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THE ISOSPIN BOUNDS AND PHASE CONTOURS IN PION-NUCLEON SCATTERING

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Изоспиновые ограничения и фазовые контуры в пион-нуклонном рассеянии

В работе изучаются изоспиновые ограничения на дифференциальные и интегральные (поляризованные и неполяризованные) сечения пион-нуклонного рассеяния. Определяя интегральное сечение $\bar{\Sigma}^{(n)}$ и используя классические неравенства Минковского и Гелдера, получаем широкий класс изоспиновых ограничений. Насыщение ограничений изучено с использованием предсказаний, полученных из дисперсионных соотношений и теоретических фазовых сдвигов, рассчитанных в ЦЕРНе.

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The Isospin Bounds and Phase Contours in Pion-Nucleon Scattering

In this paper we investigate the isospin constraints on differential and integrated (unpolarized and polarized) cross-sections in the pion-nucleon scattering. Defining $\bar{\Sigma}^{(n)}$ -integrated cross-sections and using classical Minkowski's and Hölder's inequalities we obtain a large class of isospin bounds. The saturation of the isospin bounds is investigated using the dispersion relation predictions and CERN-theoretic solution for the phase shifts.

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Introduction

The isospin bounds for differential (unpolarized or polarized) cross-sections have been investigated by many authors [1-7]. It was suggested by Höhler et al. [4] that the exact saturation of the isospin bounds is in particular connected with the zeros of different scattering amplitudes and $0, \pi$ -phase contours. On the other hand in ref.[7] we have investigated the most stringent isospin bounds on polarization and spin rotation parameters using a set of bilinear forms which can be constructed with the scatterring amplitudes for two reactions connected by the isospin invariance. Then, we have obtained that the isospin invariance implies much more restrictive conditions than those derived by Doncel et al. [7] and that the exact saturation of the isospin bounds are strictly connected with the zeros of the real and imaginary part of these bilinear forms.

In this paper, in Sect. 2 we investigate the isospin constraints on the differential (polarized and unpolarized) cross-sections, in a similar way, defining the bilinear forms $M_{ij}(\Sigma)$ (12.a.b.c). The saturation of the isospin bounds in terms of the $[\mathrm{Re}\,M_{ij}(\Sigma), \mathrm{Im}\,M_{ij}(\Sigma)]$-zero trajectories or $(n\pi;(n+\frac{1}{2})\pi)$ -phase contours is discussed in Sect. 3. In Sect. 4 we define $\bar{\Sigma}^{(n)}$ -integrated cross-sections and we obtain a large class of isospin bounds using the results of Sect. 2 and classical Minkowski's and Hölder's inequalities.

The saturation of the isospin bounds, using the dispersion relation predictions [9] and CERN-theoretic solution [10] for phase shifts are presented in Figs. (1-8).

2. The Isospin Inequalities for Differential (Polarized and Unpolarized) Cross-Sections

In order to discuss the isospin inequalities for differential (polarized and unpolarized) cross-sections in the pion-nucleon scattering, in a systematical way, we introduce the following scattering amplitudes:

$$\vec{G}_i \equiv f_i \vec{m} + g_i \vec{\ell} \quad ; \quad \vec{G}'_i \equiv f_i^{++} \vec{m}' + f_i^{+-} \vec{\ell}' , \tag{1a}$$

$$K_i^{(\pm)} \equiv f_i \pm i g_i \quad ; \quad K_i'^{(\pm)} \equiv f_i^{++} \pm i f_i^{+-} , \tag{1b}$$

$$H_i^{(\pm)} \equiv f_i \pm g_i \quad ; \quad H_i'^{(\pm)} \equiv f_i^{++} \pm f_i^{+-} , \tag{1c}$$

where $\vec{m}$, $\vec{\ell}$ and $\vec{m}'$, $\vec{\ell}'$ are orthogonal unit vectors, f_i and g_i are the usual pion-nucleon spin-non-flip and spin-flip scattering amplitudes; f_i^{++} and f_i^{+-} are the helicity non-flip and helicity-flip scattering amplitudes, respectively. The indices i refer to the charge (+,−, CE) or to s,t,u -isospin ($i=2I_s, 2I_t, 2I_u$) channels.

In terms of these amplitudes we can write

$$\sigma_i \equiv \frac{d\sigma_i}{d\Omega} = \vec{G}_i^* \cdot \vec{G}_i = \vec{G}_i'^* \vec{G}_i' \tag{2a}$$

$$(1 \pm P_i)\sigma_i = |K_i^{(\pm)}|^2 = |K_i'^{(\pm)}|^2 , \tag{2b}$$

$$(1 \pm T_i)\sigma_i = |H_i^{(\pm)}|^2 , \tag{2c}$$

$$(1 \pm A_i)\sigma_i = |H_i'^{(\pm)}|^2 , \tag{2c'}$$

$$(1 + S_i)\sigma_i = 2|f_i|^2 ; \quad (1 - S_i)\sigma_i = 2|g_i|^2 , \tag{2d}$$

$$(1 + R_i)\sigma_i = 2|f_i^{++}|^2 ; \quad (1 - R_i)\sigma_i = 2|f_i^{+-}|^2 . \tag{2d'}$$

The results obtained in this section may be conveniently summarized by the following inequalities.

Inequalities 1. Let $\Sigma_i = \sigma_i, (1 \pm X_i)\sigma_i$, $X_i = P_i, T_i, S_i$ or (P_i, A_i, R_i) be the differential (unpolarized and polarized) cross-sections, $i = +, -, CE$. The most weak isospin bounds on Σ_i implied by the isospin invariance are

$$0 \le \Sigma_+ \le 3(\Sigma_- + \Sigma_{CE}) , \tag{3a}$$

$$0 \le \Sigma_- \le 3(\Sigma_+ + \Sigma_{CE}) , \tag{3b}$$

$$0 \le \Sigma_{CE} \le \Sigma_+ + \Sigma_- . \tag{3c}$$

These inequalities are derived from s,t,u -isospin channel decomposition and positivity conditions for Σ_i , $i = 2I_s, 2I_t, 2I_u$.

Inequalities 2. The isospin invariance alone implies that the differential cross-sections for πN -scattering at any c.m. energy $\sqrt{s}$ and any c.m. angle θ, must obey the inequalities:

$$(\Sigma_+ + \Sigma_- - 2\Sigma_{CE})^2 \le 4\Sigma_+ \Sigma_- , \tag{4a}$$

$$(\Sigma_+ - \Sigma_- + 2\Sigma_{CE})^2 \le 8\Sigma_+ \Sigma_{CE} , \tag{4b}$$

$$(\Sigma_+ - \Sigma_- - 2\Sigma_{CE})^2 \le 8\Sigma_- \Sigma_{CE} , \tag{4c}$$

$$(\Sigma_+ + 3\Sigma_- - 6\Sigma_{CE})^2 \le 8\Sigma_+ (3\Sigma_- + 3\Sigma_{CE} - \Sigma_+) , \tag{4d}$$

$$(\Sigma_- - \Sigma_+)^2 \le 4\Sigma_{CE}(\Sigma_+ + \Sigma_- - \Sigma_{CE}), \quad (4e)$$

$$(\Sigma_- + 3\Sigma_+ - 6\Sigma_{CE})^2 \le 8\Sigma_-(3\Sigma_+ + 3\Sigma_{CE} - \Sigma_-), \quad (4f)$$

which are all equivalent to

$$-\lambda(\Sigma_+, \Sigma_-, 2\Sigma_{CE}) \ge 0 \quad (5)$$

and also to

$$(\Sigma_+)^{1/2} \le (\Sigma_-)^{1/2} + (2\Sigma_{CE})^{1/2}, \quad (6a)$$

$$(\Sigma_-)^{1/2} \le (\Sigma_+)^{1/2} + (2\Sigma_{CE})^{1/2}, \quad (6b)$$

$$(2\Sigma_{CE})^{1/2} \le (\Sigma_+)^{1/2} + (\Sigma_-)^{1/2}, \quad (6c)$$

or

$$[(\Sigma_+)^{1/2} - (\Sigma_-)^{1/2}]^2 \le 2\Sigma_{CE} \le [(\Sigma_+)^{1/2} + (\Sigma_-)^{1/2}]^2, \quad (7)$$

where

$$\lambda(x, y, z) \equiv x^2 + y^2 + z^2 - 2xy - 2xz - 2yz. \quad (8)$$

These inequalities follow from the triangle inequalities applied to the scattering amplitudes: $\vec{G}_i$, $K_i^{(\pm)}$, $H_i^{(\pm)}$, $\sqrt{2} f_i$, $\sqrt{2} g_i$ and (or) $\vec{G}'_i$, $K_i'^{(\pm)}$, $H_i'^{(\pm)}$, $\sqrt{2} f_i^{+-}$, $i = +, -, CE$.

Now, from the inequalities (4a,b,c,d,e,f) for $\Sigma_i^{(+)} \equiv (1 \pm X_i)\sigma_i$ combined with those for $\Sigma_i^{(-)} \equiv (1 \pm X_i)\sigma_i$ we obtain the inequalities 3.

Inequalities 3.

$$(\sigma_+ + \sigma_- - 2\sigma_{CE})^2 + (X_+\sigma_+ + X_-\sigma_- - 2X_{CE}\sigma_{CE})^2 \le 4(1 + X_+X_-)\sigma_+\sigma_-, \quad (9a)$$

$$(\sigma_+ - \sigma_- + 2\sigma_{CE})^2 + (X_+\sigma_+ - X_-\sigma_- + 2X_{CE}\sigma_{CE})^2 \le 8(1 + X_+X_{CE})\sigma_+\sigma_{CE}, \quad (9b)$$

$$(\sigma_+ - \sigma_- - 2\sigma_{CE})^2 + (X_+\sigma_+ - X_-\sigma_- - 2X_{CE}\sigma_{CE})^2 \le 8(1 + X_-X_{CE})\sigma_-\sigma_{CE}, \quad (9c)$$

$$(\sigma_+ + 3\sigma_- - 6\sigma_{CE})^2 + (X_+\sigma_+ + 3X_-\sigma_- - 6X_{CE}\sigma_{CE})^2 \le$$
$$\le 8(1 + X_+X_{1s})\sigma_+(3\sigma_- + 3\sigma_{CE} - \sigma_+), \quad (9d)$$

$$(\sigma_- - \sigma_+)^2 + (X_-\sigma_- - X_+\sigma_+)^2 \le 4(1 + X_{CE}X_{0t})\sigma_{CE}(\sigma_+ + \sigma_- - \sigma_{CE}), \quad (9e)$$

$$(\sigma_- + 3\sigma_+ - 6\sigma_{CE})^2 + (X_-\sigma_- + 3X_+\sigma_+ - 6X_{CE}\sigma_{CE})^2 \le$$
$$\le 8(1 + X_-X_{1u})\sigma_-(3\sigma_+ + 3\sigma_{CE} - \sigma_-), \quad (9f)$$

where

$$X_{1s} = \frac{3X_-\sigma_- + 3X_{CE}\sigma_{CE} - X_+\sigma_+}{3\sigma_- + 3\sigma_{CE} - \sigma_+}, \quad (10a)$$

$$X_{0t} = \frac{X_+\sigma_+ + X_-\sigma_- - X_{CE}\sigma_{CE}}{\sigma_+ + \sigma_- - \sigma_{CE}}, \quad (10b)$$

$$X_{1u} = \frac{3X_+\sigma_+ + 3X_{CE}\sigma_{CE} - X_-\sigma_-}{3\sigma_+ + 3\sigma_{CE} - \sigma_-}. \quad (10c)$$

The isospin bounds (9a,b,c,d,e,f) are all equivalent to

$$\lambda(X_+\sigma_+, X_-\sigma_-, 2X_{CE}\sigma_{CE}) \le -\lambda(\sigma_+, \sigma_-, 2\sigma_{CE}). \quad (11)$$

Next, if we define

$$M_{ij}(\sigma) \equiv \vec{G}_i^* \cdot \vec{G}_j = \vec{G}_i'^* \cdot \vec{G}_j , \quad (12a)$$

where $\vec{G}_i^* \vec{G}_j$ is the scalar product of the scattering amplitudes $\vec{G}_i^*$ and $\vec{G}_j$

$$M_{ij}(\Sigma^{(\pm)}) \equiv [F_i^{(\pm)}]^* F_j^{(\pm)} \quad (12b)$$

$$F^{(\pm)} \equiv K^{(\pm)}, H^{(\pm)} (\sqrt{2} f, \sqrt{2} g); K'^{(\pm)}, H'^{(\pm)}, (\sqrt{2} f^{++}, \sqrt{2} f^{+-}) \quad (12c)$$

then the isospin invariance implies the following equalities:

$$\Sigma_+ \Sigma_- - [\mathrm{Re}\, M_{+-}(\Sigma)]^2 = 2\{\Sigma_+ \Sigma_{CE} - [\mathrm{Re} M_{+CE}(\Sigma)]^2\} =$$

$$= 2\{\Sigma_- \Sigma_{CE} - [\mathrm{Re}\, M_{-CE}(\Sigma)]^2\} = \frac{4}{9}\{\Sigma_{1s}\Sigma_{3s} - [\mathrm{Re} M_{13s}(\Sigma)]^2\} =$$

$$= 4\{\Sigma_{0t}\ \Sigma_{2t} - [\mathrm{Re} M_{02t}(\Sigma)]^2\} = \frac{4}{9}\{\Sigma_{1u}\ \Sigma_{3u} - [\mathrm{Re}\, M_{13u}(\Sigma)]^2\} =$$

$$= -\frac{1}{4}\lambda(\Sigma_+, \Sigma_-, 2\Sigma_{CE}) \geq 0, \quad (13)$$

for any $\Sigma = \sigma$, $(1 \pm X)\sigma$.
Now, from the condition $[\mathrm{Re} M_{ij}(\Sigma)]^2 \geq 0$ we obtain

$$-\lambda(\Sigma_+, \Sigma_-, 2\Sigma_{CE}) \leq A(\Sigma), \quad (14a)$$

where

$$A(\Sigma) \equiv 4 \min\{\Sigma_+\Sigma_-, 2\Sigma_+ \Sigma_{CE}, 2\Sigma_-\Sigma_{CE}, \frac{4}{9}\Sigma_+\Sigma_{1s},$$
$$4\Sigma_{0t}\Sigma_{2t}, \frac{4}{9}\Sigma_{1u}\Sigma_-\} \quad (14b)$$

and

$$\Sigma_{1s} = \frac{1}{2}[3\Sigma_- + 3\Sigma_{CE} - \Sigma_+] \geq 0, \quad \Sigma_{3s} = \Sigma_+, \quad (15a)$$

$$\Sigma_{0t} = \frac{1}{2}[\Sigma_+ + \Sigma_- - \Sigma_{CE}] \geq 0, \quad \Sigma_{2t} = \frac{1}{2}\Sigma_{CE}, \quad (15b)$$

$$\Sigma_{1u} = \frac{1}{2}[3\Sigma_+ + 3\Sigma_{CE} - \Sigma_-] \geq 0, \quad \Sigma_{3u} = \Sigma_- . \quad (15c)$$

Hence, combining (14a) with (5) we obtain

Inequalities 2'

$$0 \leq -\lambda(\Sigma_+, \Sigma_-, 2\Sigma_{CE}) \leq A(\Sigma), \quad (16)$$

valid at any energy and any c.m. scattering angle.
Next, from the conditions

$$0 \leq [\mathrm{Re} M_{ij}(\sigma)]^2 \leq |M_{ij}(\sigma)|^2 = \frac{1}{2}[1 + \vec{P}_i \cdot \vec{P}_j']\sigma_i \sigma_j, \quad (17)$$

we obtain the most stringent lower bound on the triangle function λ /7/.

Inequalities 4

$$4H_{+-} \leq -\lambda(\sigma_+, \sigma_-, 2\sigma_{CE}) \leq A(\sigma), \quad (18a)$$

where

$$H_{ij} \equiv \frac{1}{2}(1 - \vec{P}_i \cdot \vec{P}_j)\sigma_i \sigma_j \geq 0, \quad (18b)$$

$$\vec{P}_i \vec{P}_j \equiv P_i P_j + A_i A_j + R_i R_j = P_i P_j + T_i T_j + S_i S_j = \ldots \quad (18c)$$

The isospin invariance alone implies that /6/

$$H_{+-} = 2H_{+CE} = 2H_{-CE} = \frac{4}{9}H_{13u} = 4H_{02t} = \frac{4}{9}H_{13s} . \quad (19)$$

The lower isospin bound (18a) can be written in the following equivalent forms:

$$[\sigma_+ +\sigma_- -2\sigma_{CE}]^2 \le 2(1+\vec{P}_+\vec{P}_-)\sigma_+\sigma_- \le 4\sigma_+\sigma_- , \quad (20a)$$

$$[\sigma_+ -\sigma_- +2\sigma_{CE}]^2 \le 4(1+\vec{P}_+\vec{P}_{CE})\sigma_+\sigma_{CE} \le 8\sigma_+\sigma_{CE}, \quad (20b)$$

$$[\sigma_+ -\sigma_- -2\sigma_{CE}]^2 \le 4(1+\vec{P}_-\vec{P}_{CE})\sigma_-\sigma_{CE} \le 8\sigma_-\sigma_{CE}, \quad (20c)$$

$$[\sigma_+ +3\sigma_- -6\sigma_{CE}]^2 \le 8(1+\vec{P}_+\vec{P}_{1s})\sigma_+\sigma_{1s} \le 16\sigma_+\sigma_{1s}, \quad (20d)$$

$$[\sigma_- -\sigma_+]^2 \le 4(1+\vec{P}_{CE}\vec{P}_{0t})\sigma_{CE}\sigma_{0t} \le 8\sigma_{CE}\sigma_{0t}, \quad (20e)$$

$$[\sigma_- +3\sigma_+ -6\sigma_{CE}]^2 \le 8(1+\vec{P}_-\vec{P}_{1u})\sigma_-\sigma_{1u} \le 16\sigma_-\sigma_{1u}. \quad (20f)$$

Therefore, assuming the isospin invariance, we have improved the upper bounds (4e), (9e) and (20e) on the difference between $\pi^- P$ and $\pi^+ P$ differential (polarized and unpolarized) elastic cross-sections. These results enable us to understant the small differences between elastic differential cross-sections, at high energies for fixed momentum transfer, in terms of the small charge-exchange differential cross-sections.

The isospin bounds on differential cross-sections can be investigated from experimental data, or using the available amplitude analyses, in many equivalent forms. In particular, one can use the Tórnquist-type diagrams [1] or other equivalent diagrams defining the complex numbers $Z_{ij}(\Sigma)$ by the relations:

$$\mathrm{Re}\,Z_{ij}(\Sigma) = \frac{\Sigma_i - \Sigma_j}{\Sigma_i + \Sigma_j}, \quad \mathrm{Im}\,Z_{ij}(\Sigma) = \frac{2\,\mathrm{Re}\,M_{ij}(\Sigma)}{\Sigma_i + \Sigma_j}, \quad (21)$$

where $M_{ij}(\Sigma)$ is defined by (12a) and (12b). Then, the data are represented, for each (s,t) -value, by a point inside of a circle $|Z_{ij}| \le 1$. The lower bound (16) is saturated when $|Z_{ij}| = 1$, and the upper bounds for $\mathrm{Re}\,M_{ij}(\Sigma) = 0$.

Also, we can use the isospin bounds for $\mathrm{Re}\,Z_{ij}$ (see Fig. 1). In this case we see that the isospin bounds

$$(\Sigma_j)_{min} \le \Sigma_j \le (\Sigma_j)_{max} \quad (22a)$$

are equivalent to:

$$a_{ij} \le \mathrm{Re}\,Z_{ij}(\Sigma) \le b_{ij}, \quad (22b)$$

where

$$a_{ij} = \frac{\Sigma_i - (\Sigma_j)_{max}}{\Sigma_i + (\Sigma_j)_{max}}, \quad b_{ij} = \frac{\Sigma_i - (\Sigma_j)_{min}}{\Sigma_i + (\Sigma_j)_{min}}. \quad (22c)$$

The quantities $\mathrm{Re}\,Z_{ij}(\Sigma)$ and their isospin bounds a_{ij}, b_{ij} corresponding to the zeros of the s,t,u -channel isospin-exchange amplitudes are given in Table I. Therefore, these forms of presentation of the isospon bounds are in particular connected with the determination of: (i) the zeros-positions of different isospin-exchange amplitudes, (ii) the crossover points $\Sigma_i = \Sigma_j$, (iii) the conditioned average quantum numbers [8] and its correlation with the relative phases of two scattering amplitudes and (iv) $0, \pi, \frac{\pi}{2}, \frac{3\pi}{2}$ -relative phases of two scattering amplitudes.

We note that $\mathrm{Re}\,Z_{+-}(\Sigma)$ and a_{+-}, b_{+-} can be also used in order to investigate the validity of the Pomeranchuk-type theorems in the high energy limit.

As an example, we give in Fig. 1 the quantities $\mathrm{Re}Z_{+-}(\sigma)$ $a_{+-}(\sigma)$ and $b_{+-}(\sigma)$ determined from the dispersion relations data [9] for the pion-nucleon scattering at zero transfer momentum and beam momenta up to 70 GeV/c. From Fig. 1 we see that the isospin bounds (22b,c) are saturated in the intervals: 0.01 - 0.3 GeV/c, 1.03 -

Fig. 1. $\mathrm{Re}\,Z_{+-}(\sigma)$ and its isospin bounds at $\theta = 0°$ and laboratory momenta up to 70 GeV/c, obtained from πN - forward dispersion relations [9].

1.07 and 1.430 - 1.470 GeV/c. Using table I we see that the saturation of the isospin bounds at $p_{LAB} = 0.088$, 0.130 and 0.300 GeV/c are due to the zeros of σ_{3s}, σ_{2t} and σ_{1s} respectively. A zero of σ_{0t} is expected at threshold since $\mathrm{Re}\,Z_{+-}(\sigma) = 0$, $a_{+-} = -0.8$ and $b_{+-}(\sigma) = 0$ at $p_{LAB} = 0$ seems to be possible. Similar results are obtained from Figs. 2 and 3, (and table I), where $\mathrm{Re}\,Z_{+CE}$, $\mathrm{Re}\,Z_{-CE}$ and their bounds, in the forward direction, are determined also from the dispersion relation predictions [9].

3. The Saturation of the Isospin Bounds and Phase Contours

In general the isospin bounds (16) on differential polarized cross-sections are saturated on the zeros-trajectories of $\mathrm{Re}\,M_{ij}(\Sigma^{(\pm)})$ and $\mathrm{Im}\,M_{ij}(\Sigma^{(\pm)})$ where $M_{ij}(\Sigma^{(\pm)})$ are defined by the relations (12b). Indeed, from Eq. (13) we obtain

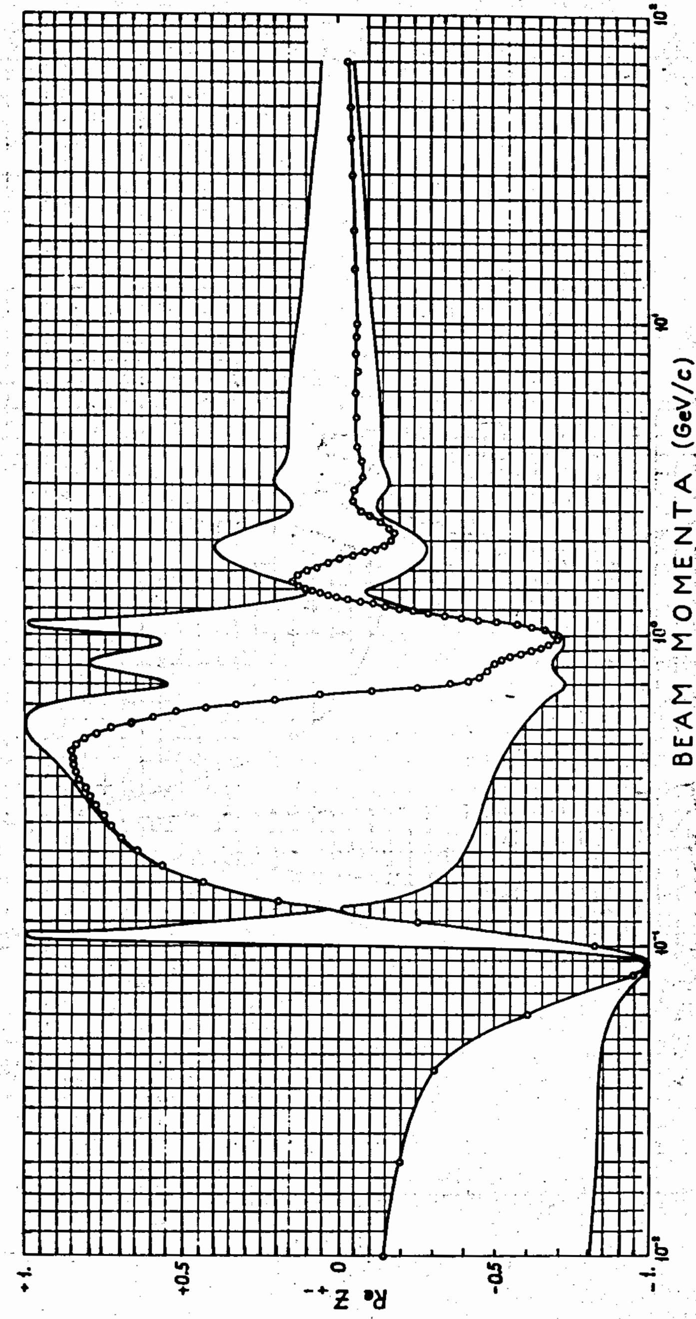
$$\lambda(\Sigma_+^{(\pm)}, \Sigma_-^{(\pm)}, 2\Sigma_{CE}^{(\pm)}) = 0 \qquad (23a)$$

when $\mathrm{Im}\,M_{ij}(\Sigma^{(\pm)}) = 0$, and

$$\lambda(\Sigma_+^{(\pm)}, \Sigma_-^{(\pm)}, 2\Sigma_{CE}^{(\pm)}) = C_{ij}\,\Sigma_i^{(\pm)}\,\Sigma_j^{(\pm)} = A(\Sigma^{(\pm)}), \qquad (23b)$$

(where $C_{+-} = 4$, $C_{+CE} = 8$, $C_{-CE} = 8$, $C_{13s} = \frac{16}{9}$, $C_{02t} = 16$, $C_{13u} = \frac{16}{9}$) when $\mathrm{Re}\,M_{ij}(\Sigma^{(\pm)}) = 0$.

Now, from the relation (13), which can be written in the form:



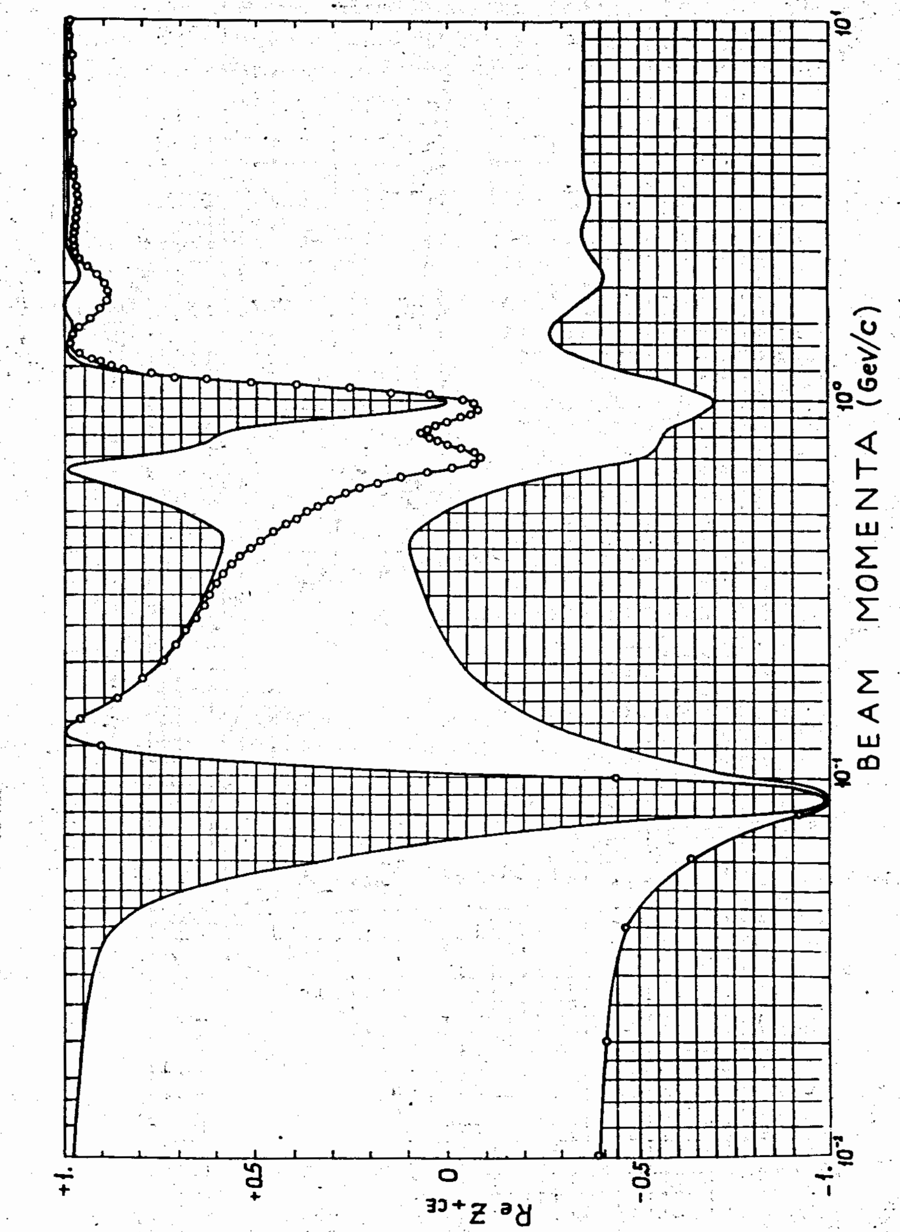


Fig. 2. $\mathrm{Re}\,Z_{+CE}(\sigma)$ and its isospin bounds at $\theta=0°$ and laboratory momenta up to 10 GeV/c, obtained from πN - forward dispersion relations [9].

$$[\mathrm{Im}\,M_{+-}(\Sigma)]^2 = 2[\mathrm{Im}\,M_{+CE}(\Sigma)]^2 = 2[\mathrm{Im}\,M_{+CE}(\Sigma)]^2 =$$

$$= \frac{4}{9}[\mathrm{Im}\,M_{13s}(\Sigma)]^2 = 4[\mathrm{Im}\,M_{02t}(\Sigma)]^2 = \frac{4}{9}[\mathrm{Im}\,M_{13u}(\Sigma)]^2 =$$

$$= -\frac{1}{4}\lambda(\Sigma_+, \Sigma_-, 2\Sigma_{CE}), \tag{24}$$

we see that the zeros-trajectories of $\mathrm{Im}\,M_{ij}(\Sigma)$ are independent of the charge or isospin indices i, j, while the zeros-trajectories of $\mathrm{Re}\,M_{ij}(\Sigma)$ are dependent on these indices.

If one of Σ_+, Σ_-, Σ_{CE}, Σ_{1s} Σ_{0t} Σ_{1u} has a zero the isospin bounds (16) are degenerated, so that the zeros of the corresponding scattering amplitudes in the physical domain must lie on the intersection of the zeros-trajectories of $\mathrm{Re}\,M_{ij}(\Sigma)=0$ and of $\mathrm{Im}\,M_{ij}(\Sigma)=0$. Therefore, in order to locate the zeros of different scattering amplitudes $F_i^{(\pm)}$, it is practical to calculate, from the available phase shift analyses, the zeros-trajectories of $\mathrm{Re}\,M_{ij}(\Sigma^{(\pm)})$ and $\mathrm{Im}\,M_{ij}(\Sigma^{(\pm)})$.

On the other hand, if we introduce the phases of the amplitudes $F_\ell^{(\pm)} = |F_\ell^{(\pm)}| \exp[i\delta_\ell^{(\pm)}(\Sigma)]$, $\ell = i,j$ and express $M_{ij}(\Sigma^{(\pm)})$ (see (12b)) in the form

$$M_{ij}(\Sigma^{(\pm)}) = [\Sigma_i^{(\pm)}\Sigma_j^{(\pm)}]^{1/2} \exp[i\delta_{ij}^{(\pm)}(\Sigma)], \tag{25}$$

then, the conditions for the saturation of the isospin bounds (16) on $\Sigma^{(\pm)}$-differential cross-sections are equivalent to

$$\delta_{ij}^{(\pm)}(\Sigma) = n\pi, \quad n=0,1,\ldots \tag{26a}$$

if $\mathrm{Im}\, M_{ij}(\Sigma^{(\pm)}) = 0$, and

$$\delta_{ij}^{(\pm)}(\Sigma) = (n + 1/2)\pi \qquad (26b)$$

if $\mathrm{Re}\, M_{ij}(\Sigma^{(\pm)}) = 0$.

If $|M_{ij}| = 0$ the relative phases $\delta_{ij}^{(\pm)}(\Sigma)$ are not determined. In table II we show the values of $\delta_{ij}^{(\pm)}(\Sigma)$ related to the zeros of the Σ -differential cross-sections.

Therefore, the $\delta_{ij} = n\pi$ -phase contours lie on the $[\mathrm{Im}\, M_{ij}(\Sigma^{(\pm)})]$ -zeros-trajectories while the $\delta_{ij}^{(\pm)} = (n+1/2)\pi$ -phase contours lie on the $[\mathrm{Re}\, M_{ij}(\Sigma^{(\pm)})]$ -zeros-trajectories.

Our calculation of the zeros-trajectories of $\mathrm{Im}\, M_{31s}(\Sigma^{(\pm)})$ and $\mathrm{Re}\, M_{31s}(\Sigma^{(\pm)})$ from CERN-theoretic phase shift solutions , are presented in Figs. (4-8). In order to identify the $n\pi$, and $(n+1/2)\pi$ -phase contours in Figs. (4-8) we have given, for each $[\mathrm{Im}\, M_{31s}(\Sigma^{(\pm)}) = 0]$ -trajectory (solid line) and $[\mathrm{Re}\, M_{31s}(\Sigma^{(\pm)}) = 0]$ -trajectory (dashed line), the signs of $\mathrm{Im}\, M_{31s}(\Sigma^{(\pm)})$ and $\mathrm{Re}\, M_{31s}(\Sigma^{(\pm)})$. Our additional input is the continuity of these trajectories in order to separate the physical regions, where $\mathrm{Re}\, M_{31s}(\Sigma^{(\pm)})$ and $\mathrm{Im}\, M_{31s}(\Sigma^{(\pm)})$ are positive from those regions, where $\mathrm{Re}\, M_{31s}(\Sigma^{(\pm)})$ and $\mathrm{Im}_{31s}(\Sigma^{(\pm)})$ respectively are negative.

We note that, at the boundaries $\cos\theta = \pm 1$ only the $\delta_{ij}^{(+)}[(1+P)\sigma]$, $\delta_{ij}^{(+)}[(1+A)\sigma]$ and $\delta^{(+)}[(1+T)\sigma]$ -phase contours are smooth continuation of the $\delta^{(-)}[(1-P)\sigma]$, $\delta^{(-)}[(1-A)\sigma]$, $\delta^{(-)}[(1-T)\sigma]$ -phase contours (see the definition (1a,b,c)). In Figs. (4-8) the zeros of the $\Sigma^{(\pm)}$ -differential cross-sections are shown by small circles.

4. The Isospin Bounds on Integrated Cross-Sections and Average Polarizations

In order to obtain a large class of isospin inequalities we define:

$$\bar{\Sigma}^{(n)} \equiv \left[\int_D \Sigma^n \, d\mu\right]^{\frac{1}{n}}, \quad \frac{1}{2} < n < +\infty, \qquad (27)$$

where Σ are the differential (polarized and unpolarized) cross-sections, D is a region in the physical domain and μ is a positive measure defined on the physical domain.

For the following discussion we start with the integrals:

$$I_p(F) \equiv \left[\int_D |F|^p \, d\mu\right]^{\frac{1}{p}} \quad \text{for } 1 < p < +\infty, \qquad (28)$$

where F are in general a linear combination of the scattering amplitudes.

The integrals $I_p(F)$ have the properties:

1°. $0 < I_p(F) < +\infty$; $I_p(F) = 0$ if and only if $|F| = 0$ in the entire region D,

2°. $I_p(aF) = |a|\, I_p(F)$, a -scalar,

3°. $I_p(F_i + F_j) \leq I_p(F_i) + I_p(F_j)$,

4°. $I_p(F_i) \leq I_p(F_j)$ if $|F_i| \leq |F_j|$ in D ,

5°. $|I_p(F_i) - I_p(F_j)| \leq I_p(F_i - F_j)$.

The properties 1°, 2°, and 4° of the $I_p(F)$ -integrals are evident while the properties 3° and 5° follow from Minkowski's inequality (a.3) (see appendix A) since $|F_i + F_j| \leq |F_i| + |F_j|$ and μ is a positive measure on D .

Now, from the isospin relations

$$F_+ = F_- + \sqrt{2}\, F_{CE} \qquad (29)$$

we obtain

$$I_p(F_+) \leq I_p(F_-) + \sqrt{2}\, I_p(F_{CE}), \qquad (30a)$$

$$I_p(F_-) \leq I_p(F_+) + \sqrt{2}\, I_p(F_{CE}), \qquad (30b)$$

$$\sqrt{2}\, I_p(F_{CE}) \leq I_p(F_+) + I_p(F_-). \qquad (30c)$$

Therefore, if we choose $p = 2n$, $\frac{1}{2} < n < +\infty$, then we obtain

$$(\bar{\Sigma}_+^{(n)})^{1/2} \leq (\bar{\Sigma}_-^{(n)})^{1/2} + (2\,\bar{\Sigma}_{CE}^{(n)})^{1/2} \qquad (31a)$$

Fig. 3. $\mathrm{Re}\, Z_{-CE}(\sigma)$ and its isospin bounds at $\theta = 0°$ and laboratory momenta up to 10 GeV/c, obtained from πN - forward dispersion relations [9].

$$(\bar{\Sigma}_{-}^{(n)})^{1/2} \leq (\bar{\Sigma}_{+}^{(n)})^{1/2} + (2\bar{\Sigma}_{CE}^{(n)})^{1/2} \qquad (31b)$$

$$\sqrt{2}(\bar{\Sigma}_{CE}^{(n)})^{1/2} \leq (\bar{\Sigma}_{+}^{(n)})^{1/2} + (\bar{\Sigma}_{-}^{(n)})^{1/2} \qquad (31c)$$

which are the triangle inequalities for $\bar{\Sigma}^{(n)}$ integrated cross-sections defined by (27). This result is equivalent to

$$[\bar{\Sigma}_{-}^{(n)} - \bar{\Sigma}_{+}^{(n)}]^2 \leq 4\bar{\Sigma}_{CE}^{(n)}(\bar{\Sigma}_{+}^{(n)} + \bar{\Sigma}_{-}^{(n)} - \bar{\Sigma}_{CE}^{(n)}) \qquad (32)$$

and also to

$$-\lambda[\bar{\Sigma}_{+}^{(n)}, \bar{\Sigma}_{-}^{(n)}, 2\bar{\Sigma}_{CE}^{(n)}] \geq 0. \qquad (33)$$

Hence the result (32) improves the theorem 1 from ref. [2] in the most general form for $\bar{\Sigma}^{(n)}$-integrated (polarized and unpolarized) cross-sections.

This result requires that if

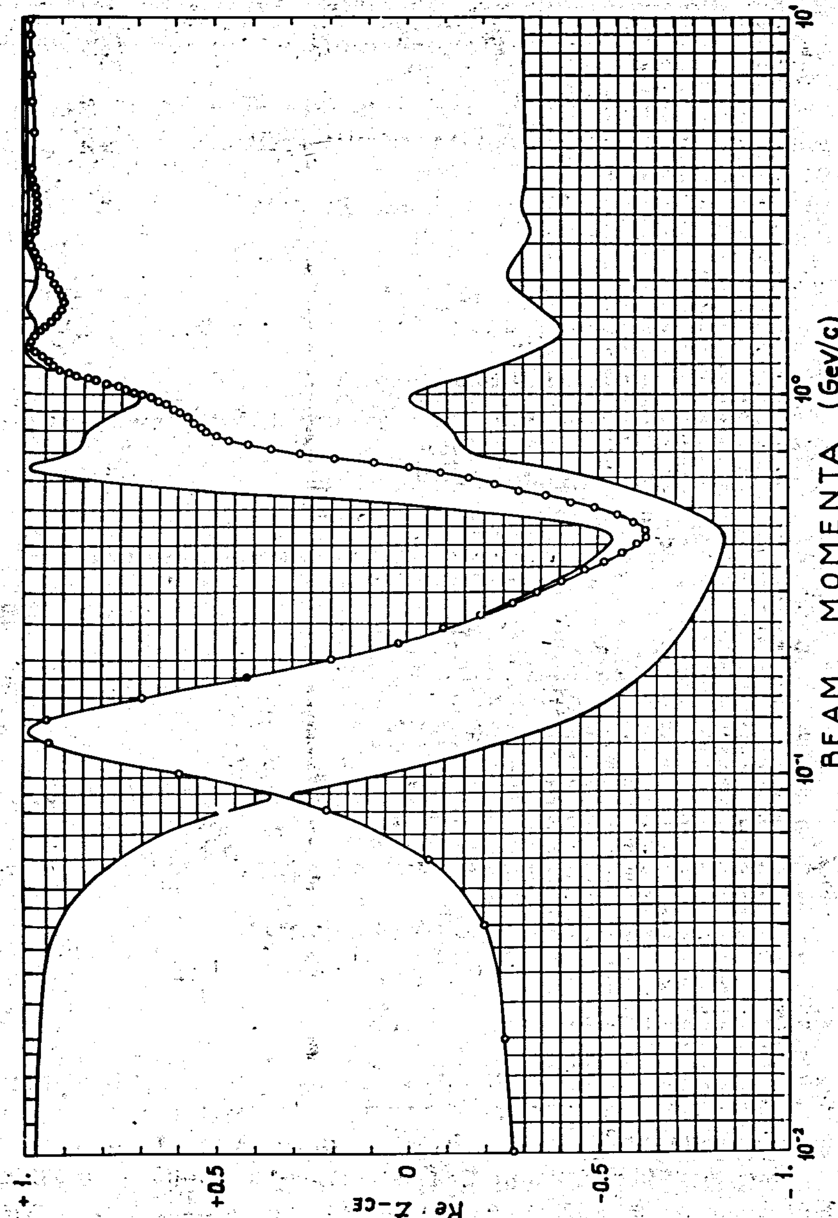
$$\bar{\Sigma}_{CE}^{(n)}(\bar{\Sigma}_{+}^{(n)} + \bar{\Sigma}_{-}^{(n)}) \xrightarrow[s\to+\infty]{} 0$$

then

$$\bar{\Sigma}_{-}^{(n)} - \bar{\Sigma}_{+}^{(n)} \xrightarrow[s\to+\infty]{} 0 \qquad (34b)$$

or, the $\bar{\Sigma}_{CE}^{(n)}$ cannot vanish for $s \to +\infty$ if a Pomeranchuk-type theorem (34b) on $\bar{\Sigma}^{(n)}$ -integrated cross-sections is violated.

It would be interesting to compare the bound (32) on $\bar{\sigma}^{(n)}$, $n = 2, 3, \ldots$ with the experimental data below the one-pion production threshold since it was remarked by



Roy [2] that in this region the upper bound (32) on integrated cross section coincides with the experimental values of the elastic integrated cross-section difference within the experimental errors.

On the other hand, it is also interesting to obtain the experimental data on polarized differential and integrated cross-sections, in this energy region, and to investigate the isospin bounds on average polarization parameters defined by

$$\bar{X} = \frac{1}{\bar{\sigma}} \int_{\Omega_0} X \sigma \, d\Omega , \quad \Omega_0 \leq 4\pi \tag{35}$$

$\bar{\sigma} \equiv \bar{\sigma}^{(1)}$ - integrated cross-section.

Therefore, let us consider the isospin inequalities (32) and (33) for $n=1$ and $\Sigma^{(\pm)} \equiv (1 \pm X)\sigma$.

From these bounds we obtain

$$(\bar{\sigma}_- - \bar{\sigma}_+)^2 + (\bar{X}_- \sigma_- - \bar{X}_+ \sigma_+)^2 \leq 4(1 + \bar{X}_{0t} \bar{X}_{CE}) \bar{\sigma}_{CE} (\bar{\sigma}_+ + \bar{\sigma}_- - \bar{\sigma}_{CE}), \tag{36}$$

where

$$\bar{X}_{0t} = \frac{\bar{X}_+ \bar{\sigma}_+ + \bar{X}_- \bar{\sigma}_- - \bar{X}_{CE} \bar{\sigma}_{CE}}{\bar{\sigma}_+ + \bar{\sigma}_- - \bar{\sigma}_{CE}} . \tag{37}$$

The bound (36) are equivalent to

$$\lambda(\bar{X}_+ \bar{\sigma}_+ , \bar{X}_- \bar{\sigma}_- , 2\bar{X}_{CE} \bar{\sigma}_{CE}) < -\lambda(\bar{\sigma}_+ , \bar{\sigma}_- , 2\bar{\sigma}_{CE}) \tag{38}$$

and also to those isospin bounds obtained from (9a,b,c,d, e,f) by the substitution: $X \to \bar{X}$, $\sigma \to \bar{\sigma}$

Next, let us consider the inequality

$$2\,|\mathrm{Re}(\vec{G}_i^* \cdot \vec{G}_j)| \leq |F_i^{(+)}||F_j^{(+)}| + |F_i^{(-)}||F_j^{(-)}| \tag{39}$$

which follows from

$$2\vec{G}_i^* \cdot \vec{G}_j = [F_i^{(+)}]^* F_j^{(+)} + [F_i^{(-)}]^* F_j^{(-)} \tag{40}$$

valid for any $F_i^{(\pm)}$, $F_j^{(\pm)}$ defined by (12c).

Now, from (39) and Hölder's inequality (A.2) we can write:

$$2\,\Big|\int_{\Omega_0} \mathrm{Re}(\vec{G}_i^* \vec{G}_j)\, d\Omega\Big| \leq I_p(F_i^{(+)}) I_q(F_j^{(+)}) + I_p(F_i^{(-)}) I_q(F_j^{(-)}), \tag{41}$$

where

$$I_\ell(F^{(\pm)}) \equiv \Big[\int_{\Omega_0} |F^{(\pm)}|^\ell d\Omega\Big]^{\frac{1}{\ell}} = \Big[\int (\Sigma^{(\pm)})^{\frac{\ell}{2}} d\Omega\Big]^{\frac{1}{\ell}}, \tag{42}$$

$\ell = p, q$, $p > 1$ and $\frac{1}{q} + \frac{1}{p} = 1$.

From the isospin relation

$$\vec{G}_+ = \vec{G}_- + \sqrt{2}\,\vec{G}_{CE}$$

we obtain

$$\mathrm{Re}(\vec{G}_+^* \vec{G}_-) = \frac{1}{2}(\sigma_+ + \sigma_- - 2\sigma_{CE}), \tag{43a}$$

$$\mathrm{Re}(\vec{G}_+^* \vec{G}_{CE}) = \frac{1}{2\sqrt{2}}(\sigma_+ - \sigma_- + 2\sigma_{CE}), \tag{43b}$$

$$\mathrm{Re}(\vec{G}_-^* \cdot \vec{G}_{CE}) = \frac{1}{2\sqrt{2}}(\sigma_+ - \sigma_- - 2\sigma_{CE}), \tag{43c}$$

$$\mathrm{Re}(\vec{G}_{1s}^* \cdot \vec{G}_{3s}) = \frac{1}{4}(\sigma_+ + 3\sigma_- - 6\sigma_{CE}), \tag{43d}$$

$$\mathrm{Re}(\vec{G}_{0t}^* \cdot \vec{G}_{2t}) = \frac{1}{4}(\sigma_- - \sigma_+), \tag{43e}$$

$$\mathrm{Re}(G_{1u}^* \cdot G_{3u}) = \frac{1}{4}(\sigma_- + 3\sigma_+ - 6\sigma_{CE}). \tag{43f}$$

So that, from (41) for $p = q = 2$ and (43a,b,c,d,e,f) we obtain:

$$0 \le \{\bar\sigma_+ \bar\sigma_-[(\bar X_+ - \bar X_-)^2 + \eta_{+-}];\ 2\bar\sigma_+ \bar\sigma_{CE}[(\bar X_+ - \bar X_{CE})^2 + \eta_{+CE}];$$

$$2\bar\sigma_- \bar\sigma_{CE}[(\bar X_- - \bar X_{CE})^2 + \eta_{-CE}];\ \frac{4}{9}\bar\sigma_{1s}\bar\sigma_{3s}[(\bar X_{1s} - \bar X_{3s})^2 + \eta_{13s}];$$

$$4\bar\sigma_{0t}\bar\sigma_{2t}[(\bar X_{0t} - \bar X_{2t})^2 + \eta_{02t}];\ \frac{4}{9}\bar\sigma_{1u}\bar\sigma_{3u}[(\bar X_{1u} - \bar X_{3u})^2 + \eta_{13u}]\} \le$$

$$\le -\lambda[\bar\sigma_+, \bar\sigma_-, 2\bar\sigma_{CE}], \qquad (44a)$$

where

$$\eta_{ij} \equiv 2 - \bar X_i^2 - \bar X_j^2 - 2(1 - \bar X_i^2)^{1/2}(1 - \bar X_j)^{1/2} \ge 0, \qquad (44b)$$

$$\bar X_{1s} = \frac{3\bar X_- \bar\sigma_- + 3\bar X_{CE}\bar\sigma_{CE} - \bar X_+ \bar\sigma_+}{3\bar\sigma_- + 3\bar\sigma_{CE} - \bar\sigma_+}, \quad \bar X_{3s} = \bar X_+, \qquad (44c)$$

$\bar X_{2t} = \bar X_{CE}$ and $\bar X_{0t}$ given by (37) (44d)

$$\bar X_{1u} = \frac{3\bar X_+ \bar\sigma_+ + 3\bar X_{CE}\bar\sigma_{CE} - \bar X_- \bar\sigma_-}{3\bar\sigma_+ + 3\bar\sigma_{CE} - \bar\sigma_-}, \quad \bar X_{3u} = \bar X_-. \qquad (44e)$$

The isospin bounds (44a) (see also relation (34a) from ref. /6/) are the best possible ones, since we can obtain the restrictive conditions for the average polarization components using only the unpolarized integrated cross-section data.

If $\lambda(\bar\sigma_+, \bar\sigma_-, 2\bar\sigma_{CE}) = 0$, the isospin bound (32) on $\bar\sigma$ is saturated, then it follows from (44a) that

$$\bar X_+ = \bar X_- = \bar X_{CE} = \bar X_{1s} = \bar X_{0t} = \bar X_{1u} \qquad (45)$$

for any $\bar X \equiv \bar P, \bar A, \bar R$ or $(\bar P, \bar T, \bar S), \dots$

Therefore, it would be interesting to obtain the experimental data for $\bar P_+$, $\bar P_-$ and $\bar P_{CE}$ below the one-pion production threshold, where the isospin bound (32) on integrated cross-sections is saturated /2/, in order to see if the equalities (45) are verified. Next, the isospin bound (36) requires that

if

$$\bar\sigma_{CE}(\bar\sigma_+ + \bar\sigma_-) \xrightarrow[s\to+\infty]{} 0 \qquad (46a)$$

then

$$\bar\sigma_- - \bar\sigma_+ \xrightarrow[s\to+\infty]{} 0, \qquad (46b)$$

$$\bar P_- - \bar P_+ \xrightarrow[s\to+\infty]{} 0, \qquad (46c)$$

$$\bar A_- - \bar A_+ \xrightarrow[s\to+\infty]{} 0, \qquad (46d)$$

$$\bar R_- - \bar R_+ \xrightarrow[s\to+\infty]{} 0, \qquad (46e)$$

$$\vdots$$

and, conversely, the πN-charge-exchange integrated cross-sections cannot vanish for $s \to +\infty$ if one of the Pomeranchuk-type theorems (46b,c,d,e) is violated.

On the other hand, if the mirror symmetry $(P_+ = -P_-)$ observed in the intermediate energy region is also preserved at high energies, then the above results imply that $|\bar P_+| = |\bar P_-| = |\bar P_{CE}| \to 0$ for $s \to +\infty$. Next, if the violation of the Pomeranchuk theorem on the total cross-sections is the largest (permitted by unitarity-analyticity bound (1) from ref. 2) the above results (45) can be improved if the isospin bound (33) on $\bar\sigma$ is exactly saturated

Table I
The values of $ReZ_{ij}(\Sigma)$, $a_{ij}(\Sigma)$ and $b_{ij}(\Sigma)$ related to the zeros of the Σ-differential cross-sections.

Zero of	ReZ_{+-}	a_{+-}	b_{+-}	ReZ_{+CE}	a_{+CE}	b_{+CE}	ReZ_{-CE}	a_{-CE}	b_{-CE}
Σ_+	-1	-1	-1	-1	-1	-1	+1/3	+1/3	+1/3
Σ_-	+1	-3/5	+1	+1/3	+1/3	+1/3	-1	-1	-1
Σ_{CE}	0	0	0	+1	-1/3	+1	+1	-1/3	+1
Σ_{1s}	+4/5	-8/17	+4/5	+7/11	+1/17	+7/11	-1/3	-7/9	-1/3
Σ_{0t}	0	-4/5	0	-1/3	-1/3	+1	-1/3	-1/3	+1
Σ_{1u}	-4/5	-4/5	0	-1/3	-7/9	-1/3	+7/11	+1/17	+7/11

Table II
The values of the relative phases for the zeros-positions of Σ-differential cross-sections

Zero of	Σ_+	Σ_-	Σ_{CE}	Σ_{1s}	Σ_{0t}	Σ_{1u}
$\delta^{(\pm)}_{+-}$	sing.	sing.	0	0	π	0
$\delta^{(\pm)}_{+CE}$	sing.	0	sing.	0	0	π
$\delta^{(\pm)}_{-CE}$	π	sing.	sing.	0	π	π
$\delta^{(\pm)}_{13s}$	sing.	π	0	sing.	π	0
$\delta^{(\pm)}_{02t}$	0	π	sing.	π	sing.	0
$\delta^{(\pm)}_{13u}$	π	sing.	0	0	π	sing.

for $s \to +\infty$ or, if theorem 2 from ref. [2] is extended such that the bound

$$[\bar{\sigma}_- - \bar{\sigma}_+]^2 + [\bar{X}_- \bar{\sigma}_- - \bar{X}_+ \bar{\sigma}_+]^2 \leq 4(1+\bar{X}_{0t}\bar{X}_{CE})[\bar{\sigma}_{CE} - \frac{2m_\pi^2}{\pi^3}(\Delta\sigma_{tot})^2] \times$$

$$\times (\bar{\sigma}_+ + \bar{\sigma}_- - \bar{\sigma}_{CE}) \qquad (47)$$

improves both the unitarity-analyticity bound (1) ref. [2] and the isospin bound (36) for $s \to \infty$.

We note that the saturation of the isospin bounds on $\bar{\Sigma}^{(n)}$ -integrated cross-sections can be investigated by analogy with the isospin bounds for Σ -differential cross-sections. This is possible since the isospin bounds (3a, b,c) and (16) are preserved when we substitute Σ_i by $\bar{\Sigma}_i^{(n)}$, $i = +, -, CE$. In this case the results presented in Tables I and II are also preserved.

Finally, we remark that other interesting results can be obtained if we write explicitly the isospin bounds on $\bar{\Sigma}^{(n)}$ -integrated cross-sections for $n = 2,3,\ldots$, or if we investigate the isospin bounds on $\bar{\Sigma}^{(n)}$ for $n < 1/2$ using Hölder's and Minkowski's inequalities for $p < 1$ [11].

5. Conclusions

In this paper we have derived all the isospin bounds on differential and $\bar{\Sigma}^{(n)}$-integrated (polarized and unpolarized) cross-sections. So in Sect. 2 we have introduced [6] the scattering amplitudes (1a,b,c) in terms of which the differential (polarized and unpolarized) cross-sections are expressed as the square modules of these amplitudes. The introduction of the $(\vec{G}, K^{(\pm)}, H^{(\pm)}_{\ldots})$-scattering amplitudes allows one to discuss all the isospin bounds on differential (polarized and unpolarized) cross-sections, at any angle (or transfer momentum), in analogy to the simple cases of backward and forward scattering. The isospin bounds have been derived using a set of bilinear forms

$M_{ij}(\Sigma)$ (12a,b), which can be constructed from the scattering amplitudes of two charge (or s, t, u -isospin)-channels. This is the most simple form of presentation of the isospin bounds on differential (polarized and unpolarized) cross-sections which have the advantage that: (i) the exact saturation of these bounds can be obtained in terms of the $[\mathrm{Im}\, M_{ij}(\Sigma), \mathrm{Re}\, M_{ij}(\Sigma)]$ -zeros-trajectories or equivalently, in terms of $[n\pi, (n+\frac{1}{2})\pi]$ -phase contours (see Sect. 3), (ii) the zeros-positions of different scattering amplitudes are simple and unambiguously determined from the intersection of these $[\mathrm{Im}\, M_{ij}(\Sigma), \mathrm{Re}\, M_{ij}(\Sigma)]$-zeros-trajectories (see Figs. (4-8)). So that in Sect. 2, we have obtained the isospin inequalities (3a,b,c), (4a,b,c,d,e,f), (9a,b,c,d,e,f), (16), (18a), (20a,b,c, d,e,f). These results (see (9e) and (20e)) enable us to understand the small differences between elastic differential cross-sections, at high energies and fixed transfer momentum, in terms of the small charge-exchange differential cross-sections. On the other hand, if $\sigma_{CE}(\sigma_+ + \sigma_-) \to 0$ when $s \to +\infty$, t -fixed, then, from (9e) or (20e) written in the equivalent form

$$(\sigma_- - \sigma_+)^2 + (1 - \vec{P}_+ \vec{P}_-)\sigma_+ \sigma_- \leq 4\, \sigma_{CE}(\sigma_+ + \sigma_- - \sigma_{CE}) \qquad (48)$$

(we have used (18b), (18c) and (19)), we obtain that $\sigma_- - \sigma_+ \to 0$, $\vec{P}_- - \vec{P}_+ \to 0$ for $s \to +\infty$, t -fixed, in any spin reference frame. Conversely, if one of the Pomeranchuk-type theorems on differential (polarized and unpolarized) elastic cross-sections is violated then σ_{CE} -differential cross-section cannot vanish for $s \to +\infty$, t -fixed. If the mirror symmetry, observed in intermediate region of energy, $(P_+ = -P_-)$ is also preserved at high energies, then $\sigma_{CE} \to 0$ for $s \to +\infty$, t -fixed, implies $|P_+| = |P_-| = |P_{CE}| \to 0$ for $s \to +\infty$ and t -fixed. The similar conclusions on $\bar{\Sigma}^{(n)}$ -integrated cross-sections are obtained in Sect. 4. Next, the saturation of the isospin bounds on differential cross-sections in the forward direction, below one-pion threshold, presented in Fig. (1-3) should be compared with the recent results of Roy [2] and Tornquist [1]. If the isospin bound (4e) on σ -differential cross-sections is saturated in the entire $\cos\theta$ - region, then the final polarizations satisfy the relation

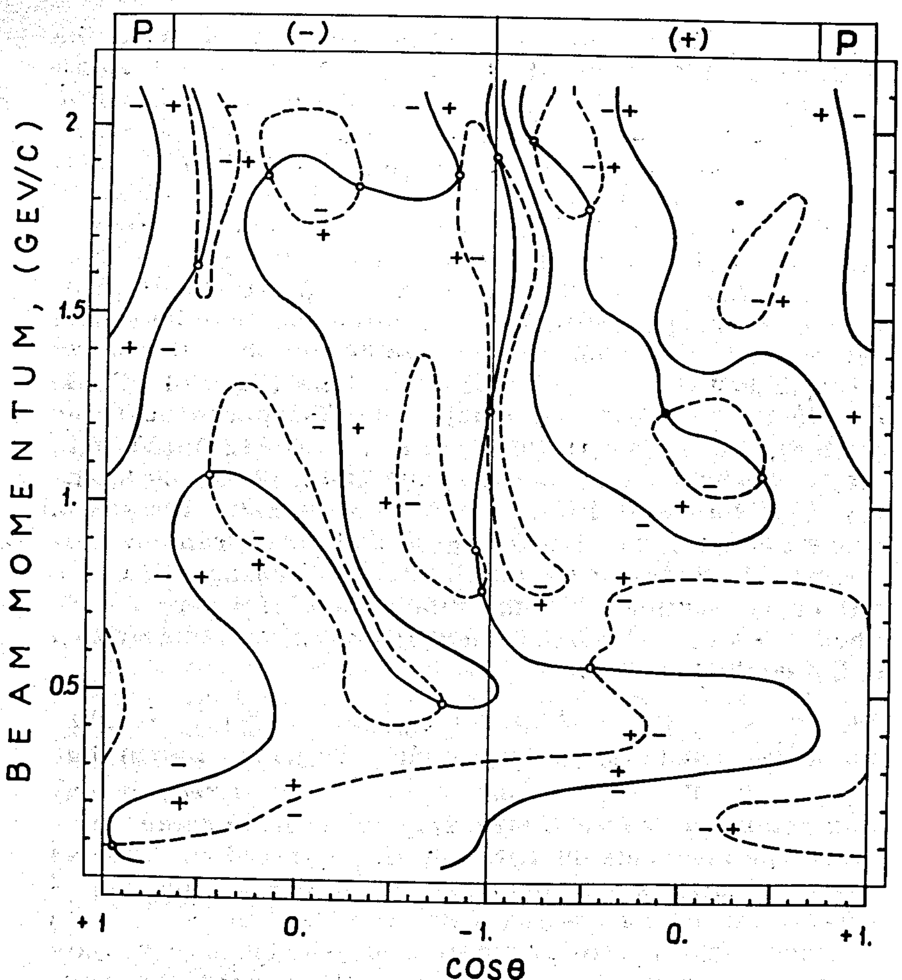
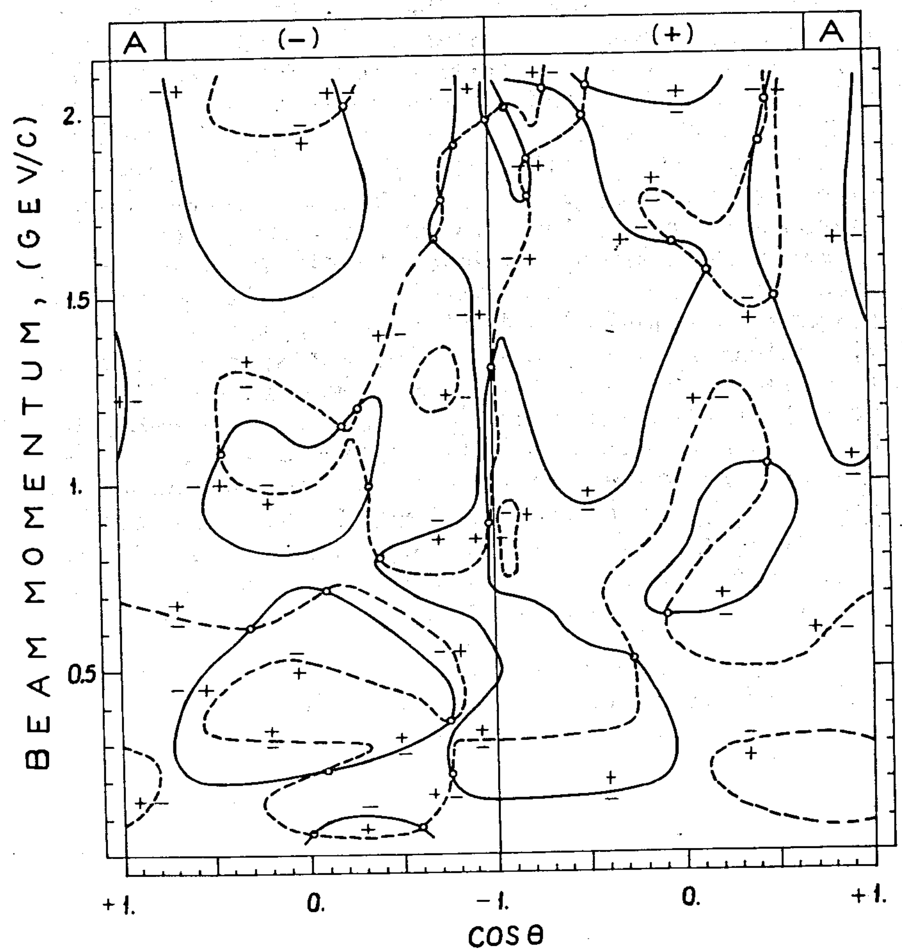


Fig. 4. The zeros-trajectories of: $\mathrm{Im}M_{31s}[(1\pm P)\sigma]$ (solid line) and $\mathrm{Re}M_{31s}[(1\pm P)\sigma]$ (dashed line) respectively, in the $(p_{LAB}, \cos\theta)$ -plane, determined from CERN-theoretic solution /10/ for the πN -phase shifts.

Fig. 5. The zeros-trajectories of: $\mathrm{Im}\,M_{31s}[(1\pm A)\sigma]$ (solid line) and $\mathrm{Re}M_{31s}[(1\pm A)\sigma]$ (dashed line), respectively, in the $(p_{LAB}, \cos\theta)$-plane, determined from CERN-theoretic solution /10/ for the πN -phase shifts.

$\vec{P}_+ = \vec{P}_- = \vec{P}_{CE}$ in the entire $\cos\theta$ -region (see also (34) from ref. /6/). If these equalities are verified experimentally then the isospin invariance is exact in this region. Under these conditions similar results will be obtained for the final average polarizations (see Sect. 4).

Next, our results on the zeros-trajectories of $\mathrm{Im}M_{31s}[(1+P)\sigma]$ (see Fig. 4 (solid line)) can be compar-



ed with the results of Höhler et al.[4] on the $0, \pi$ -phase contours. We find an approximate agreement with Fig. 1 from ref. [4] only for $\mathrm{Im}M_{31s}[(1-P)\sigma]$ in the entire $(p_{LAB}, \cos\theta)$ -plane, while for $\mathrm{Im}M_{31s}[(1+P)\sigma]$ these results are quite different only for $p_{LAB} > 1.3$ GeV/c at the forward and backward c.m. angles. But both results are in agreement with our result (fig. 1) obtained from the dispersion relation predictions[9].

It would be useful to obtain the $\mathrm{Im}M_{31s}[(1 \pm X)\sigma]$ -zeros-trajectories, for $X = A, R$ or (S, T), from the recent CERN[12] and Saclay[13] phase shifts and to compare these results with our results presented in Fig. 5-8, since it was pointed out, recently by Höhler et al.[14], that these phase shifts lead to quite different predictions for spin rotation parameters in certain kinematic regions.

Finally, we remark that, defining $\Sigma^{(n)}$ -integrated cross-sections, in Sect. 4 we obtain a large class of isospin-constraints. These constraints can be very useful to check the isospin invariance directly from the coefficients obtained by polynomial fits of the differential (polarized and unpolarized) cross-sections, since $\Sigma^{(n)}$ -integrated cross-sections can be expressed as functions of these coefficients. The isospin bounds on $\Sigma^{(n)}$-integrated cross-sections $(n > \frac{1}{2})$ are derived using the classical inequalities of Hölder and Minkowski[11] (see Appendix A). We note that our class of the isospin inequalities on $\Sigma^{(n)}$ obtained in Sect. 4 can be extended to the case $n < \frac{1}{2}$ if we use Minkowski's and Hölder's inequalities for $p < 1$, and also, the recent extensions of Minkowski's inequality[15].

I would like to thank C.A.Gheorghe for useful suggestions and discussions. I also wish to thank my colleagues S.Berceanu, S.Holan and F.Nichitiu who helped me in so many ways.

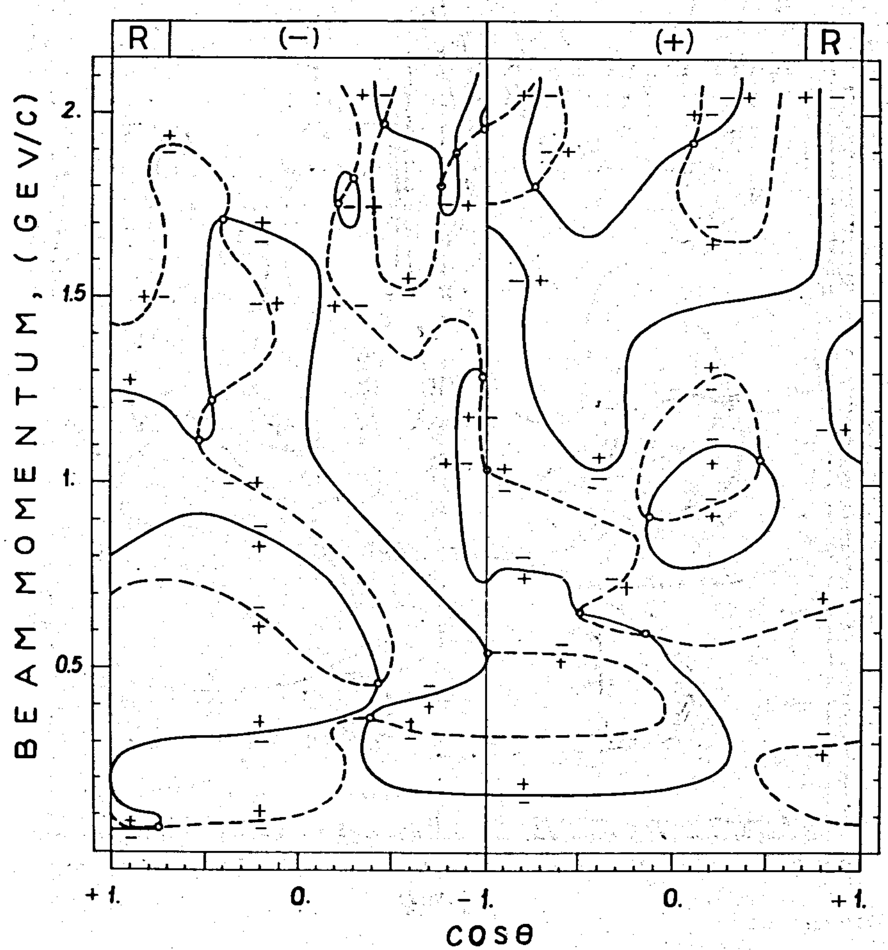


Fig. 6. The zeros-trajectories of: $\mathrm{Im}M_{31s}[(1 \pm R)\sigma]$ (solid line) and $\mathrm{Re}M_{31s}[(1 \pm R)\sigma]$ (dashed line), respectively, in the $(p_{LAB}, \cos\theta)$ -plane, determined from CERN-theoretic solution[10] for the πN -phase shifts.

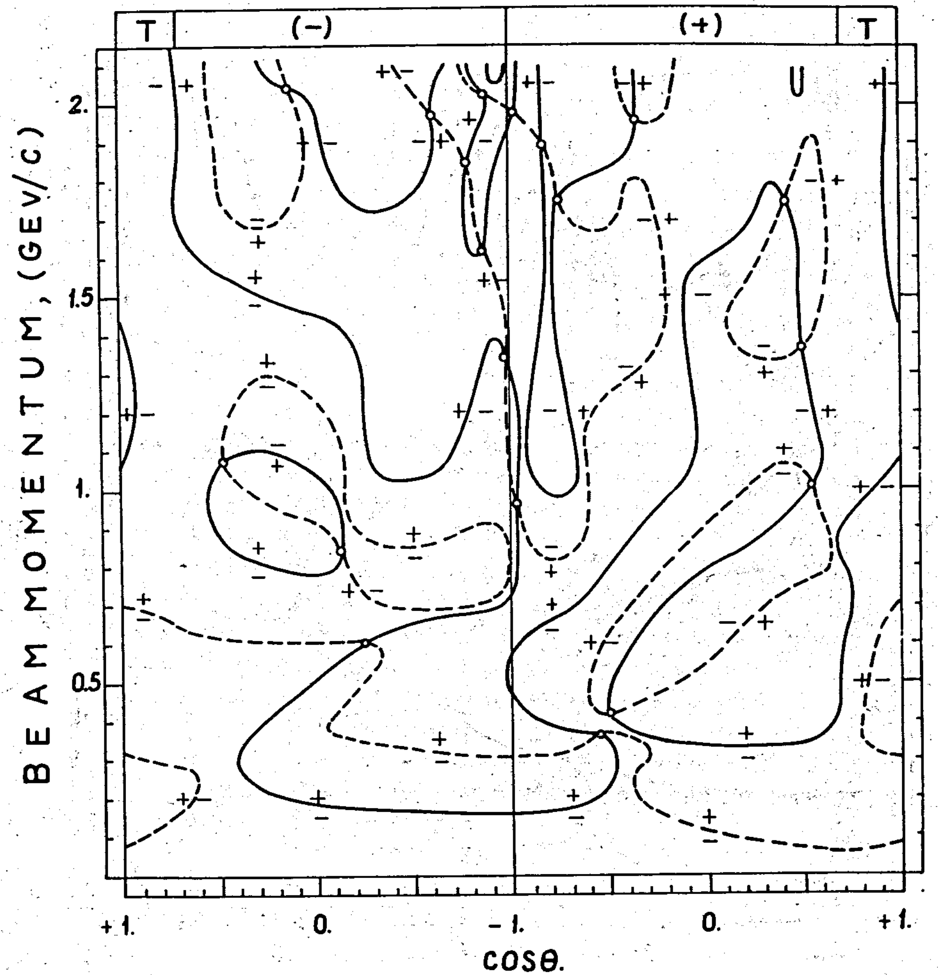


Fig. 7. The zeros-trajectories of: $\mathrm{Im}M_{31s}\,[(1\pm T)\sigma]$ (solid line) and $\mathrm{Re}M_{31s}\,[(1\pm T)\sigma]$ (dashed line), respectively, in the $(p_{LAB},\cos\theta)$ -plane, determined from CERN-theoretic solution /10/ for the πN -phase shifts.

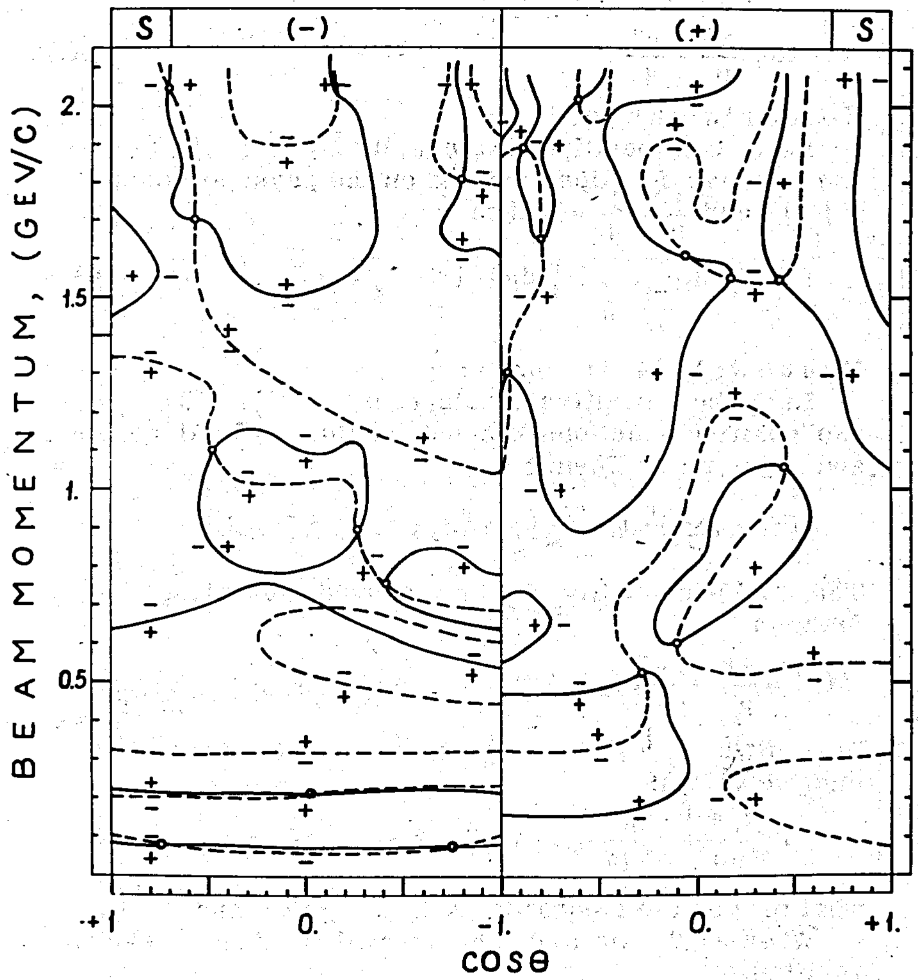


Fig. 8. The zeros-trajectories of: $\mathrm{Im}M_{31s}\,[(1\pm S)\sigma]$ (solid line) and $\mathrm{Re}M_{31s}\,[(1\pm S)\sigma]$ (dashed line), respectively, in the $(p_{LAB},\cos\theta)$-plane, determined from CERN-theoretic solution /10/ for the πN -phase shifts.

Appendix A

We shall prove the following inequalities /11/ .

Let $p>1$, $q>1$, $\frac{1}{p}+\frac{1}{q}=1$ and $a\geq 0, b\geq 0$ then

$$ab\leq \frac{a^p}{p}+\frac{b^q}{q} \qquad (A.1)$$

Hölder's inequality

Let μ be a positive measure, $0\leq \Sigma_1<+\infty$, $0\leq\Sigma_2<+\infty$ two positive functions defined on the physical domain D . If $p>1$ and $\frac{1}{p}+\frac{1}{q}=1$, then

$$\int_D \Sigma_1\Sigma_2 d\mu\leq \left[\int_D \Sigma_1^p d\mu\right]^{\frac{1}{p}}\left[\int_D \Sigma_2^q d\mu\right]^{\frac{1}{q}} \qquad (A.2)$$

Minkowski's inequality

Let μ be a positive measure, $0\leq \Sigma_1<+\infty$, $0\leq\Sigma_2<+\infty$, two positive functions defined on the physical region D and $1\leq p<+\infty$. Then,

$$\left[\int_D (\Sigma_1+\Sigma_2)^p d\mu\right]^{\frac{1}{p}}\leq\left[\int_D \Sigma_1^p d\mu\right]^{\frac{1}{p}}+\left[\int_D \Sigma_2^p d\mu\right]^{\frac{1}{p}} \qquad (A.3)$$

The (A.1)-inequality can be proved observing that the function

$$\phi(x)=\frac{x^p}{p}+\frac{x^{-q}}{q}\geq \phi(1)=1 \qquad \text{for } x>0, p>1, q>1, \frac{1}{p}+\frac{1}{q}=1.$$

Now, since $\frac{p}{q}=p-1$, $\frac{q}{p}=q-1$, we choose $x=a^{\frac{1}{q}}b^{-\frac{1}{p}}$, then we obtain:

$$1\leq\frac{a^{p-1}}{pb}+\frac{b^{q-1}}{qa}, \qquad a>0, \ b>0$$

what proves the inequality (A.1) for $a>0$ and $b>0$.

When $a=0$ or $b=0$ the inequality (A.1) is evidently verified.

Next, we see that Hölder's inequality (A.2) is verified if

$$\int_D \Sigma_1^p d\mu=0 \qquad \text{or} \qquad \int_D \Sigma_2^q d\mu=0.$$

Therefore, we assume that

$$0<\int_D \Sigma_1^p d\mu<+\infty, \qquad 0<\int_D \Sigma_2^q d\mu<+\infty$$

and we can define

$$a\equiv\frac{\Sigma_1}{\left[\int_D \Sigma_1^p d\mu\right]^{\frac{1}{p}}} \ ; \ b\equiv\frac{\Sigma_2}{\left[\int_D \Sigma_2^q d\mu\right]^{\frac{1}{q}}}$$

for each point of D . Then, from (A.1) we obtain

$$\frac{\Sigma_1\Sigma_2}{\left[\int_D \Sigma_1^p d\mu\right]^{\frac{1}{p}}\left[\int_D \Sigma_2^q d\mu\right]^{\frac{1}{q}}}\leq\frac{\Sigma_1^p}{p\left[\int_D \Sigma_1^p d\mu\right]}+\frac{\Sigma_2^q}{q\left[\int_D \Sigma_2^q d\mu\right]}$$

for each point of D . Now, if we integrate over D, since $\frac{1}{p}+\frac{1}{q}=1$, we obtain Hölder's inequality (A.2).

Next, we see that (A.3) is verified for $p=1$ and for $\int_D(\Sigma_1+\Sigma_2)^p d\mu=0$. Therefore, let $p>1$ and

$$0<\int_D (\Sigma_1+\Sigma_2)^p d\mu<+\infty.$$

Then, we can write

$$\int_D (\Sigma_1+\Sigma_2)^p d\mu=\int_D \Sigma_1(\Sigma_1+\Sigma_2)^{p-1} d\mu+\int_D \Sigma_2(\Sigma_1+\Sigma_2)^{p-1} d\mu.$$

Now, using Hölder's inequality (A.2) we obtain

$$\int_D (\Sigma_1+\Sigma_2)^p d\mu\leq\left\{\left[\int_D \Sigma_1^p d\mu\right]^{\frac{1}{p}}+\right.$$

$$\left.+\left[\int_D \Sigma_2^p d\mu\right]^{\frac{1}{p}}\right\}\left[\int_D (\Sigma_1+\Sigma_2)^{q(p-1)} d\mu\right]^{\frac{1}{q}}.$$

Therefore, Minkowski's inequality (A.3) is proved since

$$q(p-1)=p \qquad \text{and} \qquad 1-\frac{1}{q}=\frac{1}{p}.$$

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