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SOME REMARKS ON LORENTZ INVARIANT VARIABLES FOR MANY-PARTICLE REACTIONS

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INTRODUCTION

At the present time the question of appropriate variables for the description of many-particle reactions is becoming more and more important. Let the n 4-momenta of a system of n particles be $p_1, \ldots, p_n$. When investigating Lorentz invariant variables for describing the state of this system one has to consider, in addition to the simple invariants (the Minkowski scalar products), the determinants of the components of the sets of four 4-momenta. This is a consequence of the fundamental role of the 4-dimensional space-time. We emphasize that the simplest structure of space-time which underlies any physical theory, a causal one, is the flat Minkowski space /2/.

The determinant of components of four 4-momenta of a reaction with only four particles in a final state denoted by Δ_{1234} is shown to have a simple geometrical interpretation, reducing essentially to a volume in the 3-dimensional $\vec{p}$ space. Therefore,groups of three particles may play a special role in the invariant description of four particle groups in a final state. Some details are given and a digression on a particular case of the general problem of invariance /3/ is included in Sect. 2, only * to illustrate as an example how this determinant may appear.

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*The theoretical significance will be discussed in another paper .

The aim of this note is to suggest to try if Δ_{1234} plays any exquisite role in the "mechanism" of interaction. The best way for testing it, for the beginning, is to get the hystogram for Δ_{1234} computed from experimental data using the components of the 4-momenta of the four particles in final state and compare it with the phase space hystogram for the reaction with respect to Δ_{1234}.

Finally we wish to point out that the determinant is, as a polynomial of the momentum components an observable so it permits a physical interpretation in the frame of quantum mechanics. It corresponds (its values) to the spectrum of one observable projected on a subspace of the "four particle" Hilbert space $(H_{out})_4$.

2. INVARIANT DETERMINANT FOR FOUR MOMENTA

We consider a set of four 4-momenta p_1, p_2, p_3, p_4. These are vectors in a Minkowski space.

$p_i p_k = p_i^0 p_k^0 - \vec{p}_i \vec{p}_k$.

Let us for a moment look at p_1, p_2, p_3 and p_4 as 4-vectors in a real 4-dimensional space R^4 each p_i having the components $p_i^0, p_i^1, p_i^2, p_i^3$, $(i = 1,2,3,4)$. From the components we can form the determinant

$$\Delta_{1234} = \begin{pmatrix} p_1^0 & p_2^0 & p_3^0 & p_4^0 \\ p_1^1 & p_2^1 & p_3^1 & p_4^1 \\ p_1^2 & p_2^2 & p_3^2 & p_4^2 \\ p_1^3 & p_2^3 & p_3^3 & p_4^3 \end{pmatrix} \qquad (1)$$

We wish to recall that Δ_{1234} is invariant with respect to all linear homogeneous transformations of the vectors which have the determinant equal to unity. Indeed, let A be the matrix of an arbitrary linear homogeneous transformation with the real coefficients and P the matrix from (1). Denote the matrix product $\bar{P} = A \times P$ then for those transformations for which $\det A = 1$ one has $\det \bar{P} = \det A \det P$. So $\Delta_{1234} = \det P$ is invariant under all the linear transformations represented by 4x4 matrices with the real coefficients and determinant equals 1, i.e., to the group $SL(4,R)$.

The invariance with respect to proper Lorentz transformations follows as a special case (being a subgroup of $SL(4,R)$). So the invariance of Δ_{1234} is evidently independent of the metrics (Euclidean or Minkowski) defined on p -space.

The determinant Δ_{1234} is also directly connected with the Gram determinant attached to the set p_1, p_2, p_3, p_4. It is the determinant of the matrix with the coefficients being the scalar product of four-vectors, which we denote by G and by $D_{1234} = \det G$ so

$$D_{1234} = \begin{pmatrix} p_1^2 & p_1 p_2 & p_1 p_3 & p_1 p_4 \\ p_1 p_2 & p_2^2 & p_2 p_3 & p_2 p_4 \\ p_1 p_3 & p_2 p_3 & p_3^2 & p_3 p_4 \\ p_1 p_4 & p_2 p_4 & p_3 p_4 & p_4^2 \end{pmatrix} \qquad (1')$$

As it is immediately seen, if the scalar product $p_i p_j$ $(i,j = 1,2,3,4)$ would mean an Euclidean one $p_i p_j = p_i^0 p_j^0 + \vec{p}_i \vec{p}_j$ then the matrix $G = P^T \times P$ (T means transposed) and in this case D_{1234} (Euclidean case) would be always positive, being the square of $\det P$.

For the pseudo-Euclidean scalar $p_i p_j = p_i^0 p_j^0 - \vec{p}_i \vec{p}_j$ we must multiply the space components by $\sqrt{-1} = i$ then we have *

$$D_{1234} = i^6 (\Delta_{1234})^2 \qquad (2)$$

Now since the elements of Δ_{1234} are real, the D_{1234} is always negative, which is one of the constraints for defining the physical domain /4/. Though (2) establishes a relation between Δ_{1234} and D_{1234} one can not say that (2) permits to express Δ_{1234} in the simple Lorentz invariants through D_{1234} since the signum** of D_{1234} appears. It is well known that the Gram determinant, and thus also the determinant of the components vanishes if and only if p_1, p_2, p_3 and p_4 are linearly dependent. A rather complete treatment of the Lorentz invariant variables, algebraical and geometrical properties of the physical regions for the exclusive and inclusive processes with n particles, one can find in ref. /5/.

Let us now have the elementary particle reactions with four particles in a final state

$$a + b \to a_1 + a_2 + a_3 + a_4 . \qquad (3)$$

In the center of mass system (c.m.) of the colliding

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* Indeed,

$$G = \begin{pmatrix} p_1^o & ip_1^1 & ip_1^2 & ip_1^3 \\ p_2^o & ip_2^1 & ip_2^2 & ip_2^3 \\ p_3^o & ip_3^1 & ip_3^2 & ip_3^3 \\ p_4^o & ip_4^1 & ip_4^2 & ip_4^3 \end{pmatrix} \begin{pmatrix} p_1^o & p_2^o & p_3^o & p_4^o \\ ip_1^1 & ip_2^1 & ip_3^1 & ip_4^1 \\ ip_1^2 & ip_2^2 & ip_3^2 & ip_4^2 \\ ip_1^3 & ip_2^3 & ip_3^3 & ip_4^3 \end{pmatrix}$$

** It is obvious since Δ_{1234} is a pseudoscalar and not a scalar.

particles, according to the conservation law of four 4-momenta, we have:

$$\sum_{i=1}^{4} p_i^{\nu} = 0 , \qquad \nu = 1,2,3$$

$$\sum_{i=1}^{4} p_i^{\circ} = \sqrt{s}$$

then the determinant of the components of the momenta of particles in final states (we have not specified "in center of mass system" since Λ_{1234} is invariant) can be written

$$\Lambda_{1234} = \begin{pmatrix} p_1^{\circ} & p_2^{\circ} & p_3^{\circ} & \sqrt{s} \\ p_1^1 & p_2^1 & p_3^1 & 0 \\ p_1^2 & p_2^2 & p_3^2 & 0 \\ p_1^3 & p_2^3 & p_3^3 & 0 \end{pmatrix} = \sqrt{s}\ (\vec{p}_1 \times \vec{p}_2) \cdot \vec{p}_3 \qquad (4)$$

Here $\vec{p}_1, \vec{p}_2, \vec{p}_3$ *are three-dimensional momenta, taken in the (c.m.) system.* The mixed product represents the geometrical volume defined by the three dimensional vectors. So Λ_{1234} for a fixed s has a simple geometrical meaning (not a "physical" one yet?).

Let us denote the absolute value of this volume $V_{123}^{(c.m)}$ $|(\vec{p}_1 \times \vec{p}_2) \cdot \vec{p}_3|$ then

$$|\Lambda_{1234}| = \sqrt{s}\, V_{123}^{(c.m)} . \qquad (5)$$

The condition of the stability of the particles, i.e., the fact that the components of p_i satisfy the relation $(p_i^{\circ})^2 - |\vec{p}_i|^2 = m_i^2$ was not used. So the same relation holds when a_4 plays the role of "anything" in inclusive reactions.

The ordering of the momenta (of the particles) in (3) is, of course, nonessential. Thus, in fact any combination of the three momenta can be chosen. So it is worth noticing that all the volumes $V_{i_1 i_2 i_3}$ for any combination $i_1 \neq i_2 \neq i_3 \neq i_4$ with $i_1, i_2, i_3, i_4 = 1,2,3,4$ are equal * to each other.

It is obvious that for a resonance decaying into four particles the mass of the resonance appears instead of $\sqrt{s}$ in (5).

Now if we choose the rest system of one particle which is stable, for instance the particle corresponding to the momentum p_4, and which has the mass denoted by m_4 we get analogously $\Lambda_{1234} = -m_4(\vec{p}_1 \times \vec{p}_2)\cdot\vec{p}_3$. Evidently, $\vec{p}_1, \vec{p}_2, \vec{p}_3$ must be taken in the rest mass of the particle a_4 and $|\Lambda_{1234}| = m_4 V_{123}^{(4)}$; (4) just indicates that the volume $V_{123}^{(4)}$ has to be determined in the rest system of particle a_4. Now using the fact that Λ_{1234} is invariant we get the transformation formula for volumes

$$V_{123}^{(4)} = \frac{\sqrt{s}}{m_4} V_{123}^{(c.m)} . \qquad (6')$$

We note again that we could get any three vectors, so in general the following relation holds

$$\frac{\sqrt{s}}{m_{i_4}} = \frac{V_{i_1 i_2 i_3}^{(i_4)}}{V_{i_1 i_2 i_3}^{(c.m)}} . \qquad (6)$$

The same relation holds for an inclusive reaction. In this case in general the "particle" a_{i_4} corresponding to "anything" is not a stable one and then m_{i_4} in (6) is not a constant, it depends on the individual event.

It also might be interesting to mention the relation which follows when some of the values are taken in the laboratory system for instance in the rest system of the particle a or b from reaction (3). Then we have:

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* A well known geometrical property.

$$\epsilon \sqrt{s}\; V_{123}^{(c.m)} + \epsilon' E^{(L)} V_{123}^{(L)} = p^{(L)} \begin{pmatrix} p_1^{\circ} & p_2^{\circ} & p_3^{\circ} \\ p_1^{2} & p_2^{2} & p_3^{2} \\ p_1^{3} & p_2^{3} & p_3^{3} \end{pmatrix} \qquad (7)$$

where $\epsilon, \epsilon' = \pm 1$. $E^{(L)}$ is the energy in the laboratory system (of the rest mass of one colliding particle) in which the components of p_1, p_2, p_3 in the determinant from (7) must be taken.

Let us now make a digress. We consider a certain process with arbitrary number of particles $n \geq 4$ in final state. The mechanism of reaction is expected to be described by a function f of $p_1, \dots, p_n$. Assuming Lorentz invariance and that the function $f(p_1, \dots, p_n)$ is invariant we have

$$f(p_1, \dots, p_n) = f(\Lambda p_1, \dots, \Lambda p_n) \qquad (8)$$

for any transformation Λ belonging to the restricted Lorentz group. By definition, Λ leaves invariant the scalar product $p_i p_j = p_i^{\circ} p_j^{\circ} - \vec{p}_i \vec{p}_j$. Then one would like to write the function f explicitly in an invariant way, i.e., as a function of the scalar products $p_i p_j$, $i, j = 1, \dots, n$ which actually means, as function of the symmetrical matrices $n \times n$ (Gram matrices).

Let us denote the mapping which carries any set of n four vectors $\underline{p} \equiv (p_1, \dots, p_n)$ in a Gram matrix by π compactly written $\pi(\underline{p}) = G$. For the simplest case when f is a polynomial it was stated by Weyl /1/ that any Lorentz invariant polynomial $P(\underline{p})$ admits the following decomposition

$$P(\underline{p}) = Q(\pi(\underline{p})) + \sum_{1 \leq i_1 < i_2 < i_3 < i_4 \leq n} \Delta_{i_1 i_2 i_3 i_4} Q_{i_1 i_2 i_3 i_4}(\pi(\underline{p}))$$

Thus a Lorentz invariant polynomial $P(\underline{p})$ can be represented explicitly by the polynomials Q and $Q_{i_1 i_2 i_3 i_4}$

of the scalar products and by $n(n-1)(n-2)(n-3)/24$ determinants of the type $\Delta_{i_1 i_2 i_3 i_4}$. They are not independent for $n > 4$ satisfying, the following constraints rank of the matrix $\pi(p) \leq 4$, and

$$\Delta_{i_1 i_2 i_3 i_4}(\underline{p})\, \Delta_{j_1 j_2 j_3 j_4}(\underline{p}) +$$

$$+ \det\, (p_i\, p_j)_{i=i_1, i_2, i_3, i_4;\ j=j_1, j_2, j_3, j_4} = 0 .$$

For the general case when $f(p_1, \dots p_n)$ is not polynomial several details can be found in ref. /6/.

Here we would like only to recall the content of the equation (8). Let us consider a certain point $\underline{p} = (p_1, \dots, p_n)$ and apply all the transformation Λ to it. All the points $(p_1', \dots, p_n') = (\Lambda p_1, \dots, \Lambda p_n)$ obtained in this way form a so-called orbit. The relation (8) means that $f(p_1, \dots, p_n)$ does not change as far as the point $p_1', \dots, p_n'$ describes an orbit. It can change only if we pass from one orbit to the other. In other words, (8) says actually that we define a function $\hat{f}$ on the space of orbits. As to an orbit there corresponds a Gram matrix *eq.(8)leads in a natural way to the necessity of defining a function $\hat{f}$ on the space of the Gram matrices. After this short digress let us give some kinematical relations which come immediately from the usual kinematical constraints.

3. "CONSERVATION" RELATION FOR THE DETERMINANTS

The quotient mark is used just because the determinant we defined above can not be defined for the initial state where there are only two particles, so it is not

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* Gram matrix is not always enough to characterize an orbit, and in general the problem is much more complicated /6/.

a relation of the conservation of a physical value from initial to final state, what we are going to speak about.

Let us consider a reaction

$$a + b \rightarrow a_1 + \dots + a_n \qquad (n \geq 4).$$

As before we form the determinant Δ_{1234}. We replace the fourth column according to the conservation law by $p_4 = p_a + p_b - \sum_{i=1}^{3} p_i - \sum_{k=5}^{n} p_k$. This gives immediately "a conservation relation" for the four-dimensional determinants

$$\sum_{i=a,b,4}^{n} \alpha_i \, \Delta_{123i} = 0 \qquad \alpha_i = \begin{cases} 1 & \text{if} \quad i = a, b \\ -1 & \text{if} \quad 4 \leq i \leq n \end{cases}$$

This relation can also be expressed by using the three dimensional volume using successively the rest system of the particles in which respectively the corresponding volumes are computed. Thus we have

$$\sum_{i=a,b,4}^{n} \epsilon_i \, m_i \, V_{123}^{(m_i)} = 0,$$

where ϵ_i states for ± 1, depending on the ordering of the momenta. It is obvious that instead of p_1, p_2, p_3, p_4 we could take any other four momenta.

Remark: This note is particularly referred to the physicists-experimentalists who possess the experimental data for the reactions with four particles in a final state, both the exclusive and inclusive ones as well as the decaying process in four particles, or five particles in ae final state, and also three particles in a final state.

One can also use the variable Δ for the reactions with three particles in a final state. Then the determinant Δ_{ab12} is directly expressed in azimuthal angle and the perpendicular components of the momenta $p_{1\perp}$ and $p_{2\perp}$. We mention that the correlation with respect to Δ_{ab12} is a new information to those correlations which are already established in azimuthal angle (in the sense that it gives an independent correlation).

It is also interesting for reactions with five particles in a final state to have the plot - which correlates the parallel momentum of one particle with the values of determinant of the remaining four particles.

Finally, it is tempting to help the reason for doing so. The reason is that one gets the new correlations which are strongly connected with the creation-multiparticle reactions, for two in two processes obviously this variable does not exist. Besides, recent studies of high energy multiparticle processes have exploited several model independent properties, just because not even one model for multiple particle processes can hope so far for a theoretical background. Therefore, it becomes rather clear that it is imperatively to look for empirically "dynamical" constraints of the interaction. This arises actually from the fact that the theoretical frame in which one might in a proper sense do the physical interpretation, see /7/ and the references, leads to the results which are very far /8/ from a possible direct connection with experimental data.

The authors like to emphasize also that it is very easy to get the hystogram on Λ and compare them with the phase space hystogram. But, obviously, from a single experiment no one conclusion can follow. So it should be desirable or rather necessary to have such hystogram for different multiproduction reactions: annihilation processes, electro-production, photo-production and hadrono-production, and in every case to perform it for different energies of colliding particles and different sets of the particles in final state. Then it might be possible to get a gess of how much the "mechanism" of production depends manifestly on this variable.

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