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# SELECTION RULES FOR DUAL RESONANCE STATES

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1. Recently a method has been proposed [1-4] for factorization of the N-point amplitude in the dual narrow-resonance model [5,6] ( the Veneziano model). The factorization is achieved through the use of a finite set of 5-dimensional U(4,1) invariant oscillators. Its main peculiarity is that the level degeneracy increases not faster than some power of the intermediate resonance mass [7]. This peculiarity follows from the equation for the resonance wave functions

$$H|\psi_n\rangle = n|\psi_n\rangle. \tag{1.1}$$

Here $H$ is the Hamiltonian of the system containing N independent, identical U(4,1) invariant oscillators:

$$H = \sum_{i=1}^{N} H_i . \tag{1.2}$$

As is known, equation (1.1) leads to degeneracy of the levels, and the number of the states, $f$, with $n$ fixed, is

$$f = \frac{(n+1)(n+2)(n+3)\ldots(n+5N-1)}{(5N-1)!} . \tag{1.3}$$

It is just the equality, that ascertains the above-mentioned peculiarity of the considered factorization. However, note should be made that the power increase of f, which stems from eq. (1.1), is only the maximum possible one. In fact, the resonance wave functions satisfy eq. (1.1), but they do not exhaust the whole set of linearly independent solutions of this equation ( for n fixed). In other words, the factorization of the dual amplitude produces as resonance wave functions only some of all possible linearly independent solutions of eq. (1.1). This sets one thinking, that there are some selection *rules* ( or auxiliary conditions), which being considered together with eq. (1.1) could eliminate all extra solutions and lead to the unique determination of *resonance states*.

The purpose of this paper is to reveal these auxiliary conditions and to consider them in detail.

2. The (N+1) particle state of the intermediate resonance with energy $n$, which is obtained through factorization of the dual multiparticle amplitude, has the form:

$$|\psi_n^{N+1}\rangle = \sum_{n_1}\sum_{n_2}\dots\sum_{n_N} B_{N+2}\big(\alpha(s_1)-n_1;\ \alpha(s_2)-n_1-n_2;\ \dots \tag{2.1}$$

$$\dots\ \alpha(s_{N+1})-\sum_{i=1}^{N-1} n_i\big)\prod_{k=1}^{N}\frac{1}{n_k}\left(i\sqrt{\alpha'}\,p_k^{\mu}a^{+}_{k\mu}+a^{+}_{k4}\right)^{n_k}|0\rangle_k .$$

Here we have adopted the following notations:

a) $p_k^{\mu}$ is the four-momentum of the k-th particle ;

b) $a^{+}_{k\mu}$ and $a^{+}_{k4}$ are, respectively, the vector and scalar creation operators of the k-th 5-dimensional oscillator. Analogously, $a_{k\mu}$ and $a_{k4}$ are the annihilation operators. These operators satisfy the commutation relations:

$$[a_{k\mu}, a^+_{k'\mu'}] = -\delta_{kk'} g_{\mu\mu'}, \quad [a_{k4}, a^+_{k'4}] = \delta_{kk'} \tag{2.2}$$

$$k, k' = 1, 2, \dots N; \quad \mu, \mu' = 0, 1, 2, 3 \;;$$

c) $|0\rangle_k$ is the vacuum state of the k-th oscillator ;

d) $S_i = \left(\sum_{\ell=0}^{i} p_\ell\right)^2$ is the total energy of the resonance channels and $\alpha(s_i) = \alpha' s_i + \alpha(0)$ - the corresponding linear Regge trajectory ;

e) $B_{N+2}$ is the (N+2) point Veneziano amplitude in the Bardakci-Ruegg form, in which we have written only a part of all invariant variables, namely $s_i$.

Expression (2.1) shows that the state $|\psi^{N+1}_n\rangle$ is composed of eigenvectors of the Hamiltonian $H_k = -a^+_{k\mu} a_k^\mu + a^+_{k4} a_{k4}$ and each of such vectors has the form

$$|n_k\rangle = \frac{1}{n_k!} \left( i\sqrt{\alpha'}\, p_k^\mu a^+_{k\mu} + a^+_{k4} \right)^{n_k} |0\rangle_k . \tag{2.3}$$

Obviously, such expressions correspond to the two-particle links in the multiperipheral diagrams, which are connected with (2.1). As the presence of $B_{N+2}$ in the right-hand side of eq. (2.1) is more or less natural, one meets the necessity to understand the conditions which give rise to the remaining factors of type (2.3).

The states $|n_k\rangle$ are eigenvectors of the operator $H_k$ with eigenvalues $n_k$. The general form of such an eigenvector is

$$|f_n\rangle = \sum_{\sum_{i=0}^{4} n_i = n} f_{n_0 n_1 n_2 n_3 n_4} \prod_{i=0}^{4} \frac{(a_i^+)^{n_i}}{\sqrt{n_i!}} |0\rangle. \tag{2.4}$$

Here and later on, for the sake of simplicity, we will omit the index $K$.

First of all one can see that the vectors $|n_K\rangle$, eq.(2.3), are Lorentz-invariant with respect to the simultaneous transformations of $p^\mu$ and $a^+_\mu$. This invariance is a consequence of the fact, that the external particles constituting the resonance in state (2.1), are scalars. The generators of the present Lorentz-transformation are

$$M_{\mu\nu} = -i(a^+_\mu a_\nu - a^+_\nu a_\mu) + M^{(p)}_{\mu\nu}, \tag{2.5}$$

where

$$M^{(p)}_{\mu\nu} = \begin{cases} i\left(p_\mu \frac{\partial}{\partial p_\nu} - p_\nu \frac{\partial}{\partial p_\mu}\right) & \mu,\nu = 1,2,3 \\ i\sqrt{\vec{p}^2 + m^2}\, \frac{\partial}{\partial p_\nu} & \mu = 0,\ \nu = 1,2,3. \end{cases}$$

Then the Lorentz-invariance condition can be written in the following way:

$$M_{\mu\nu} |n_K\rangle = 0. \tag{2.6}$$

As $M^{(p)}_{\mu\nu}$ define the orbital momentum of the external particle with four-momentum p, and $-i(a^+_\mu a_\nu - a^+_\nu a_\mu)$ the resonance spin, eq. (2.6) means, that in this case spin arises due to the external particle orbital momentum. In the narrow resonance model for scattering of scalar particles just this situation is realized and, therefore, eq. (2.6) could be treated as an auxiliary condition, which singles out of the whole set (2.4) only the part with the above-mentioned property. The general form of those vectors (2.4), which obey condition (2.6) is:

$$|f_n(p)\rangle = \sum_{n_1+2n_2+n_3=n} f_{n_1 n_2 n_3}(p^\mu a^+_\mu)^{n_1}(a^+_\mu a^{+\mu})^{n_2}(a^+_4)^{n_3}|0\rangle. \tag{2.7}$$

Thus, the Lorentz invariance itself does not result in the unique determination of states (2.3). In particular, we see that eq. (2.3) does not contain at all the factor $(a^+_\mu a^{+\mu})$. In order to rule it out we note that the operators

$$\frac{p^\mu a^+_\mu}{m^2}, \quad \frac{p^\mu a_\mu}{m^2} \tag{2.8}$$

are the creation and annihilation operators of one-dimensional oscillations, collinear with p in the four-dimensional space-time. Eq. (2.3) shows, that the transversal oscillations

$$a^+_\mu - \frac{p^\mu a^+_\mu}{m^2} p_\mu, \quad a_\mu - \frac{p^\nu a_\nu}{m^2} p_\mu \tag{2.9}$$

do not contribute to the (?) coherent states and therefore the following equation holds:

$$\left(a_\mu - \frac{p^\nu a_\nu}{m^2} p_\mu\right) |n_\kappa\rangle = 0 \qquad (2.10)$$

$$H_t |n_\kappa\rangle = 0 ,$$

where

$$H_t \equiv -\left(a^+_\mu - \frac{p^\nu a^+_\nu}{m^2} p_\mu\right)\left(a^\mu - \frac{p^\nu a_\nu}{m^2} p^\mu\right) = -a^+_\mu a^\mu + \frac{(p^\nu a^+_\nu)(p^\mu a_\mu)}{m^2} .$$

It is obvious, that gauge condition (2.10) is the second auxiliary condition, which is satisfied by vectors (2.3). Being applied to vectors (2.7) the factor $a^+_\mu a^\mu$ disappears and instead of (2.7) we get

$$|f_n(p)\rangle = \sum_{\ell=0}^{n} f_\kappa \, (p^\mu a^+_\mu)^\ell (a^+_4)^{n-\ell} \, |0\rangle \qquad (2.11)$$

In this manner from the whole set of Hamiltonian eigenvectors Lorentz-invariance (2.6) and longitudinality condition (2.10) select only vectors (2.11). Though they do not contain extra structures, they still differ from vectors (2.3). In order to get complete coincidence a third condition is required which should affect the fifth component of the oscillator. It is clear that any obvious physical meaning cannot be expected for such a condition.

Let us suppose that the third condition has a form of a linear homogeneous equation:

$$(\alpha^{\mu} a_{\mu} + \beta a_4)|f_n\rangle = 0. \qquad (2.12)$$

Substituting eq. (2.3) into (2.12), for $\alpha^{\mu}$ and $\beta$ we get

$$\alpha^{\mu} = \frac{p^{\mu}}{m}, \quad \beta = i\sqrt{\alpha'}\, m. \qquad (2.13)$$

( As far as (2.12) is a homogeneous equation, $\alpha^{\mu}$ and $\beta$ are determined only up to an arbitrary common factor, which value has been taken to be equal to unity). Thus, (2.12) takes the form

$$\left(\frac{p^{\mu} a^{\mu}}{m} + i\sqrt{\alpha'}\, m a_4\right)|f_n\rangle = 0. \qquad (2.14)$$

Now it is easy to be convinced, that within a normalization constant the latter condition transforms (2.11) into (2.3). In this way we have found three auxiliary conditions (2.6), (2.10) and (2.14). When applied to the eigenvectors of the five dimensional oscillator Hamiltonian, they select in a unique way only the dual resonance states.

3. As we have already mentioned, because of the presence of the fifth oscillator component, condition (2.14) has no obvious physical meaning. That is why a more intent consideration of this

condition is needed. We shall show that eq. (2.14) considered as an equation for the states

$$\left(\frac{p^\mu a_\mu}{m} + i\sqrt{\alpha'}\, m a_4\right)|f\rangle = 0 \qquad (3.1)$$

is an invariant equation with respect to the representation of $SL(2,R)$ group which has been determined in ref.[8]. Let us briefly recall the main results of this paper. Let

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \qquad \alpha\delta - \beta\gamma = 1 \qquad (3.2)$$

be an arbitrary element of the group of all real unimodular two-dimensional matrices ( the group SL(2,R) ). It is known, that the projective transformations

$$z \longrightarrow \frac{\alpha z + \beta}{\gamma z + \delta} \qquad (3.3)$$

realize a conformal transformation of the complex $z$ plane and their action is transitive. And the whole plane decomposes into two homogeneous invariant subspaces: the upper and the lower half-planes.

On the other hand, the one-dimensional oscillator coherent states are uniquely characterised by the complex variable $z$;

$$|z\rangle = e^{za^+}|0\rangle, \tag{3.4}$$

i.e., one-to-one correspondence exists between coherent states and points in the complex $z$ plane. Therefore transformation (3.3) of this plane induces the corresponding transformation in the coherent state space, which has the form

$$|z\rangle \longrightarrow \left|\frac{\alpha z+\beta}{\gamma z+\delta}\right\rangle. \tag{3.5}$$

This transformation realizes a representation of SL(2,R) group in the linear covering of vectors (3.4). As well as in the case of the complex plane, the total set of states (3.4) splits into two subsets: with $\mathrm{Im}\, z > 0$ and $\mathrm{Im}\, z < 0$. Similarly, in the linear coverings of these subsets the representation, induced by (3.5), splits into two representations. One of them does not take the vectors out of the subset $\mathrm{Im}\, z > 0$, another one- out of the subset $\mathrm{Im}\, z < 0$. As both the representations have similar structures, we will restrict ourselves to considering only one of them, e.g., for $\mathrm{Im}\, z > 0$. The subspace $\mathrm{Im}\, z = 0$ is not invariant and should be treated as a boundary case of the first two cases.

It is also possible to define a representation $\theta(g)$ of SL(2,R) in the five-dimensional oscillator coherent state space. This has been made in ref.[8] and the action of its operators $\theta(g)$ is given by:

$$\theta(g)\,|z_\mu; z_\nu\rangle = \left|\frac{\alpha z_\nu+\beta}{\gamma z_\nu+\delta}\cdot\frac{z_\mu}{z_\nu}; \frac{\alpha z_\nu+\beta}{\gamma z_\nu+\delta}\right\rangle, \qquad (3.6)$$

where

$$|z_\mu; z_\nu\rangle = e^{z^\mu a_\mu^+ + z_\nu a_\nu^+}\,|0\rangle. \qquad (3.7)$$

Is is easy to see, that the subspace $\Omega(c)$ ( $c$ - fixed four-vector) which is a linear covering of the vectors $|z_\nu c_\mu; z_\nu\rangle$ is invariant with respect to representation (3.6), i.e., $c_\mu = \frac{z_\mu}{z_\nu}$ is an invariant of this representation. By analogy with the one-dimensional case, $\Omega(c)$ decomposes into two invariant subspaces: $\Omega^+(c)$ for $\mathcal{J}m\, z_\nu > 0$ and $\Omega^-(c)$ for $\mathcal{J}m\, z_\nu < 0$. The representation $\theta(g)$ in the subspace $\Omega(c)$ and representation (3.5) of SL(2,R) in the one-dimensional oscillator coherent state space are equivalent.

An important property of these representation is the possibility to define in $\Omega^+(c)$ an operator, inverse to the annihilation operator ( the same could be made in $\Omega^-(c)$). In fact, let us take the transformation:

$$S = \theta(s) \qquad (3.8)$$

where

$$s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in SL(2,R). \qquad (3.9)$$

On the other hand, it is known that the coherent states are eigenstates of the annihilation operaotrs, i.e.,

$$a_4 |z_4 c_\mu ; z_4\rangle = z_4 |z_4 c_\mu ; z_4\rangle. \tag{3.10}$$

Combining two equations (3.9) and (3.10) gives:

$$-S a_4 S |z_4 c_\mu ; z_4\rangle = \frac{1}{z_4} |z_4 c_\mu ; z_4\rangle. \tag{3.11}$$

This relation shows that the operator $-Sa_4 S$ in its essence is inverse with respect to $a_4$ . Eq. (3.11) is well defined, as $\mathcal{J}m\, z_4 > 0$ and $z_4$ cannot take the value zero . For brevity we will adopt the notation

$$\frac{1}{a_4} = -S a_4 S \tag{3.12}$$

so far as for an arbitrary coherent state with $\mathcal{J}m\, z_4 > 0$ the identity

$$-S a_4 S a_4 = -a_4 S a_4 S = 1 \tag{3.13}$$

holds.

The operation $a_\mu$ action (3.11) from the left, with the help of (3.12), gives

$$\frac{a_\mu}{a_4} |z_4 c_\mu ; z_4\rangle = c_\mu |z_4 c_\mu ; z_4\rangle. \tag{3.14}$$

This equation shows, that the operator $\frac{a_\mu}{a_4}$ is invariant under the representation $\theta(g)$ as its eigenvalues are invariant. ( The same could be proved directly, showing that $\left[\frac{a_\mu}{a_4}, \theta(g)\right]=0$ for arbitrary $g \in$ SL(2,R) ).

Let us now return to equation (3.1). Being multiplied from the left by the above-defined operator $\frac{1}{a_4}$ it takes the form:

$$\left(\frac{a_\mu}{a_4}\cdot\frac{p_\mu}{m}+i\sqrt{\alpha'}\,m\right)|f\rangle=0. \tag{3.15}$$

This is possible if one supposes, that $|f\rangle$ belongs to the covering of the five-dimensional oscillator coherent states with $\mathrm{Im}\, z_4 \neq 0$ . Form (3.15) of eq . (3.1) shows explicitly the SL(2,r) invariance of this equation with respect to the representation $\theta(g)$ which accomplishes the proof of our statement. ( This statement could be proved in a common manner without dividing by the operator $a_4$.) Because of the presence of two representations $\theta^+(g)$ and $\theta^-(g)$ in $\Omega^+(c)$ and $\Omega^-(c)$ , respectively, it is obvious, that there will be two equations (3.15) too, which are invariant with respect to $\theta^+(g)$ and $\theta^-(g)$ . ( The operators $\frac{1}{a_4}$ in the spaces $\Omega^+(c)$ and $\Omega^-(c)$ do not coincide).

Thus we see that the third condition, which fixes the form of the dual resonance states, is connected with the presence of definite inner symmetry, described by SL(2,R) group.

4. In this section we shall consider the general solution of the SL(2,R) invariant equation, as well as some mathematical questions connected with it. We shall solve the equation (3.1) in the space $\Omega^+(c)$ . If one realizes the operators $a^+$ and $a$ in the form of the operators of multiplication by $a^+$ and differentiation with respect to $a^+$ correspondingly, eq. (3.1) takes the form of a first order partial differential equation, which general solution is:

$$|\psi\rangle = \int\limits_{\operatorname{Im} z^\mu p_\mu > 0} d^4 \operatorname{Re} z_\mu \, d^4 \operatorname{Im} z_\mu \, \psi(z_\mu, \bar{z}_\mu) \, e^{z_\mu \left( i\sqrt{2}\, a^{+\mu} + \frac{p^\mu a_4^+}{m^2} \right)} |0\rangle \qquad (4.1)$$

( $\bar{z}_\mu$ is complex conjugate to $z_\mu$ ). We see that in order to exploit eq. (3.1) in the framework of the invariant space $\Omega^+(c)$ we have to deal with integrals of the coherent states, which are taken not over the whole complex z-plane, but only over a part of it. From this point of view all sums of type (4.1) essentially differ from similar ones, in which the summation is extended over the whole complex plane. In other words, expressing all in terms of the one-dimensional oscillator, here we have to deal with integrals of the type:

$$I = \int\limits_{\operatorname{Im} z > 0} d\operatorname{Re} z \, d\operatorname{Im} z \, \psi(z, \bar{z}) \, e^{z a^+} |0\rangle , \qquad (4.2)$$

which have the SL(2,R) covariant meaning. It is easy to see that the common methods which provide results in the case of integrals over the whole complex plane, now are useless. Moreover, even their meaning here becomes unclear. As a matter of fact, the usual factor $e^{-\lambda \bar{z}}$ which provides the convergence of the common type integrals, is not invariant with respect to the representation $\theta(g)$. After an SL(2,R) transformation it is no more cut-off factor. In order to clarify this question, let us introduced a new parametrisation of SL(2,R) group:

$$g(x_1, x_2, x_3) = \frac{1}{\sqrt{\Delta}} \begin{pmatrix} x_2 - x_3 & x_1(x_3 - x_2) \\ x_2 - x_1 & x_3(x_1 - x_2) \end{pmatrix}, \quad x_1 < x_2 < x_3, \tag{4.3}$$

where

$$\Delta = (x_1 - x_2)(x_2 - x_3)(x_3 - x_1).$$

In this parametrization transformation (3.3) takes the form:

$$z \longrightarrow \left(g(x_1, x_2, x_3) z\right) = \frac{(z - x_1)(x_2 - x_3)}{(z - x_3)(x_2 - x_1)} . \tag{4.4}$$

We see, that the points $x_1$, $x_2$ and $x_3$ are transformed into 0,1 and $\infty$ , correspondingly. The transitivity of the transformation (4.4) assures that $x_1, x_2, x_3$ could be chosen arbitrary ( as they are real, we have imposed ordering condition (4.3)).

On the other hand, the upper half-plane $\mathrm{Im}\, z > 0$ is homogeneous and, consequently, every point $z_0$ with $\mathrm{Im}\, z_0 > 0$ could be obtained from $z = i$ with the help of suitable transformation (4.4):

$$z_0 = \left( g(x_1, x_2, x_3)\, i \right). \tag{4.5}$$

Let us consider now the integrals:

$$I(a^+; x_1, x_2, x_3) = \int\limits_{\mathrm{Im}\, z_0 > 0} \frac{d\mathrm{Re}\, z\, d\mathrm{Im}\, z}{(z-\bar{z})^2} f\left(g^{-1}(x_1, x_2, x_3) z\right). \tag{4.6}$$

$$\exp\left\{ 4 \frac{[z - (g(x)i)][\bar{z} - \overline{(g(x)i)}]}{[z - \bar{z}][(g(x)i) - \overline{(g(x)i)}]} \right\} e^{\lambda a^+} |0\rangle .$$

Here $f(z)$ is an arbitrary function and the dash denotes complex conjugation. After some simple calculations one can show, that the operator $\theta(h)$ ( $h \in SL(2,R)$ ) transforms vector (4.6) in the following way:

$$\theta(h)\, I(a^+; x_1, x_2, x_3) = I(a^+; x_1', x_2', x_3'), \tag{4.7}$$

where $x_1'$, $x_2'$ and $x_3'$ are given by

$$g(x_1', x_2', x_3') = h g(x_1, x_2, x_3). \tag{4.8}$$

Thus the form of integral (4.6) is invariant with respect to the representation $\theta(g)$ .

Consider now the argument of the nonoperator exponent in the r.h.s. of (4.6). For $z = u + iv$ it takes the form:

$$-\frac{1}{\sqrt{\Delta}}\left|z-(g(x)i)\right|^{2}\left(1+x_3^2\right)\left(x_2-x_1\right)^2 < 0.$$

Consequently, in integral (4.6) this exponent is a *cut-off factor*, which exponentially tends to zero on the whole boundary of the integration domain. Hence it follows, that if $f(z)$ grows on this boundary slower than exponent, then integral (4.6) will converge for arbitrary $x_1, x_2, x_3$.

Now one should remember, that transformation (4.8) is transitive in the space of the parameters of the group. This means that it is sufficient to determine (4.6) for some fixed values of the parameters $x_1$, $x_2$ and $x_3$, and then, by means of the representation $\theta(g)$ one can pass to other values. It is convenient to choose as such fixed values the following ones: $x_1 = 0$, $x_2 = 1$, $x_3 = \infty$ and then $g(x) =$ unit matrix. In this way, instead of (4.6) we get

$$I(a^+)|0\rangle = \int\limits_{\mathrm{Im}\,z>0} \frac{d\mathrm{Re}\,z\, d\mathrm{Im}\,z}{(z-\bar z)^2} f(z,\bar z)\, e^{-2i\frac{z\bar z+1}{z-\bar z}}\, e^{za^+}|0\rangle. \qquad (4.9)$$

The last formula gives the general expansion of an arbitrary state from $\Omega^+(c)$ in coherent states, which belong to the same space. In what follows we will list some particular cases of this expansion, which proof is left to Appendix.

The first particular case is for $f(z,\bar{z}) = 1$ . The calculation of the integral gives:

$$\frac{e^2}{\pi} \int\limits_{Jm\, z>0} \frac{d\,Re\, z\, d\,Jm\, z}{(z-\bar{z})^2} e^{-2i\frac{z\bar{z}+1}{z-\bar{z}}} e^{za^+} |0\rangle = e^{ia^+} |0\rangle . \tag{4.10}$$

If one applies to this equality an arbitrary element of the representation $\theta(g)$ one gets also the identity:

$$e^{z'a^+} |0\rangle = \frac{-4}{\pi} \int\limits_{Jm\, z>0} \frac{d\,Re\, z\, d\,Jm\, z}{(z-\bar{z})^2} e^{4\frac{(z-z')(\bar{z}-\bar{z}')}{(z'-\bar{z}')(z-\bar{z})}} e^{za^+} |0\rangle . \tag{4.11}$$

Obviously, (4.11) determines $\delta$ - function in the space $\Omega^+(c)$. Formulae (4.10) and (4.11) open the possibility to obtain some other their modifications, given in Appendix.

The authots express their deep gratitude to V.A.Matveev and R.M.Muradyan for many fruitful discussions.

## APPENDIX

A. Let us calculate the integral (4.9) for $f(z,\bar{z}) = 1$. To simplify the notations we shall drop out the vacuum state and instead of $a^+$ we shall write down the letter $\alpha$, treating it as an arbitrary complex number. If $z = x+iy$ then in this case (4.9) can be rewritten in the form

$$I(\alpha) = \int_{-\infty}^{\infty} dx \int_{0}^{\infty} \frac{dy}{y^4}\, e^{-\frac{x^2+y^2+1}{y}}\; e^{(x+iy)\alpha} . \tag{A.1}$$

After simple calculations one can integrate over $x$ and get the result:

$$I(\alpha) = \sqrt{\pi} \int_{0}^{\infty} dy\, y^{-3/2}\, e^{-\left(1-\frac{i\alpha}{2}\right)^2 y - \frac{1}{y}} . \tag{A.2}$$

Thus, we have to calculate the integral

$$\varphi(z) = \int_{0}^{\infty} dy\, y^{-3/2}\, e^{-z^2 y - \frac{1}{y}} , \tag{A.3}$$

where $z$ is an arbitrary complex number. The presence of the factor $e^{-1/y}$ guarantees the convergence at the point zero, but to get convergence at infinity one has to demand

$$Re\, z^2 > 0. \tag{A.4}$$

As $\varphi(z)$ is an analytical function of $z$ in this region for $Re z^2<0$ integral (A.3) can be regularised through analytical continuation. In order to calculate the integral in region (A.4) we first differentiate (A.3) with respect to $z$ :

$$\frac{d\varphi}{dz} = -2z\int_0^\infty dy\, y^{-1/2} e^{-z^2 y - \frac{1}{y}} \tag{A.5}$$

The substitution

$$y = \frac{1}{z^2 u} \tag{A.6}$$

leads to

$$\frac{d\varphi}{dz} = -2\int_C du\, u^{-3/2} e^{-z^2 u - \frac{1}{u}}, \tag{A.7}$$

where the contour $C$ is a ray from the coordinate origin with the slope $tg\, \varphi_0 = -\frac{Im\, z^2}{Re\, z^2}$. Because of the integrand analyticity in the sector between $C$ and the real axis and due to condition (A.4), we have

$$\int_C du\, u^{-3/2} e^{-z^2 u - \frac{1}{u}} = \int_0^\infty du\, u^{-3/2} e^{-z^2 u - \frac{1}{u}} \tag{A.8}$$

Then from (A.7) we get the differential equation

$$\frac{d\varphi}{dz} = -2\varphi, \tag{A.9}$$

which together with the initial condition

$$\varphi(0) = \int_0^{\infty} dy\, y^{-3/2} e^{-1/y} = \Gamma(1/2) = \sqrt{\pi} \tag{A.10}$$

gives the following result

$$\varphi(z) = \sqrt{\pi}\, e^{-z} e^{-i\alpha}. \tag{A.11}$$

Thus, for integral (A11) we obtain

$$I(\alpha) = \pi e^{-2} e^{-i\alpha}, \tag{A.12}$$

which coincides with formula (4.9).

B. Consider the integral

$$\mathcal{J}(\alpha) = \int_{\operatorname{Im} z>0} \frac{d\operatorname{Re} z\, d\operatorname{Im} z}{(z-\bar z)^2}\, 2i\frac{z\bar z+1}{z-\bar z}\, e^{-2i\frac{z\bar z+1}{z-\bar z}}\, e^{z\alpha} =$$

$$= \int_{-\infty}^{\infty} dx \int_0^{\infty} \frac{dy}{y^2} \cdot \frac{x^2+y^2+1}{y}\, e^{-\frac{x^2+y^2+1}{y}}\, e^{(x+iy)\alpha}. \tag{B.1}$$

Using the identity

$$\frac{1}{y}\left(x^2+y^2+1\right)=y\left(-i\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)\left(-\frac{x^2+y^2+1}{y}\right)-2i(x+iy) \qquad \text{(B.2)}$$

we can integrate by parts and bring integral (B.1) to the form

$$\mathcal{J}(\alpha)=-2i\int_{-\infty}^{\infty}dx\int_{0}^{\infty}\frac{dy}{y^2}(x+iy)e^{-\frac{x^2+y^2+1}{y}}\cdot e^{(x+iy)\alpha}+$$

$$+\int_{-\infty}^{\infty}dx\int_{0}^{\infty}\frac{dy}{y^2}\,e^{-\frac{x^2+y^2+1}{y}}\cdot e^{(x+iy)\alpha} \qquad \text{(B.3)}$$

Thus, we have proved the identity:

$$\int_{\mathcal{J}m z>0}\frac{d\mathcal{R}e z\, d\mathcal{J}m z}{(z-\bar{z})^2}\, z\, e^{-2i\frac{z\bar{z}+1}{z-\bar{z}}}\cdot e^{\alpha z}= \qquad \text{(B.4)}$$

$$=\frac{1}{2i}\int_{\mathcal{J}m z>0}\frac{d\mathcal{R}e z\, d\mathcal{J}m z}{(z-\bar{z})^2}\left(1-2i\frac{z\bar{z}+1}{z-\bar{z}}\right)e^{-2i\frac{z\bar{z}+1}{z-\bar{z}}}\cdot e^{\alpha z}.$$

The integral in the l.h.s. of (3.4) can be calculated, taking into account, that his values is $\frac{dI(\alpha)}{d\alpha}$, where $I(\alpha)$ is given by A.1),

In an analogous way one can prove the more general identity:

$$J_k(\alpha) = \int\limits_{Im z>0} \frac{dRe z\, dIm z}{(z-\bar z)^2} z^k e^{-2i\frac{z\bar z+1}{z-\bar z}} e^{z\alpha} =$$

$$= \frac{1}{(2i)^k} \int\limits_{Im z>0} \frac{dRe z\, dIm z}{(z-\bar z)^2} F_k\left(2i\frac{z\bar z+1}{z-\bar z}\right) e^{-2i\frac{z\bar z+1}{z-\bar z}} e^{z\alpha}. \qquad \text{(B.5)}$$

In order to find out the function $F_k$ we first differentiate (B.5) with respect to $\alpha$ and get

$$J_{k+1}(\alpha) = \frac{1}{(2i)^k} \int\limits_{Im z>0} \frac{dRe z\, dIm z}{(z-\bar z)^2} z F_k\left(2i\frac{z\bar z+1}{z-\bar z}\right) e^{-2i\frac{z\bar z+1}{z-\bar z}} e^{z\alpha}. \qquad \text{(B.6)}$$

Now in (B.6) we can use identity (B.2) to perform $\ell$ times integration by parts. In this manner we obtain:

$$J_{k+1}(\alpha) = \frac{1}{(2i)^{k+1}} \int\limits_{Im z>0} \frac{dRe z\, dIm z}{(z-\bar z)^2} \left\{\sum_{n=0}^{\ell} F_k^{(n)}\left(2i\frac{z\bar z+1}{z-\bar z}\right) - 2i\frac{z\bar z+1}{z-\bar z} F_k\left(2i\frac{z\bar z+1}{z-\bar z}\right)\right\} e^{-2i\frac{z\bar z+1}{z-\bar z}+i\alpha} - \qquad \text{(B.7)}$$

$$- \frac{4i}{(2i)^{k+1}} \int\limits_{Im z>0} \frac{dRe z\, dIm z}{(z-\bar z)^2} F_k^{(\ell+1)}\left(2i\frac{z\bar z+1}{z-\bar z}\right) e^{-2i\frac{z\bar z+1}{z-\bar z}} e^{z\alpha} \frac{\partial}{\partial \bar z} \cdot \frac{z\bar z+1}{z-\bar z},$$

where

$$F_k^{(n)}(x) = \frac{d^n}{dx^n} F_k(x). \qquad \text{(B.8)}$$

It follows from here and from definition (B.5), that $F_k(x)$ is a polynomial of degree $k$ . Actually, we have already seen, that $F_1 = 1-x$, i.e., it is a first-degree polynomial ( see eq. (B.4) ). If in eq. (B.7) for $k = 1$ we take $\ell = 1$ and compare this expression with the r.h.s. of (B.5) for $k = 2$, we get

$$F_2(x) = F_1^{(0)}(x) + F_1^{(1)}(x) - x F_1(x), \tag{B.9}$$

i.e., $F_2(x)$ is a second-degree polynomial.

$$F_2(x) = x^2 - 2x. \tag{B.10}$$

The continuation of this procedure leads us to

$$F_k(x) = (-1)^{k-1}\left(k x^{k-1} - x^k\right). \tag{B.11}$$

This accomplishes the proof of the identity (B.5).

As the integrals in the l.h.s. of eq. (B.5) are $k$ - th derivatives of $I(\alpha)$ with respect to $\alpha$ it is easy to get

$$\mathcal{J}_k(\alpha) = \pi e^{-2} i^k e^{i\alpha}. \tag{B.12}$$

Therefore, if $f(z)$ is an entire function of the complex variable $z$ , then the identity

$$\int\limits_{\mathcal{J}m\, z>0} \frac{d\mathcal{R}e\, z\, d\mathcal{J}m\, z}{(z-\bar{z})^2} f(z) e^{-2i\frac{z\bar{z}+1}{z-\bar{z}}} e^{z\alpha} =$$

$$= \frac{-1}{(2i)^k} \int\limits_{\mathcal{J}m\, z>0} \frac{d\mathcal{R}e\, z\, d\mathcal{J}m\, z}{(z-\bar{z})^2} \left[ f'\left(-2i\frac{z\bar{z}+1}{z-\bar{z}}\right) + f\left(-2i\frac{z\bar{z}+1}{z-\bar{z}}\right) \right] e^{-2i\frac{z\bar{z}+1}{z-\bar{z}}+z\alpha} = \tag{B.13}$$

$$= \pi e^{-2} f(i) e^{i\alpha}$$

holds, where

$$f'(x) = \frac{df}{dx}.$$

This result shows, that on the half-plane $\mathcal{J}m\, z > 0$ the coherent states present an overcomplete basis , as well as on the whole plane. The formulae which we have obtained in this Appendix, are easily transformed by complex conjugation into the corresponding formulae for the case $\mathcal{J}m\, z < 0$.

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