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RELATIVISTIC PHASE EQUATIONS AND NUCLEON-NUCLEON SCATTERING

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ЛАБОРАТОРИЯ ВЫЧИСЛИТЕЛЬНОЙ ТЕХНИКИ И АВТОМАТИЗАЦИИ

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RELATIVISTIC PHASE EQUATIONS AND NUCLEON-NUCLEON SCATTERING

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1. Introduction

Recently a considerable interest has arisen in the study of nucleon-nucleon scattering in the nonrelativistic energy region on the foundation of the relativistic wave equations.

For example, Fortes and Jackson [1] have recently demonstrated the importance of relativistic corrections even at low energies, using some phenomenological potentials in the framework of the Bethe-Salpeter equation in the ladder approximation (BSE).

But the complexity and the well-known drawbacks of the Bethe-Salpeter equation [2] make it desirable to use simpler wave equations to describe a relativistic two-body scattering. An equation of this type has been derived by Logunov and Tavkhelidze [3] and Kadyshevsky [4] using the quasi-potential approach to the quantum field theory.

The Kadyshevsky equation in the momentum space has been used by Dedonder [5] to describe pion- nucleus scattering and also in refs. [6,7] to describe nucleon-nucleon scattering with some field-theoretic potentials. However it is possible to examine the Kadyshevsky equation in the configuration space [8] as well. Using this equation it is not difficult to obtain the relativistic phase equations. But, in contrast to refs. [11], where the phase equations are the finite-difference nonlinear ones, we [12] have obtained ordinary nonlinear first order differential equations for the scattering amplitide and for the phase-shifts (RPE). Point out, that these equations are the analogs of the well-known nonrelativistic phase equations (NRPE) [13,14].

In this context RPE is used to describe nucleon-nucleon scattering.

Section 2 briefly reviews the Kadyshevsky equation in the configuration space. Section 3 describes the relativistic phase equations used for the numerical calculations. In section 4 we illustrate the importance of relativistic effects by comparing RPE-phase shifts with NRPE-phase shifts for different phenomenological potentials. We then compare the s -wave phase shifts obtained with the RPE-equation to the corresponding phase shifts obtained in refs. /1/ with the Bethe-Salpeter and Blankenbecler-Sugar equations (BBSE) /15/. Section 5 contains our conclusions.

2. The Kadyshevsky Equation in the Relativistic Three-Dimensional Configuration Space

As has been shown in /10/, the scattering problem of two spinless particles with the masses m_1 and m_2 and the momenta $\vec{P}_1$ and $\vec{P}_2$, respectively, is reduced to that of one effective particle having the mass $m_{(12)} = \sqrt{m_1 m_2}$ and the relative momentum $\vec{q}_{(12)}$ in the quasi-potential field. The total energy * of two particles in c.m.-system ($\vec{P}_1 = -\vec{P}_2 = \vec{P}$) is

$$W = \sqrt{\vec{P}^2 + m_1^2} + \sqrt{\vec{P}^2 + m_2^2} = \frac{m_1 + m_2}{m_{(12)}} \sqrt{m_1 m_2 + \vec{q}^{\,2}_{(12)}} . \quad (2.1)$$

Considering the scattering problem it is convenient to use the following parametrization

$$|\vec{P}_1| = m_1 \operatorname{sh} \chi_1 , \quad P_{10} = m_1 \operatorname{ch} \chi_1 ,$$

$$|\vec{P}_2| = m_2 \operatorname{sh} \chi_2 , \quad P_{20} = m_2 \operatorname{ch} \chi_2 ,$$

$$q_{(12)} \equiv |\vec{q}_{(12)}| = m_{(12)} \operatorname{sh} \chi_{(12)} , \quad q_{(12)0} = m_{(12)} \operatorname{ch} \chi_{(12)} . \quad (2.2)$$

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* We shall deal with the $\hbar = c = 1$ system of units.

Then the total energy W and the relative momentum $q_{(12)}$ may be simply expressed:

$$W = m_1 \operatorname{ch} \chi_1 + m_2 \operatorname{ch} \chi_2 = (m_1 + m_2) \operatorname{ch} \chi_{(12)} ,$$

$$m_1 \operatorname{sh} \chi_1 = m_2 \operatorname{sh} \chi_2 , \quad q_{(12)} = 2\mu \operatorname{sh} \frac{\chi_1 + \chi_2}{2} , \quad (2.3)$$

$$\mu = m_1 m_2 / (m_1 + m_2) .$$

To simplify matters, let us consider only the case of a local spherically symmetric real quasipotential $V(r)$. Then the partial wave function $\Psi_\ell^{(+)}(r, q_{(12)})$ of the continuous spectrum describing the scattering on a potential $V(r)$, satisfies the integral equation (which can be considered as a relativistic analog of the Lippmann-Schwinger equation) /8-10/:

$$\Psi_\ell^{(+)}(r, q_{(12)}) = S_\ell(r, \chi_{(12)}) + \int G_\ell^{(+)}(r, r', W)\, V(r', W)\, \Psi_\ell^{(+)}(r', q_{(12)})\, dr' . \quad (2.4)$$

Here $G_\ell^{(+)}(r, r', W)$ is the partial wave Green's function.

$$G_\ell^{(+)} = w_\ell(r') [\, e_\ell^{(1)}(r, \chi_{(12)})\, S_\ell(r', \chi_{(12)}) + D_\ell^{(+)}] ,$$

$$w_\ell(r') = - \frac{2\mu\, \nu_\ell(r')}{m_{(12)} \operatorname{sh} \chi_{(12)}} , \quad (2.5)$$

$$D_\ell^{(+)} = w_\ell(r') \{ \hat{\theta}(r' - r) [\, e_\ell^{(1)}(r', \chi_{(12)})\, S_\ell(r, \chi_{(12)}) -$$

$$- S_\ell(r', \chi_{(12)}) \, e_\ell^{(1)}(r, \chi_{(12)})] + \hat{\theta}(-r-r') \times \tag{2.6}$$

$$\times [2i S_\ell(r, \chi_{(12)}) S_\ell(r', \chi_{(12)}) - e_\ell^{(1)}(r, \chi_{(12)}) S_\ell(r', \chi_{(12)}) -$$

$$- e_\ell^{(1)}(r', \chi_{(12)}) S_\ell(r, \chi_{(12)})] \}$$

$$\nu_\ell(r) = (-1)^{\ell+1} \frac{\Gamma(-irm_{(12)}+\ell+1)\,\Gamma(irm_{(12)})}{\Gamma(-irm_{(12)})\,\Gamma(irm_{(12)}+\ell+1)}, \tag{2.7}$$

$$\hat{\theta}(r'-r) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} (e^{-iz(r'-r)m_{(12)}})/(e^{z-i\epsilon} - 1)\, dz . \tag{2.8}$$

S_ℓ, C_ℓ and $e_\ell^{(1,2)}$ are relativistic analogues of the spherical Bessel, Neumann and Hankel functions. Let us write them explicitly:

$$S_\ell = \sqrt{\frac{\pi}{2} \operatorname{sh} \chi_{(12)}}\, (-i)^{\ell+1} \frac{\Gamma(irm_{(12)}+\ell+1)}{\Gamma(irm_{(12)})} \times$$

$$\times P^{-\ell-\frac{1}{2}}_{irm_{(12)}-\frac{1}{2}}(\operatorname{ch} \chi_{(12)}), \tag{2.9}$$

$$C_\ell = \sqrt{\frac{\pi}{2} \operatorname{sh} \chi_{(12)}}\, (-i)^{-\ell} \frac{\Gamma(irm_{(12)}-\ell)}{\Gamma(irm_{(12)})} \times$$

$$\times P^{\ell+\frac{1}{2}}_{irm_{(12)}-\frac{1}{2}}(\operatorname{ch} \chi_{(12)}), \tag{2.10}$$

$$e_\ell^{(1,2)} = C_\ell \pm i S_\ell . \tag{2.11}$$

The asymptotic form of $\Psi_\ell^{(+)}(r, q_{(12)})$ at $r \to \infty$ can be found by using (2.4), (2.5), (2.6) and (2.9).

$$\Psi_\ell^{(+)}(r, q_{(12)}) \approx \sin\left(rm_{(12)} \chi_{(12)} - \frac{\pi\ell}{2}\right) + f_\ell(W)\, e^{i(rm_{(12)}\chi_{(12)} - \frac{\pi\ell}{2})} \tag{2.12}$$

where

$$f_\ell(W) = - \frac{2\mu}{m_{(12)} \operatorname{sh} \chi_{(12)}} \int S_\ell^*(r, \chi_{(12)}) \times V(r, W)\, \Psi_\ell^{(+)}(r, q_{(12)})\, dr . \tag{2.13}$$

is the partial scattering amplitude.

It is connected with the phase shifts δ_ℓ and with the s-matrix element s_ℓ by the simple relation hips:

$$f_\ell(W) = \frac{s_\ell(W) - 1}{2i} = e^{i\delta_\ell(W)} \sin\delta_\ell(W). \tag{2.14}$$

The total elastic scattering cross-section is expressed by $f_\ell(W)$ and $\delta_\ell(W)$

$$\sigma = \int \frac{d\sigma}{d\Omega} = 4\pi C(m_1, m_2, W^2) \frac{1}{q^2_{(12)}} \sum_{\ell=0}^{\infty} (2\ell+1)|f_\ell(W)|^2 =$$

$$= 4\pi C(m_1, m_2, W^2) \frac{1}{q_{(12)}} \sum_{\ell=0}^{\infty} (2\ell+1) \sin^2 \delta_\ell (W), \tag{2.15}$$

where

$$C(m_1, m_2, W^2) = 2\mu/(m_1+m_2)(1-(m_1-m_2)^2/W^2).$$

3. The Relativistic Phase Equations

First of all we should remember that in the variable phase approach to nonrelativistic potential scattering /13,14/ the functions are introduced, which satisfy simple non-linear first order differential equations, and those asymptotic values give directly the values of the scattering amplitude, the s_ℓ -matrix element or the phase shift.

To obtain these equations it is essential, that the asymptotic value of the nonrelativistic wave function should begin where the potential is "cut off". On the other hand, the asymptotic of the relativistic wave function begins only at $r \to \infty$ (see Appendix). Thus for the relativistic partial scattering amplitude, on the basis of (2.4), we obtain the system of the equations /12/:

$$\frac{d}{dr} f_\ell(W,r) = w_\ell(r) V(r,W) [\Phi_\ell^{(1)}(r, q_{(12)}) +$$

$$+ f_\ell(W,r) \Phi_\ell^{(2)}(r, q_{(12)})]^2,$$

$$\Phi_\ell^{(1)} = S_\ell + \int_0^r D_\ell^{(+)}(r,r',W) V(r',W) \Phi_\ell^{(1)}(r', q_{(12)}) dr',$$

$$\Phi_\ell^{(2)} = e_\ell^{(1)} + \int_0^r D^{(+)}(r,r',W) V(r',W) \Phi_\ell^{(2)}(r', q_{(12)}) dr'. \tag{3.1}$$

However, as it was shown in /12/ in the example of the exactly soluble model with a square well potential /9/ the sufficiently accurate approximation to (3.1) is the following equation:

$$\frac{d}{dr} f_\ell(W,r) = w_\ell(r) V(r,W)[S_\ell + f_\ell e_\ell^{(1)}]^2. \tag{3.2}$$

It is possible to derive the analogous equation for the phase shift by using (2.14). At $\ell = 0$ it looks quite simple

$$\frac{d}{dr} \delta_0(W,r) = -\frac{2\mu}{q_{(12)}} V(r) \sin^2[r m_{(12)} \chi_{(12)} + \delta_0(r,W)]. \tag{3.3}$$

It is not difficult to show that in the nonrelativistic limit ($r m_{(12)} \gg 1$, $\chi_{(12)} \ll 1$) we obtain from (3.3) the well-known phase equation /13, 14/

$$\frac{d}{dr} \delta_0(r,k) = -\frac{2\mu}{k} V(r) \sin^2(kr + \delta_0(r)), \tag{3.4}$$

where

$$k = \sqrt{2\mu E}.$$

4. Results

In our calculations we used the ordinary Yukawa potential

$$V^{NK}(r) = -\lambda \exp(-\sigma r)/r, \tag{4.1}$$

(or the superposition of (4.1)) for the NRPE-equation (3.4) and a relativistic analog of the Yukawa potential /8/

$$
V^{R}(r) = \begin{cases} -\lambda \dfrac{1}{4\pi r} \dfrac{\mathrm{ch}[r m_{(12)} \operatorname{Arcos}(\sigma^2 - 2m^2_{(12)})/m^2_{(12)}]}{\sin(\pi r m_{(12)})} & \text{if } \sigma^2 < 4m^2_{(12)} \\[2ex] -\lambda \dfrac{1}{4\pi r} \dfrac{\cos[r m_{(12)} \operatorname{Arch}(\sigma^2 - 2m^2_{(12)})/m^2_{(12)}]}{\mathrm{sh}(\pi r m_{(12)})} & \text{if } \sigma > 4m^2_{(12)} \end{cases} \tag{4.2}
$$

(or the superposition of (4.2)) for the RPE-equation (3.3). Let us emphasize at once the fact that $V^R(r)$ is more singular than the Yukawa potential ($V^R(r) \sim r^{-2}$ when $r \to 0$).

Both (3.3) and (3.4) equations were solved numerically by means of the Runge-Kutt method with $m_1 = m_2 = 4{,}758\ \mathrm{fm}^{-1}$. In Fig. 1. we plot the phase shifts as a functions of laboratory energy for the attractive potential with $\lambda = 0.54583$ and $\sigma = 0.7\ \mathrm{fm}^{-1}$.

We note that the phase shifts calculated with the RPE are generally larger than those calculated with NRPE even at the laboratory energy which might be well considered "non-relativistic". The average difference in the phase shifts is 0.4%. The direction of the relativistic effect is quite different from that of Fortes and Jackson /1/ found in their comparison of the Lippmann-Schwinger phase shift with the BSE and BBSE phase shifts.

The phase shifts (a) and (b) in Fig. 2 represent a similar calculation for the $^1S_0$ soft-core potential of Reid /16/, but (c) and (d) are taken from refs. /1/. We can see that σ_{NRPE} is larger than δ_{RPE} by 19.8%, 26.5% and 46.4% at 25, 105 and 185 MeV, respectively, while δ_{BSE} and δ_{BBSE} are smaller than δ_{NRPE} at an average by 30%.

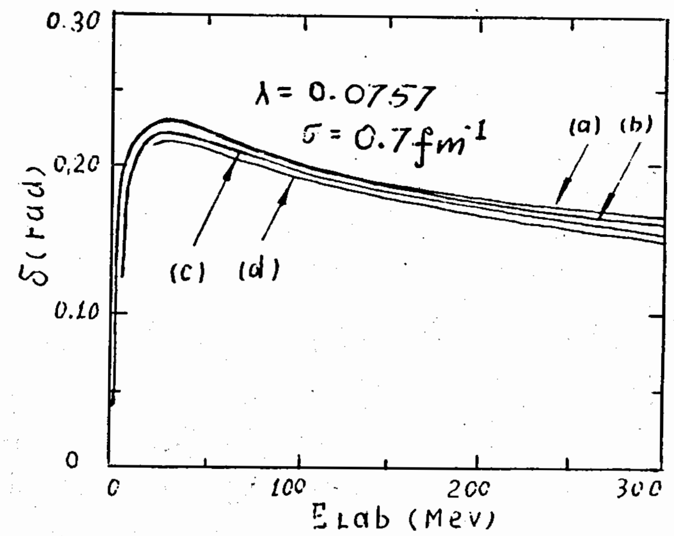

Fig. 1. S-wave phase shifts as a function of lab. energy for one-Yukawa attractive potential. Curves (a) and (b) correspond to RPE and NRPE, respectively, while curve (c) is the BBSE phase shifts and curve (d) is the BSE phase shifts from ref. /1/.

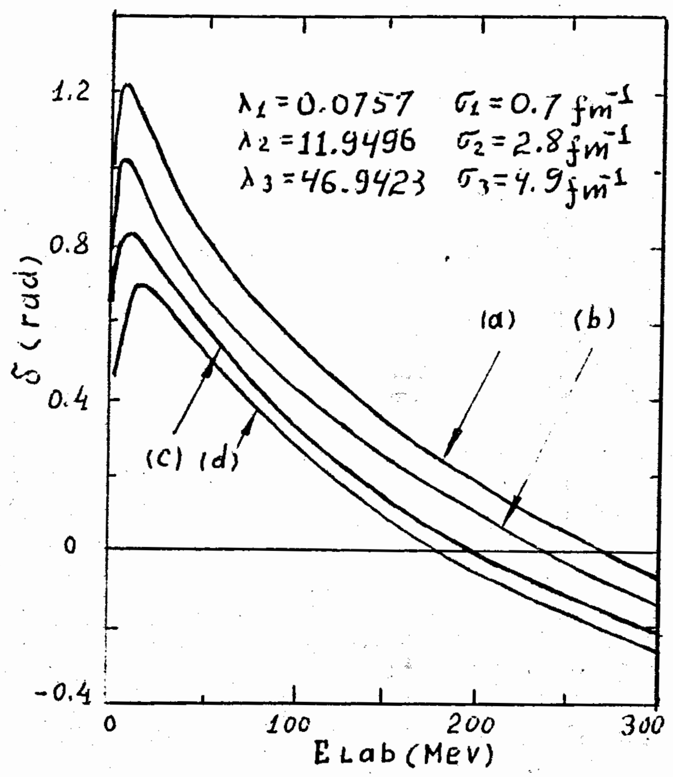

Fig. 2. Phase shifts calculated with the parameters of Reid's soft-core potential in the $^1S_0$ channel. Curves (a) and (b) correspond to RPE and NRPE, respectively, while curve (c) is the BBSE phase shifts and curve (d) is the BSE phase shifts from ref. /1/.

Finally, Fig. 3 represents our results for the phase shifts versus energy for the $^1S_0$ soft-core potential of Malfliet and Tjon /17/. For this potential again δ_{RPE} is larger than δ_{NRPE} and differs by 5.98%, 11, 59% and 33.09% at 25, 105 and 185 MeV, respectively. It is seen that the curves (a) and (b) (see Fig. 2 and Fig. 3) describe the experimental data well /18/.

Conclusion

Relativistic generation of the variable phase approach given in /12/ on the foundation of the quasi-potential approach, has many important features common to the non-relativistic potential theory. The types of phase equations have much in common. They are simple differential equations of the first order and they can be solved numerically with any parameters. In order to illustrate the method which is developed in this article, the solutions are given here of some problems of nucleon-nucleon scattering in the most simple case $\ell = 0$. For this purpose equation (3.3) has been numerically solved. This equation is a first order approximation to the system of relativistic phase equations. The results of computation were compared with the corresponding data of other studies in this field. The fact that relativistic phases of scattering for all values of energy are larger than the non-relativistic phases can be explained by the great power of relativistic potential of interaction as compared with the non-relativistic ones. It has been shown (see Appendix) that in principle it is not difficult to achieve sufficiently accurate results for the relativistic phase equations. Here it should be pointed out that consistency in expressions for relativistic wave function (A.2) in solving the problems of scattering can give favourable results not only in making the relativistic phase equations more accurate but also in the deeper understanding of the interdependence of the differential and integral formulations of the Kadyshevsky equation in the configuration space.

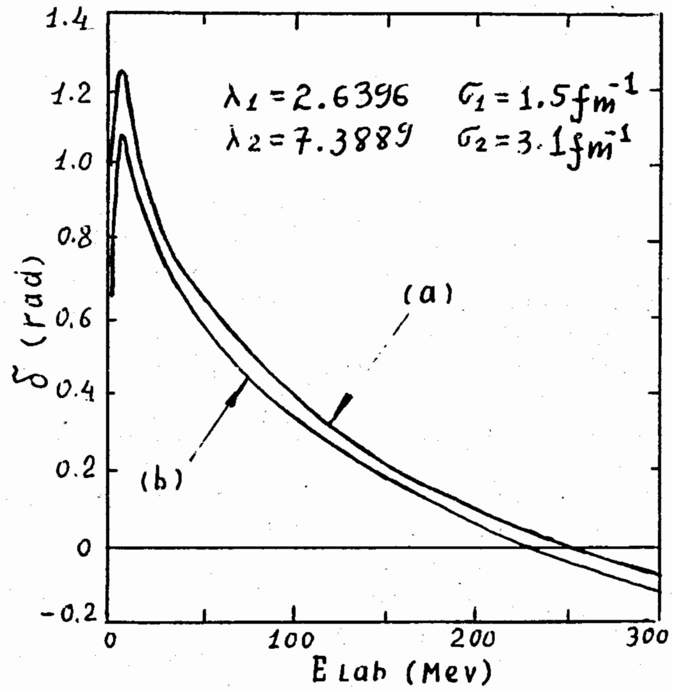

Fig. 3. Phase shifts as a function of lab. energy for the parameters of the $^1S_0$ soft-core potential of Malfliet and Tjon. Curves (a) and (b) correspond to RPE and NRPE, respectively.

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Appendix

Let us investigate formula (2.4) for the purpose of finding some differences between relativistic and non-relativistic asymptotics of wave function. For this, using the explicit form of $\hat{\theta}$-function (to simplify matters here $m_{(12)} = 1$)*

$$\hat{\theta}(r - r') = 1/(1 - e^{-2\pi(r-r')}) = \sum_{n=0}^{\infty} e^{-2\pi n(r-r')} , \quad (A.1)$$

we rewrite equation (2.4) in the following form

$$\psi_\ell^{(+)}(r, q_{(12)}) = S_\ell(r, \chi_{(12)}) + f_\ell(W)\, e_\ell^{(1)}(r, \chi_{(12)}) +$$

$$+ \sum_{n=0}^{\infty} \{ S_\ell(r, \chi_{(12)}) [\phi_{n\ell}^{(1)}(r) + \phi_{n\ell}^{(3)}(r) + e^{-2\pi(n+1)r} A_{n\ell}^{(1)}] +$$

$$+ e_\ell^{(1)}(r, \chi_{(12)}) [\phi_{n\ell}^{(2)}(r) + \phi_{n\ell}^{(4)}(r) + e^{-2\pi(n+1)r} A_{n\ell}^{(2)}] \}, \quad (A.2)$$

where

$$\phi_{n\ell}^{(1)}(r) = e^{-2\pi(n+1)r} \int_0^r dr' \, e^{2\pi(n+1)r'} e_\ell^{(1)}(r', \chi_{(12)}) \times$$

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* Representation of $\hat{\theta}$ -function from (2.4) in a form of infinite sum (A.1) is valid for all r, r' except $r' \neq r$. However, it may be shown, that this limitation is not essential because $D(r, r', \chi_{(12)})$ has no singularities when $r = r'$.

$$\times W_\ell(r')\,\Psi_\ell^{(+)}(r',q_{(12)}),$$

$$\phi^{(2)}_{n\ell}(r) = e^{-2\pi(n+1)r}\int_0^r dr'\, e^{2\pi(n+1)r'} \times$$

$$\times S_\ell(r',\chi_{(12)})\, W_\ell(r')\,\Psi_\ell^{(+)}(r',q_{(12)}),$$

$$\phi^{(3)}_{n\ell}(r) = -e^{2\pi n r}\int_r^\infty dr'\, e^{-2\pi n r'}\, e^{(1)}_\ell(r',\chi_{(12)}) \times$$

$$\times W_\ell(r')\,\Psi_\ell^{(+)}(r',q_{(12)}), \qquad (A.3)$$

$$\phi^{(4)}_{n\ell}(r) = e^{2\pi n r}\int_r^\infty dr'\, e^{-2\pi n r'}\, S_\ell(r',\chi_{(12)}) \times$$

$$\times W_\ell(r')\,\Psi_\ell^{(+)}(r',q_{(12)}),$$

$$A^{(1)}_{n\ell} = \int_0^\infty dr'[\,2iS_\ell(r',\chi_{(12)}) - e^{(1)}_\ell(r',\chi_{(12)})\,] \times$$

$$\times e^{-2\pi(n+1)r'}\, W_\ell(r')\,\Psi_\ell^{(+)}(r',q_{(12)}),$$

$$A^{(2)}_{n\ell} = \int_0^\infty dr'\, S_\ell(r',\chi_{(12)})\, e^{-2\pi(n+1)r'} \times$$

$$\times W_\ell(r')\,\Psi_\ell^{(+)}(r',q_{(12)});$$

$$W_\ell(r) = (\nu_\ell(r)\, V(r))/\mathrm{sh}\,\chi_{(12)}. \qquad (A.4)$$

In potential "cut off" point $(r=\lambda)$ we obtain from (A.2)

$$\Psi_\ell^{(+)}(\lambda, q_{(12)}) = S_\ell(\lambda,\chi_{(12)}) + f_\ell\, e^{(1)}_\ell(\lambda,\chi_{(12)}) +$$

$$+ \sum_{n=0}^{\infty} e^{-2\pi(n+1)\lambda}\,[\,B^{(1)}_{n\ell}\, S_\ell + B^{(2)}_{n\ell}\, e^{(1)}_\ell\,],$$

Here

$$B^{(1)}_{n\ell} = A^{(1)}_{n\ell} + \int_0^\lambda dr\, e^{(1)}_\ell(r,\chi_{(12)})\, e^{2\pi r(n+1)} \times$$

$$\times W_\ell(r)\,\Psi_\ell^{(+)}(r, q_{(12)}),$$

$$B^{(2)}_{n\ell} = A^{(2)}_{n\ell} - \int_0^\lambda dr\, S_\ell(r,\chi_{(12)})\, e^{2\pi r(n+1)} \times$$

$$\times W_\ell(r)\,\Psi_\ell^{(+)}(r, q_{(12)}).$$

It follows from here that for nonambiguous setting of asymptotics of relativistic wave function $\Psi_\ell^{(+)}$ one should know not only scattering amplitude $f_\ell(W)$ but also all the constants $B^{(1)}_{n\ell}$ and $B^{(2)}_{n\ell}$ at $n, e = 1,2,\ldots,\infty$. And only at $r \to \infty$

$$\Psi_\ell^{(+)}(r, q_{(12)}) \approx S_\ell(r,\chi_{(12)}) + f_\ell(W)\, e^{(1)}_\ell(r,\chi_{(12)}).$$

Let us point out one of the opportunities of finding solution of relativistic equation (2.4).

Having differentiated equations (A.3) over r we have

$$\frac{d}{dr}\phi^{(1)}_{n\ell} = -2\pi(n+1)\,\phi^{(1)}_{n\ell} +$$

$$+ W_\ell(r)\, e^{(1)}_\ell(r, \chi_{(12)})\, \Psi^{(+)}_\ell(r, q_{(12)}),$$

$$\frac{d}{dr}\phi^{(2)}_{n\ell} = 2\pi(n+1)\,\phi^{(2)}_{n\ell} + \tag{A.5}$$

$$+ W_\ell(r)\, S_\ell(r, \chi_{(12)})\, \Psi^{(+)}_\ell(r, q_{(12)}),$$

$$\frac{d}{dr}\phi^{(3)}_{n\ell} = -2\pi n\,\phi^{(3)}_{n\ell} +$$

$$+ W_\ell(r)\, e^{(1)}_\ell(r, \chi_{(12)})\, \Psi^{(+)}_\ell(r, q_{(12)}),$$

$$\frac{d}{dr}\phi^{(4)}_{n\ell} = 2\pi n\,\phi^{(4)}_{n\ell} + W_\ell(r)\, S_\ell(r, \chi_{(12)})\, \Psi^{(+)}_\ell(r, q_{(12)}).$$

Sunstituting (A.2) into (A.5), (A.4) and (2.13) we obtain a system of coupled differential and algebraic equations for function in search $\phi^{(1)}_{n\ell}$, $\phi^{(2)}_{n\ell}$, $\phi^{(3)}_{n\ell}$, $\phi^{(4)}_{n\ell}$, $A^{(1)}_{n\ell}$, $A^{(2)}_{n\ell}$ and $f_\ell(W)$. Thus, we have substituted equation (2.4) by an infinite system of equations. Being limited in expansion of (A.1) by N terms we will get, according to the above scheme, the finite system of $7N$ equations, which is easily integrated numerically and might appear to be useful for the qualitative study of relativistic equations.