ОБЪЕДИНЕННЫЙ
ИНСТИТУТ
ЯДЕРНЫХ
ИССЛЕДОВАНИЙ
ДУБНА

E2 - 7732

D.B.Ion

THE ISOSPIN BOUNDS ON POLARIZATION AND SPIN ROTATION PARAMETERS

1974

ЛАБОРАТОРИЯ
ТЕОРЕТИЧЕСКОЙ ФИЗИКИ

E2 - 7732

D.B.Ion*

THE ISOSPIN BOUNDS ON POLARIZATION AND SPIN ROTATION PARAMETERS

* On leave of absence from Institute for Atomic Physics, Bucharest, Romania.

The bound (1), which is more stringent than the triangle inequalities, was recently investigated by Tornqvist /7/ using the phase shift analyses /8/.

In this paper we derive all the possible upper and lower bounds on H imposed by the isospin invariance. In Section 2 we define the lower and upper bounds on H in terms of the real and imaginary parts of some bilinear forms which can be formed with the scattering amplitudes of two spin $(0\frac{1}{2}\to 0'\frac{1'}{2})$ reactions. In Section 3 we determine these bounds from the isospin invariance conditions. The isospin bounds on polarization and spin rotation parameters in terms of differential cross sections alone are also introduced in Section 3. The bounds (30), (30a,b) obtained here are more stringent than the bound (1).

2. Bounds on Final Polarization Independent of Any Hypothesis

It is interesting to determine the quantitative limits on $H_{ij}\equiv\frac{1}{2}(1-\vec{P}_i\cdot\vec{P}_j)\,\sigma_i\sigma_j$ independent of any internal symmetry hypothesis. For this we observe that the differential cross sections, polarizations and spin rotation parameters are particular cases of the following bilinear forms:

$$Z^{(0)}_{ij}= f^*_i f_j + g^*_i g_j , \tag{3a}$$

$$Z^{(1)}_{ij}= i(f^*_i g_j - g^*_i f_j) , \tag{3b}$$

$$Z^{(2)}_{ij}= f^*_i g_j + g^*_i f_j , \tag{3c}$$

$$Z^{(3)}_{ij}= f^*_i f_j - g^*_i g_j , \tag{3d}$$

or in the helicity frame

$$Z'^{(0)}_{ij}= [f^{++}_i]^* f^{++}_j+[f^{+-}_i]^* f^{+-}_j , \tag{4a}$$

$$Z'^{(1)}_{ij}= i\{[f^{++}_i]^* f^{+-}_j - [f^{+-}_i]^* f^{++}_j\}, \tag{4b}$$

$$Z'^{(2)}_{ij}= [f^{++}_i]^* f^{+-}_j + [f^{+-}_i]^* f^{++}_j , \tag{4c}$$

$$Z'^{(3)}_{ij}= [f^{++}_i]^* f^{++}_j - [f^{+-}_i]^* f^{+-}_j , \tag{4d}$$

where

$$f^{++}_k = f_k \cos\frac{\theta}{2} - g_k \sin\frac{\theta}{2} , \tag{5a}$$

$$f^{+-}_k = f_k \sin\frac{\theta}{2} + g_k \cos\frac{\theta}{2}, \quad k = i, j , \tag{5b}$$

where f_k and g_k are the non-spin-flip and spin-flip scattering amplitudes for spin $(0^-\frac{1}{2}^+)$ scattering, and θ is the centre-mass scattering angle.

We see that

$$Z^{(0)}_{kk} = Z'^{(0)}_{kk} = \sigma_k = | f_k|^2 + |g_k|^2, \tag{6a}$$

$$Z^{(1)}_{kk} = Z'^{(1)}_{kk} = P_k \sigma_k = 2\,\mathrm{Im}\,(f_k g^*_k) , \tag{6b}$$

$$Z^{(2)}_{kk} = T_k \sigma_k = 2\,\mathrm{Re}\,(f^*_k g_k) , \tag{6c}$$

$$Z^{(3)}_{kk} = S_k \sigma_k = | f_k|^2 - |g_k|^2, \tag{6d}$$

$$Z'^{(2)}_{kk} = -A_k \sigma_k = 2\,\mathrm{Re}\,([f^{++}_k]^* f^{+-}_k) , \tag{6e}$$

$$Z'^{(3)}_{kk} = R_k \sigma_k = | f^{++}_k |^2 - | f^{+-}_k |^2. \tag{6f}$$

Also, we define

$$Y^{(0)}_{ij} = f_i g_j - g_i f_j , \tag{7a}$$

$$Y^{(1)}_{ij} = f_i f_j + g_i g_j , \qquad (7b)$$

$$Y^{(2)}_{ij} = f_i f_j - g_i g_j , \qquad (7c)$$

$$Y^{(3)}_{ij} = f_i g_j + g_i f_j . \qquad (7d)$$

We obtain

$$|Z^{(0)}_{if}|^2 = |Z'^{(0)}_{ij}|^2 = \frac{1}{2} [1 + \vec{P}_i \cdot \vec{P}_j]\sigma_i \sigma_j , \qquad (8a)$$

$$|Y^{(0)}_{ij}|^2 = |Y'^{(0)}_{ij}|^2 = \frac{1}{2}[1 - \vec{P}_i \cdot \vec{P}_j]\sigma_i \sigma_j \equiv H_{ij} , \qquad (8b)$$

$$|Z^{(1)}_{ij}|^2 = |Z'^{(1)}_{ij}|^2 = H_{ij} + P_i P_j \sigma_i \sigma_j , \qquad (9a)$$

$$|Y^{(1)}_{ij}|^2 = |Y'^{(1)}_{ij}|^2 = -H_{ij} + (1 - P_i P_j) \sigma_i \sigma_j , \qquad (9b)$$

$$|Z^{(2)}_{ij}|^2 = H_{ij} + T_i T_j \sigma_i \sigma_j , \qquad (10a)$$

$$|Y^{(2)}_{ij}|^2 = -H_{ij} + (1 - T_i T_j) \sigma_i \sigma_j , \qquad (10b)$$

$$|Z'^{(2)}_{ij}|^2 = H_{ij} + A_i A_j \sigma_i \sigma_j , \qquad (11a)$$

$$|Y'^{(2)}_{ij}|^2 = -H_{ij} + (1 - A_i A_j) \sigma_i \sigma_j , \qquad (11b)$$

$$|Z^{(3)}_{ij}|^2 = H_{ij} + S_i S_j \sigma_i \sigma_j , \qquad (12a)$$

$$|Y^{(3)}_{ij}|^2 = -H_{ij} + (1 - S_i S_j) \sigma_i \sigma_j , \qquad (12b)$$

$$|Z'^{(3)}_{ij}|^2 = H_{ij} + R_i R_j \sigma_i \sigma_j , \qquad (13a)$$

$$|Y'^{(3)}_{ij}|^2 = -H_{ij} + (1 - R_i R_j) \sigma_i \sigma_j , \qquad (13b)$$

from which we have

$$|Z^{(n)}_{ij}|^2 + |Y^{(n)}_{ij}|^2 = |Z'^{(n)}_{ij}|^2 + |Y'^{(n)}_{ij}|^2 = \sigma_i \sigma_j , \; n=0,1,2,3. \qquad (14)$$

Now, from the relations (8-13) we see that the following bounds on H_{ij} can be introduced:

$$\max \{ W_\ell (ij) \} \leq H_{ij} \leq \min \{ W_u (ij) \} , \qquad (15)$$

where

$$W_\ell (ij) \equiv 0 ; -Z^{(n)}_{ii} Z^{(n)}_{jj} ; -Z'^{(n)}_{ii} Z'^{(n)}_{jj} ; \ldots ; \; n = 1,2,3, \qquad (15a)$$

$$W_u (ij) \equiv \sigma_i \sigma_j ; \; \sigma_i \sigma_j - Z^{(n)}_{ii} Z^{(n)}_{jj} ; \; \sigma_i \sigma_j - Z'^{(n)}_{ii} Z'^{(n)}_{jj} ; \ldots ; \; n = 1, 2, 3, \qquad (15b)$$

and the most stringent bounds

$$\max \{ L_{ij} \} \leq H_{ij} \leq \min \{ U_{ij} \} , \qquad (16)$$

where

$$L_{ij} \equiv 0 ; [\mathrm{Re}\, Z^{(n)}_{ij}]^2 - Z^{(n)}_{ii} Z^{(n)}_{jj} ; [\mathrm{Im}\, Z^{(n)}_{ij}]^2 - Z^{(n)}_{ii} Z^{(n)}_{jj} ; \ldots ; \; n = 1,2,3, \qquad (16a)$$

$$U_{ij} \equiv \sigma_i \sigma_j - [\mathrm{Re}\, Z^{(0)}_{ij}]^2 ; \; \sigma_i \sigma_j - [\mathrm{Im}\, Z^{(0)}_{ij}]^2 ; \ldots \qquad (16b)$$

In general, either the lower or the upper bound (16) will be saturated when one of the conditions

$$\mathrm{Re}\,Z_{ij}^{(n)} = 0\ ;\ \mathrm{Im}\,Z_{ij}^{(n)} = 0\ ;\ \mathrm{Re}\,Z'^{(n)}_{ij} = 0\ ;\ \mathrm{Im}\,Z'^{(n)}_{ij} = 0\ ;\ldots$$

$$\mathrm{Re}\,Y_{ij}^{(n)} = 0\ ;\quad \mathrm{Im}\,Y_{ij}^{(n)} = 0\ ;\quad \mathrm{Re}\,Y'^{(n)}_{ij} = 0\ ;\quad \mathrm{Im}\,Y'^{(n)}_{ij} = 0\ ;\ldots$$

$$n = 0,1,2,3. \tag{17}$$

is satisfied.

The lower or the upper bound (15) will be saturated when $|Z_{ij}^{(n)}|$ or $|Y_{ij}^{(n)}|$, $n = 0,1,2,3$, vanishes.

Unfortunately, for each of the two reactions (i,j) the measurements of the cross section and final polarization components determine the scattering amplitudes f_k, g_k up to an unobservable common phase factor, so, that the real or imaginary part of any $Z_{ij}^{(n)}$ bilinear form cannot be determined independently of any amplitude analysis or any internal symmetry hypothesis.

We note that all the isospin, U -spin, V -spin or full unitary spin bounds follow from the bounds (16) when L_{ij} and U_{ij} are expressed in terms of unpolarized differential cross sections and final polarization components, using the linear relations implied by these internal symmetries on the transition matrices of different reactions.

3. Determination of the Most Stringent Bounds from the Isospin Invariance Conditions

Let us consider the implications of the isospin invariance and crossing symmetry on the bilinear forms $Z_{ij}^{(n)}$, $n = 0,1,2,3$. The problem can be presented in a general form, but we prefer to discuss here the particular case of the pion-nucleon scattering. Therefore, let (f_+, g_+), (f_-, g_-) and (f_{CE}, g_{CE}) be the non-spin-flip and spin-flip scattering amplitudes for $\pi^\pm P \to \pi^\pm P$ and $\pi^- P \to \pi^0 n$ reactions.

From the isospin invariance condition

$$f_+ = f_- + \sqrt{2}\, f_{CE}\ ;\quad g_+ = g_- + \sqrt{2}\, g_{CE} \tag{18}$$

and s, t, u -channel isospin decomposition of the scattering amplitudes we obtain:

$$\mathrm{Re}\,Z_{+-}^{(n)} = \frac{1}{2}\,[\,Z_{++}^{(n)} + Z_{--}^{(n)} - 2Z_{CE\,CE}^{(n)}\,], \tag{19a}$$

$$\mathrm{Re}\,Z_{+CE}^{(n)} = \frac{1}{2\sqrt{2}}\,[\,Z_{++}^{(n)} - Z_{--}^{(n)} + 2Z_{CE\,CE}^{(n)}\,], \tag{19b}$$

$$\mathrm{Re}\,Z_{-CE}^{(n)} = \frac{1}{2\sqrt{2}}\,[\,Z_{++}^{(n)} - Z_{--}^{(n)} - 2Z_{CE\,CE}^{(n)}\,], \tag{19c}$$

$$\mathrm{Re}\,Z_{13s}^{(n)} = \frac{1}{4}\,[\,Z_{++}^{(n)} + 3Z_{--}^{(n)} - 6Z_{CE\,CE}^{(n)}\,], \tag{19d}$$

$$\mathrm{Re}\,Z_{02t}^{(n)} = \frac{1}{4}\,[\,Z_{--}^{(n)} - Z_{++}^{(n)}\,], \tag{19e}$$

$$\mathrm{Re}\,Z_{13u}^{(n)} = \frac{1}{4}\,[\,Z_{--}^{(n)} + 3Z_{++}^{(n)} - 6Z_{CE\,CE}^{(n)}\,], \tag{19f}$$

and

$$Z_{33s}^{(n)} = Z_{++}^{(n)}\ ;\quad Z_{11s}^{(n)} = \frac{1}{2}\,[\,3Z_{--}^{(n)} + 3Z_{CE\,CE}^{(n)} - Z_{++}^{(n)}\,], \tag{20a}$$

$$Z_{22t}^{(n)} = \frac{1}{2}\,Z_{CE\,CE}^{(n)}\ ;\quad Z_{00t}^{(n)} = \frac{1}{2}\,[\,Z_{++}^{(n)} + Z_{--}^{(n)} - Z_{CE\,CE}^{(n)}\,], \tag{20b}$$

$$Z_{33u}^{(n)} = Z_{--}^{(n)}\ ;\quad Z_{11u}^{(n)} = \frac{1}{2}\,[\,3Z_{++}^{(n)} + 3Z_{CE\,CE}^{(n)} - Z_{--}^{(n)}\,], \tag{20c}$$

where we have used the notation $i,j = 2I$ for I -isospin in s,t,u -channels.

Also, from (18) and s,t,u -channel isospin relations we obtain

$$\mathrm{Im}\, Z^{(n)}_{+-} = \sqrt{2}\,\mathrm{Im}\, Z^{(n)}_{CE+} = \sqrt{2}\,\mathrm{Im}\, Z^{(n)}_{CE-} =$$

$$= \frac{2}{3}\,\mathrm{Im}\, Z^{(n)}_{13s} = 2\,\mathrm{Im}\, Z^{(n)}_{02t} = \frac{2}{3}\,\mathrm{Im}\, Z^{(n)}_{13u}. \qquad (21)$$

Now, in order to determine all $\mathrm{Im}\, Z^{(n)}_{ij}$ we introduce the amplitudes:

$$K^{(\pm)}_i = f_i \pm i g_i ; \qquad |K^{(\pm)}_i|^2 = (1 \pm P_i)\,\sigma_i , \qquad (22a)$$

$$H^{(\pm)}_i = f_i \pm g_i ; \qquad |H^{(\pm)}_i|^2 = (1 \pm T_i)\,\sigma_i , \qquad (22b)$$

$$H'^{(\pm)}_i = f^{++}_i \pm f^{+-}_i ; \qquad |H'^{(\pm)}_i|^2 = (1 \pm A_i)\,\sigma_i , \qquad (22c)$$

in terms of which the bilinear forms $Z^{(n)}_{ij}$, $n = 0,1,2,3$ and $Z'^{(n)}_{ij}$ can be written

$$Z^{(0)}_{ij} = \frac{1}{2}\{[K^{(+)}_i]^* K^{(+)}_j + [K^{(-)}_i]^* K^{(-)}_j\}$$

$$= \frac{1}{2}\{[H^{(+)}_i]^* H^{(+)}_j + [H^{(-)}_i]^* H^{(-)}_j\}$$

$$= \frac{1}{2}\{[H'^{(+)}_i]^* H'^{(+)}_j + [H'^{(-)}_i]^* H'^{(-)}_j\}, \qquad (23a)$$

$$Z^{(1)}_{ij} = \frac{1}{2}\{[K^{(+)}_i]^* K^{(+)}_j - [K^{(-)}_i]^* K^{(-)}_j\}, \qquad (23b)$$

$$Z^{(2)}_{ij} = \frac{1}{2}\{[H^{(+)}_i]^* H^{(+)}_j - [H^{(-)}_i]^* H^{(-)}_j\}, \qquad (23c)$$

$$Z'^{(2)}_{ij} = \frac{1}{2}\{[H'^{(+)}_i]^* H'^{(+)}_j - [H'^{(-)}_i]^* H'^{(-)}_j\}. \qquad (23d)$$

Now, from triangular inequalities applied to the scattering amplitudes $K^{(\pm)}$, $H^{(\pm)}$, f, g, f^{++}, f^{+-} and from relations (3a,b,c,d) (4a,b,c,d) and (21) we obtain:

$$[\mathrm{Im}\, Z^{(0)}_{+-}]^2 = 2[\mathrm{Im}\, Z^{(0)}_{+CE}]^2 = 2[\mathrm{Im}\, Z^{(0)}_{-CE}]^2$$

$$= \frac{4}{9}[\mathrm{Im}\, Z^{(0)}_{13s}]^2 = 4[\mathrm{Im}\, Z^{(0)}_{02t}]^2 = \frac{4}{9}[\mathrm{Im}\, Z^{(0)}_{13u}]^2$$

$$= \frac{1}{16}\left[\sqrt{-\lambda^{(+)}_n} + \epsilon_n \sqrt{-\lambda^{(-)}_n}\right]^2 = \frac{1}{16}\left[\sqrt{-\lambda'^{(+)}_n} + \epsilon'_n \sqrt{-\lambda'^{(-)}_n}\right]^2$$

$$= -H_{+-} - \frac{1}{4}\lambda(\sigma_+, \sigma_-, 2\sigma_{CE}) \qquad (24)$$

for any $n = 1,2,3$;

and
$$[\mathrm{Im}\, Z^{(n)}_{+-}]^2 = 2[\mathrm{Im}\, Z^{(n)}_{+CE}]^2 = 2[\mathrm{Im}\, Z^{(n)}_{-CE}]^2$$

$$= \frac{4}{9}[\mathrm{Im}\, Z^{(n)}_{13s}]^2 = 4[\mathrm{Im}\, Z^{(n)}_{02t}]^2 = \frac{4}{9}[\mathrm{Im}\, Z^{(n)}_{13u}]^2 \qquad (25)$$

$$= \frac{1}{16}\left[\sqrt{-\lambda^{(+)}_n} - \epsilon_n \sqrt{-\lambda^{(-)}_n}\right]^2 = H_{+-} - \frac{1}{4}\lambda(Z^{(n)}_{++}, Z^{(n)}_{--}, 2Z^{(n)}_{CECE})$$

for each $n = 1,2,3$, and similar relations for $\mathrm{Im}\, Z'^{(n)}_{ij}$, where $\lambda^{(\pm)}_n$ and ϵ_n are defined by

$$\lambda^{(\pm)}_n \equiv \lambda[\sigma_+ \pm Z^{(n)}_{++},\ \sigma_- \pm Z^{(n)}_{--},\ 2(\sigma_{CE} \pm Z^{(n)}_{CECE})] \le 0, \qquad (26a)$$

$$\epsilon_n \equiv \mathrm{sign}[(\mathrm{Im}\, Z^{(0)}_{+-})^2 - (\mathrm{Im}\, Z^{(n)}_{+-})^2], \qquad (26b)$$

$\lambda'^{(\pm)}_n$ and ϵ'_n are obtained from (26a) and (26b) by substitution $Z^{(n)}_{ij} \to Z'^{(n)}_{ij}$, λ is defined by (2c).

Also, $Y^{(0)}_{ij}$ satisfy the relations

$$Y^{(0)}_{+-} = \sqrt{2}\, Y^{(0)}_{CE+} = \sqrt{2}\, Y^{(0)}_{CE-} = \frac{2}{3} Y^{(0)}_{31s} = 2 Y^{(0)}_{02t} = \frac{2}{3} Y^{(0)}_{13u} \qquad (27)$$

so that

$$H \equiv H_{+-} = 2H_{+CE} = 2H_{-CE} = \frac{4}{9} H_{13s} = 4H_{02t} = \frac{4}{9} H_{13u} . \quad (28)$$

Finally, from (19) and (20) we obtain

$$(\mathrm{Re}\, Z_{+-}^{(n)})^2 - Z_{++}^{(n)} Z_{--}^{(n)} = 2[(\mathrm{Re}\, Z_{+CE}^{(n)})^2 - Z_{++}^{(n)} Z_{CECE}^{(n)}]$$

$$= 2[(\mathrm{Re}\, Z_{-CE}^{(n)})^2 - Z_{--}^{(n)} Z_{CECE}^{(n)}] = \frac{4}{9}[(\mathrm{Re}\, Z_{13s}^{(n)})^2 - Z_{11s}^{(n)} Z_{33s}^{(n)}]$$

$$= 4[(\mathrm{Re}\, Z_{02t}^{(n)})^2 - Z_{00t}^{(n)} Z_{22t}^{(n)}] = \frac{4}{9}[(\mathrm{Re}\, Z_{13u}^{(n)}) - Z_{11u}^{(n)} Z_{33u}^{(n)}]$$

$$= \frac{1}{4} \lambda(Z_{++}^{(n)}, Z_{--}^{(n)}, 2Z_{CECE}^{(n)}), \quad n = 0,1,2,3, \quad (29)$$

and similar relations for $(\mathrm{Re}\, Z'^{(n)}_{ij})^2 - Z'^{(n)}_{ii} Z'^{(n)}_{jj}$.

Therefore, we have obtained that: $\mathrm{Im}\, Z_{ij}^{(n)}$ (21), $(\mathrm{Re}\, Z_{ij}^{(n)})^2 - Z_{ii}^{(n)} Z_{jj}^{(n)}$ (29), $n = 0,1,2,3$ and $Y_{ij}^{(0)}$ (or H_{ij}) are independent of the charge or isospin indices i,j in any spin reference frame.

Now, the bounds (16) can be written as

$$\max\{L_0, L_{+-}\} \le H \le \min\{U_0, U_{+-}\}, \quad (30)$$

where

$$L_0 \equiv 0; \ \frac{1}{4}\lambda(Z_{++}^{(n)}, Z_{--}^{(n)}, 2Z_{CECE}^{(n)});$$

$$\frac{1}{4}\lambda(Z'^{(n)}_{++}, Z'^{(n)}_{--}, 2Z'^{(n)}_{CECE}); \ldots$$

$$n = 1,2,3. \quad (30a)$$

$$L_{+-} \equiv [\mathrm{Im}\, Z_{+-}^{(n)}]^2 - Z_{++}^{(n)} Z_{--}^{(n)}; \ [\mathrm{Im}\, Z'^{(n)}_{+-}]^2 - Z'^{(n)}_{++} Z'^{(n)}_{--}; \ldots$$

$$n = 1,2,3. \quad (30b)$$

$$U_0 \equiv -\frac{1}{4}\lambda(\sigma_+, \sigma_-, 2\sigma_{CE}). \quad (30c)$$

$$U_{+-} \equiv \sigma_+ \sigma_- - [\mathrm{Im}\, Z_{+-}^{(0)}]^2; \ldots \quad (30d)$$

where $\mathrm{Im}\, Z_{+-}^{(n)}$, $n=0,1,2,3$, are given by (24) and (25). So that, by inequalities (30) we have improved the most stringent isospin bounds on H in pion-nucleon scattering.

We note that

$$\lambda(Z_{++}^{(n)}, Z_{--}^{(n)}, 2Z_{CECE}^{(n)}) < 0 \quad \text{for} \quad Z_{ii}^{(n)} Z_{jj}^{(n)} > 0 \quad (31a)$$

for all indices $i \neq j = +-CE$ and n -fixed;

$$\lambda(Z_{++}^{(n)}, Z_{--}^{(n)}, 2Z_{CECE}^{(n)}) > 0 \ \text{for one of} \ Z_{ii}^{(n)} Z_{jj}^{(n)} < 0 \quad (31b)$$

$i \neq j = +, -, CE$ and n -fixed; and

$$\lambda(Z_{++}^{(n)}, Z_{--}^{(n)}, 2Z_{CECE}^{(n)}) = 0 \quad (31c)$$

when $[\mathrm{Re}\, Z_{ij}^{(n)}]^2 = Z_{ii}^{(n)} Z_{jj}^{(n)}$ for one of the pairs $(ij) = (+-), (+CE), (-CE), (13s), (02t)$ and $(13u)$, n -fixed.

Let us investigate now the conditions for the saturation of the lower and upper bounds (30). From the relations (19a,b,c,d,e,f) we obtain the following constraints:

$$Z_{CECE}^{(n)} = \frac{1}{2}[Z_{++}^{(n)} + Z_{--}^{(n)}], \quad \text{for} \quad \mathrm{Re}\, Z_{+-}^{(n)} = 0; \quad (32a)$$

$$Z^{(n)}_{CECE} = \frac{1}{2}\left[Z^{(n)}_{--} - Z^{(n)}_{++} \right], \quad \text{for} \quad \mathrm{Re}\, Z^{(n)}_{+CE} = 0, \tag{32b}$$

$$Z^{(n)}_{CECE} = \frac{1}{2}\left[Z^{(n)}_{++} - Z^{(n)}_{--} \right], \quad \text{for} \quad \mathrm{Re}\, Z^{(n)}_{-CE} = 0, \tag{32c}$$

$$Z^{(n)}_{CECE} = \frac{1}{6}\left[3Z^{(n)}_{--} + Z^{(n)}_{++} \right], \quad \text{for} \quad \mathrm{Re}\, Z^{(n)}_{13s} = 0, \tag{32d}$$

$$Z^{(n)}_{++} = Z^{(n)}_{--}, \quad \text{for} \quad \mathrm{Re}\, Z^{(n)}_{02t} = 0, \tag{32e}$$

$$Z^{(n)}_{CECE} = \frac{1}{6}\left[3Z^{(n)}_{++} + Z^{(n)}_{--} \right], \quad \text{for} \quad \mathrm{Re}\, Z_{13u} = 0, \tag{32f}$$

and

$$Z^{(n)}_{CECE} = \frac{(\sigma_+ - 2\sigma_- - 4\sigma_{CE})\, Z^{(n)}_{++} + (\sigma_- - 2\sigma_+ - 4\sigma_{CE})\, Z^{(n)}_{--}}{4(\sigma_+ + \sigma_- - \sigma_{CE})} \tag{33a}$$

for $\mathrm{Im}\, Z^{(n)}_{+-} = 0$, $n = 1,2,3$, $\sigma_+ + \sigma_- - \sigma_{CE} \neq 0$, $(\epsilon_n = +1)$. The saturation of the upper bound U_0 when $\mathrm{Im}\, Z^{(0)}_{+-} = 0$ implies also the constraints (33a) (but, then $\epsilon_n = -1$). The lower and upper bounds L_0 and U_0 are degenerated when (33a) and also

$$\lambda^{(+)}_n = \lambda^{(-)}_n = 0 \ \text{or}\ \lambda(Z^{(n)}_{++}, Z^{(n)}_{--}, 2Z^{(n)}_{CECE}) = -\lambda(\sigma_+, \sigma_-, 2\sigma_{CE}) \tag{33b}$$

holds.

All the constraints (32a,b,c,d,e,f) and (33a,b) are also valid when $\mathrm{Re}\, Z'^{(n)}_{ij} = 0$ or $\mathrm{Im}\, Z'^{(n)}_{ij} = 0$ with $Z^{(n)}_{ii} \to Z'^{(n)}_{ii}$, $i = +, -, CE$.

Finally, we give here the isospin bounds on final polarization components in terms of unpolarized differential cross sections alone:

$$\begin{aligned} 0 \le \{ & \sigma_+ \sigma_- \left[(X^{(n)}_+ - X^{(n)}_-)^2 + \eta^{(n)}_{+-} \right]; \\ & 2\sigma_+ \sigma_{CE} \left[(X^{(n)}_+ - X^{(n)}_{CE})^2 + \eta^{(n)}_{+CE} \right]; \\ & 2\sigma_- \sigma_{CE} \left[(X^{(n)}_- - X^{(n)}_{CE})^2 + \eta^{(n)}_{-CE} \right]; \\ & \frac{4}{9} \sigma_{1s} \sigma_{3s} \left[(X^{(n)}_{1s} - X^{(n)}_{3s})^2 + \eta^{(n)}_{13s} \right]; \\ & 4\sigma_{0t} \sigma_{2t} \left[(X^{(n)}_{0t} - X^{(n)}_{2t})^2 + \eta^{(n)}_{02t} \right]; \\ & \frac{4}{9} \sigma_{1u} \sigma_{3u} \left[(X^{(n)}_{1u} - X^{(n)}_{3u})^2 + \eta^{(n)}_{13u} \right] \} \le \\ & \le -\lambda(\sigma_+, \sigma_-, 2\sigma_{CE}), \quad n = 1,2,3, \end{aligned} \tag{34}$$

where

$$\eta^{(n)}_{ij} \equiv 2 - (X^{(n)}_i)^2 - (X^{(n)}_j)^2 - 2\sqrt{[1 - (X^{(n)}_i)^2][1 - (X^{(n)}_j)^2]} \ge 0 \tag{34a}$$

$$X^{(n)}_k \equiv \frac{Z^{(n)}_{ii}}{\sigma_i} \equiv P_k, T_k, S_k \quad \text{or} \quad (A_k, R_k). \tag{34b}$$

We remark that the bounds given by (34), (34a,b) are the best possible ones, since giving only the unpolarized differential cross sections we can obtain restrictive conditions for the final polarization components, in any spin reference frame, at all energies and scattering angles.

2. Conclusions

In this paper we have derived all the bounds on final polarization components imposed by the isospin invariance in pion-nucleon scattering. We show that the isospin invariance implies much more restrictive conditions than those derived by Doncel et al. [6]. So that, in Section 2 we have improved all the bounds (15) and (16) on

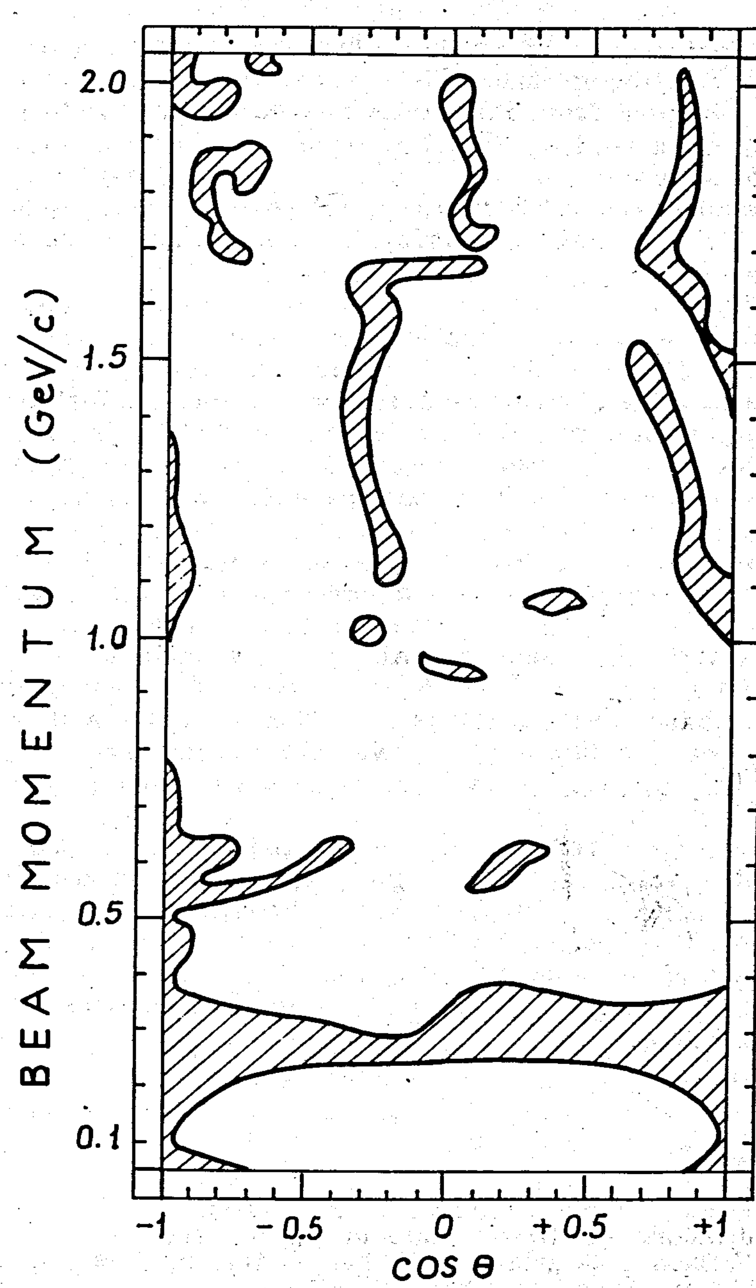
$$\frac{1}{2}[1 - \vec{P}_i\ \vec{P}_j\ ]\sigma_i\ \sigma_j$$

which can be obtained from model-independent considerations. These bounds and their saturations have been derived using a set of bilinear forms (3a,b,c,d), (4a,b,c,d), (7a,b,c,d) which can be constructed with the scattering amplitudes of two reactions. In Section 3 we have obtained all the constraints, imposed on real and imaginary parts of these bilinear forms, from isospin invariance conditions. So that, we have obtained the most stringent isospin bounds (30), (30a,b,c,d) in terms of the differential cross sections and final polarization components.

Finally, we have improved the best isospin bounds (34), (34a,b), on polarization and spin rotation parameters in terms of umpolarized differential cross sections alone, valid at all energies and scattering angles. The bounds (34) are more restrictive than the bound (4) from ref. [3] and equal to them only for $|X_k^{(n)}| \ll 1$, $k = +, -$. So that, our result (34) does not give important corrections at experimental values of $|P_+ - P_-|/2$ (since $\eta_{ij} = 0(X_k^4)$ when $|X_k| \ll 1$) estimated in ref. [3].

Also, we have obtained that the bounds L_0 (30a), U_0 (30c), as well as the bounds (34), are independent of any charge and isospin indices ij in any spin reference frame. These properties are a direct consequence of the fact that $\mathrm{Im}\, Z_{ij}^{(n)}$, $n = 0,1,2,3$ or $\mathrm{Im}\, Z'^{(n)}_{ij}$, obtained from the isospin invariance conditions, are independent

Fig. 1. The regions where the isospin bounds L_0 and U_0 on polarization parameters are simultaneously saturated.



of the charge or isospin indices. In these bounds a specific charge or isospin channel is present only by a constant factor obtained from the Clebsch-Gordan coefficients.

For a fundamental test of isospin invariance, in pion-nucleon scattering, one must know the (s,t) -region where the isospin bounds (30), (30a,b,c,d) are saturated according to the available amplitude analysis. For this it is of great interest to obtain all the zero trajectories of $\mathrm{Im}\, Z_{ij}^{(n)}$, $\mathrm{Re}\, Z_{ij}^{(n)}$, $n = 0,1,2,3$ in (s,t) plane in the low and high energies region. The most stringent constraints will be obtained at the intersection of these zeros-trajectories where two isospin bounds are simultaneously saturated. As an example, we have calculated from the CERN-phase shift analysis /8/ the regions where the isospin bounds L_0 and U_0, on polarization parameters, are degenerated. In these regions, presented in Fig. 1, the polarization parameters satisfy both the relations (33a) and (33b) so that we have $P_+ = a(\sigma) P_-$ and $P_{CE} = b(\sigma) P_-$ where the parameters $a(\sigma)$, $b(\sigma)$ are dependent only on unpolarized differential cross sections. Also we have calculated all the zeros-trajectories of $\mathrm{Re}\, Z_{ij}^{(n)}$ and $\mathrm{Im}\, Z_{ij}^{(n)}$ using the CERN-phase shift solutions /8/. These results will be discussed in a future paper. We note that the zeros-trajectories for $\mathrm{Im}\, Z_{ij}^{(0)}$ have recently been discussed by Tornqvist /7/.

Finally, we remark that all the above isospin bounds are also valid for other spin $(0\frac{1}{2} \to 0'\frac{1'}{2})$ reactions (e.g., $KN \to KN$) which are going through two channels of isospin (U-spin, V-spin) with the corresponding substitution of the constant factors.

We are pleased to acknowledge helpful discussions with Dr. El.Mihul.

References

1. L.Michel. Nuovo Cimento, 22, 203 (1961).
2. R.J.N.Phillips. Nuovo Cimento, 26, 103 (1962).
3. G.V.Dass, J.Froyland, F.Halzen, A.Martin, C.Michael, S.M.Roy. Phys.Lett., 36B, 339 (1971).

4. G.Höhler, F.Kaiser, H.Kohl, W.Schmidt. Lett. al Nuovo Cimento, 5, 413 (1972).
5. N.W.Dean, W.F.Long. Phys.Rev., D7, 3356 (1973).
6. M.G.Doncel, L.Michel, P.Minnaert. Phys.Lett., 38B, 42 (1972).
7. N.A.Tornqvist. Preprint, 26-73, Helsinki, October, 1973.
8. S.Almehed, C.Lovelace. CERN Th. 1403; Nucl.Phys., B40, 157 (1972).

Received by Publishing Department
on February 18, 1974.