquad (7)$$

Consequently, in any reference frame form factor $F(t)$ is parametrized by the square of 3-vector of momentum transfer that belongs to the Lobachevsky space **

$$F(t) = F(\vec{\Delta}^{\,2}). \qquad (8)$$

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* The equation $p_0^2 - \vec{p}^{\,2} = M^2$ defines in the Minkowski space the surface of hyperboloid on the upper sheet of which the Lobachevsky space is realized [4,5].

** The Lobachevsky space first was applied for description of form factors in [3].

The group of motion of the Lobachevsky space is the Lorentz group. Therefore in [1] it has been suggested to introduce the coordinate representation in the relativistic theory by means of expansions over the unitary irreducible representations of the Lorentz group

$$F(r) = \frac{1}{(2\pi)^3} \int \xi(\vec{\Delta},\vec{r}) F(\vec{\Delta}^2) \frac{d\vec{\Delta}}{\Delta_0} \tag{9}$$

which are realized by the functions:

$$\xi(\vec{\Delta},\vec{r}) = \left(\frac{\Delta_0 - \vec{\Delta}\vec{n}}{M}\right)^{-1-irM} \tag{10}$$

$$\vec{r} = r \cdot \vec{n}\ ; \quad \vec{n}^2 = 1\ ; \quad 0 \le r < \infty$$

explicitly found in [6]. In doing so, as is shown in [1], the parameter r plays the role of modulus of the radius-vector of the relativistic relative distance, since in the nonrelativistic limit when $\vec{\Delta} = \vec{p}(-)\vec{k} \to \vec{q} = \vec{p} - \vec{k}$ the functions (10) turn into the usual plane waves:

$$\xi(\vec{\Delta},\vec{r}) \xrightarrow[c\to\infty]{} e^{i\vec{q}\vec{r}} .$$

The modulus of the relativistic relative coordinate r is the relativistic invariant because it determines the eigenvalues of the invariant operator, the Casimir operator of the Lorentz group $\hat{C} = \vec{N}^2 - \vec{M}^2$, on functions (10):

$$\hat{C}\xi(\vec{\Delta},\vec{r}) = (\vec{N}^2 - \vec{M}^2)\xi(\vec{\Delta},\vec{r}) = \left(\frac{1}{M^2} + r^2\right)\xi(\vec{\Delta},\vec{r}) . \tag{11}$$

Thus, the function $F(r)$ defined by (9) provides in the relativistic configurational representation the invariant description of the nucleon spatial distribution in any reference frame including the rest frame.

In the nonrelativistic limit the Casimir operator of the Lorentz group becomes the Casimir operator of the group of motion of three-dimensional Euclidean momentum space, i.e., the operator $(i\frac{\partial}{\partial \vec{q}})^2$ in (14). The eigenvalue of the operator $(i\frac{\partial}{\partial \vec{q}})^2$ on the functions $e^{i\vec{q}\vec{r}^*}$, as is seen from (3), is the square of nonrelativistic relative coordinate r^2.

It appears that the invariant description of the nucleon mean-square radius (1), (2) has a group-theoretical interpretation. Employing the fact that in the spherical coordinates

$$\Lambda_0 = M \operatorname{ch}\chi ; \quad \vec{\Lambda} = M \operatorname{sh}\chi (\sin\theta \sin\phi, \sin\theta\cos\phi, \cos\phi)$$

the Casimir operator of the Lorentz group has the form of the Laplace operator on hyperboloid

$$\hat{C} = \vec{N}^2 - \vec{M}^2 = -\frac{1}{M^2}\frac{\partial^2}{\partial\chi^2} - 2\frac{\operatorname{cth}\chi}{M^2}\frac{\partial}{\partial\chi} - \frac{\Delta_{\theta,\phi}}{M^2 \operatorname{sh}^2\chi} \qquad (12)$$

(the hyperbolic angle χ in the Lobachevsky geometry is called "rapidity" [5]) it is easy to verify that in virtue of the spherical symmetry one has $F(t) = F[2M^2(1-\operatorname{ch}\chi)]$

$$\frac{[\hat{C}F(t)]|_{\chi=0}}{F(0)} = \frac{-\frac{6}{M^2}\frac{\partial F(t)}{\partial\chi^2}|_{\chi=0}}{F(0)} = \frac{6\frac{\partial F(t)}{\partial t}|_{t=0}}{F(0)}. \qquad (13)$$

Expression (13) coincides with the invariant definition of the mean-square radius (2) which can now be connected with the invariant function $F(r)$ describing, in accordance with (9), the nucleon spatial distribution in an arbitrary reference frame [1]

$$\langle r_0^2 \rangle = \frac{6\frac{\partial F(t)}{\partial t}|_{t=0}}{F(0)} = \frac{[(\vec{N}^2-\vec{M}^2)F(t)]|_{t=0}}{F(0)} =$$

$$= \frac{\int \left(\frac{h^2}{M^2 c^2} + r^2 \right) F(r) d\vec{r}}{\int F(r) d\vec{r}} = \frac{h^2}{M^2 c^2} + \langle r^2 \rangle . \qquad (14)$$

Hence it is seen that the relativistic relative coordinate describes the nucleon structure beyond its Compton wave length.

Due to the spherical symmetry of the form factor $F(t)$ transformation (9) and its inverse are as follows:

$$F(r) = \frac{1}{2\pi^2} \int \frac{\sin rM\chi}{rM \operatorname{sh}\chi} F(\chi^2) \operatorname{sh}^2\chi \, d\chi . \qquad (15)$$

$$F(\chi^2) = 4\pi \int \frac{\sin rM\chi}{rM \operatorname{sh}\chi} F(r) r^2 dr . \qquad (16)$$

For the form factor $\Phi(\chi) = \frac{\operatorname{sh}\chi}{\chi} F(t)$ with the same normalization condition as $F(t)$, i.e., $\Phi(0) = F(0) = 1$ transformations (15) and (16) take the form of the conventional transformation with the zero-order Bessel functions, where, however, the relativistic coordinate is conjugated not to the momentum transfer but to the "rapidity" $\chi = \operatorname{Arch}\left(\frac{2M^2 - t}{2M^2} \right)$, corresponding to this momentum transfer.

Choosing different forms of the distribution $F(r)$, we obtain various expressions for the form factors $\Phi(\chi)$ and $F(t)$. In particular, to the localized in r - space particle $F(r) = \frac{\delta(r)}{r^2}$ (i.e., localized in the size of Compton wave length) there corresponds, according to (17), the form factor

$$F(t) = \frac{\chi}{\operatorname{sh}\chi} = 2M^2 \frac{\operatorname{Ar\,ch}\left(\frac{2M^2 - t}{2M^2}\right)}{\sqrt{t(t - 4M^2)}}, \tag{17}$$

decreasing at large $|t|$ by the law

$$F(t) \underset{|t| >> 4M^2}{\sim} 2M^2 \frac{\ln \frac{|t|}{2M^2}}{|t|}. \tag{18}$$

To the Gaussian distribution $F(r) = e^{-ar^2}$ there corresponds the form factor

$$F(t) = \frac{\pi}{a}\sqrt{\frac{\pi}{2}} \cdot \frac{\chi}{\operatorname{sh}\chi} \cdot e^{-\frac{\chi^2 M^2}{4a}} \tag{19}$$

which has been obtained earlier by analogy with the nonrelativistic form factor in [7] and whose asymptotic expression

$$F(t) \underset{|t| >> 4M^2}{\sim} \frac{\ln \frac{|t|}{2M^2}}{|t|} e^{-M^2 \ln^2 \frac{|t|}{2M^2}}$$

coincides with that of the form factors introduced in [8]. In [7,9] it has been shown that the form factors of the type of Gaussian distribution over rapidity describe, for certain values of parameters, the experimental data better than the dipole formula.

Thus, we see that the relativistic configurational representation introduced in [1] allows one to describe in the invariant way, the nucleon spatial distribution. In doing so, the nucleon mean-square radius in an arbitrary reference frame can be connected with the spatial distribution in the new coordinate space.

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