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1. Introduction

Recently a number of papers have appeared [1-6] devoted to description of the low-energy pion-pion scattering within quantum field theory with the chiral invariant Lagrangian. In these works the corrections to the so-called "tree"-diagram approximation have been calculated allowing for a contribution from one-loop diagrams to the pion scattering amplitude. Since here the nonrenormalizable field theory is considered indefinite constants arising in calculating the expression corresponding to pion loops cannot be removed by renormalization of the finite number of physical quantities. The use, however, of superpropagator (SP) method [7-8] allows one to fix these constants and obtain unique expression for scattering amplitudes.

The first rather interesting results in this branch have been achieved by Lehmann [1-4]. However, Lehmann used a calculation method for superpropagator noncovariant with respect to the group $SU(2) \times SU(2)$ and his final results depended on a choice of chiral Lagrangian. The covariant calculation method for superpropagator has been proposed in papers by Honerkamp and Ecker [10,11]. It consists in constructing such a perturbation theory which would take account of all possible contractions of diagrams, with an arbitrary number of vertices and internal lines, into diagrams with fixed number of vertices. This contraction is achieved due to transferring two derivatives at vertices into one propagator and reducing them to δ- functions. The calculation methods approaching the covariant perturbation theory [10] in the "tree"-graph approximation have been developed even earlier in works by D.V.Volkov [12 x].

In a paper by one of the authors (V.P.) [14] the chiral-invariant Lagrangian has been reduced to such a form which makes it possible to reproduce the Honerkamp covariant perturbation theory by using the conventional one. It is also shown in this paper [14] that the use of the SP method leads in fact to normal ordering of this form of chiral Lagrangian.

One of the main results of Lehmann works was to elucidate the large effect of pion-nucleon interactions on the behaviour of low-energy pion-scattering phase shifts. Then in our work [15] and in the paper [6] there has been found an essential influence of other terms of baryon octet on the form factor behaviour as well as on that of the low-energy pion-scattering phase shifts. At the same time the interaction with kaons in this energy range (up to 700-800 MeV is not essential (see ref.[6]). In the papers by Lehmann, Ecker and Honerkamp [1-6] pions were considered to be massless particles. As all these papers are studying the low-energy region of pion scattering, then it is rather important to take into account the pion finite mass, in particular in the domain of two-pion production. It seems therefore of interest to us to reproduce all these calculations allowing for the pion mass finiteness, in the same way as made in ref.[15] making use of the calculation procedure for the massive superpropagator given in paper [8]. This will permit us to get expressions for the pion-scattering phases

[x]A covariant perturbation theory, distinct from that of refs.[10-11] was proposed also in paper of Faddeev and Slavnov [13].

with more correct behaviour in the threshold region. Furthermore, our formulas enable the pion scattering lengths to be calculated directly. In the second section an expression for the $\pi\pi$ scattering amplitude is obtained. This expression generalizes the Lehmann formula for the massless pions [1-4] into the case of massive pions. The Lehmann formula contains two indefinite parameters which could be fixed by using the SP method in calculating the pion loops. Our formula includes four indefinite parameters which will also be able to be fixed, if we will employ the SP technique in calculating the pion loops; and when calculating the baryon loops we shall require that in the limit $m=0$ our formulas coincide with those derived in works [1-6].

In the third section a contribution to the $\pi\pi$ scattering amplitude from the pion loops is computed. The use of the SP method is made both in covariant and non-covariant form.

In the fourth section a contribution to the $\pi\pi$ scattering amplitude from the baryon loops is discussed. Here we make use of the results found in refs. [3,6] . Finally, in the fifth section phase-shift analysis is performed for scattering amplitude and the low-energy pion-scattering lengths and phase shifts are obtained. It is shown how an interaction of pions with baryons or with nucleons only influences the behaviour of different scattering phases and how the phase behaviour depends upon the calculation method of the $\pi\pi$ interaction only (in the covariant or non-covariant with respect to $SU(2)\times SU(2)$ way).

2. Pion-scattering amplitude

In the general case a Lagrangian of the chiral type has the form

$$\mathcal{L}(x) = \mathcal{L}_\pi(x) + \mathcal{L}'(x), \tag{2.1}$$

where $\mathcal{L}'$ is the part of the Lagrangian responsible for the interaction of pions with nucleons (or baryons, see sect.4), and

$$\mathcal{L}_\pi(x) = \frac{1}{2} g_{ij}(\pi)\, \partial_\mu \pi^i \partial_\mu \pi^j - \frac{m^2}{2}\varphi(\vec{\pi}^2). \tag{2.2}$$

A concrete form of the functions $g_{ij}(\pi)$ and $\varphi(\vec{\pi}^2)$ is defined by a choice of one or another coordinate system in the pion isotopic space. In particular, for Gursey coordinates we have [16]

$$g_{ij}(\pi) = f(\vec{\pi}^2)\delta_{ij} - [f(\vec{\pi}^2)-1]\frac{\pi_i\pi_j}{\vec{\pi}^2}\ ; \qquad \varphi(\vec{\pi}^2) = \vec{\pi}^2\ ; \tag{2.3}$$

$$f(\vec{\pi}^2) = \left[\frac{\sin z}{z}\right]^2 = \sum_{n=0}^{\infty} C_f(n)\vec{\pi}^{2n}\ ; \quad z = \sqrt{\frac{\vec{\pi}^2}{F_\pi^2}}\ ; \quad F_\pi = 92\ \text{MeV}.$$

For $m \neq 0$ the chiral invariance of the Lagrangian (2.1) is violated.

The $\pi\pi$ scattering amplitude is written in the following form

$$i(2\pi)^4\delta^{(4)}(p_1+p_2-p_3-p_4)\left[A(s,t,u)\delta_{i_1i_2}\delta_{i_3i_4} + A(t,s,u)\delta_{i_1i_3}\delta_{i_2i_4} + A(u,t,s)\delta_{i_1i_4}\delta_{i_3i_2}\right] =$$

$$= \iiiint \prod_{k=1}^{4}\left[dx_k \frac{\delta}{\delta\pi_{i_k}(x_k)}\right] S(\pi)\exp\{i(x_1p_1 + x_2p_2 - x_3p_3 - x_4p_4)\}, \tag{2.4}$$

where

$$s = (p_1+p_2)^2\ ; \quad t = (p_1-p_3)^2\ ; \quad u = (p_1-p_4)^2\ ;$$

$$S(\pi) = S_0(\pi) + S'_\pi(\pi) + S'_B(\pi) + O\left(\frac{1}{F^6}\right). \tag{2.5}$$

$S_0(\pi)$ is the functional S-matrix in Born approximation, $S_\pi(\pi)$ and $S_B(\pi)$ those in the $1/F^4$ approximation of pion and baryon loop diagrams, respectively.

Passing to the dimensionless variables

$$\bar{\xi} = \frac{\xi}{4m^2}\ ;\ (\xi = s,t,u) \quad ; \qquad \alpha_0 = \frac{1}{3}\left[\frac{m}{2\pi F_\pi}\right]^2 = \frac{2}{103} \tag{2.6}$$

and making use of an arbitrary regularization we get the following experession for massive-pion scattering amplitude

$$\frac{A(s,t,u)}{32\pi} = \frac{\pi}{2}\alpha_0(3\bar{s}-1) + i\left(1-\frac{1}{\bar{s}}\right)^{1/2}\left[\frac{\pi}{2}\alpha_0(3\bar{s}-1)\right]^2 + \frac{\pi}{2}\alpha_0^2\,\Pi(\bar{s},\bar{t},\bar{u}). \tag{2.7}$$

Here

$$\Pi(\bar{s},\bar{t},\bar{u}) = -\mathrm{Re}\,\mathcal{J}(\bar{s})(3\bar{s}-1)^2 - \mathcal{J}(\bar{u})[3(\bar{u}-1)(\bar{u}-\bar{t}) + 3\bar{u} - 1] -$$
$$- \mathcal{J}(\bar{t})[3(\bar{t}-1)(\bar{t}-\bar{u}) + 3\bar{t} - 1] + A\bar{s}^2 + B(\bar{t}^2+\bar{u}^2) + C\bar{s} + D \tag{2.8}$$

and

$$\mathcal{J}(x) = \begin{cases} \frac{1}{2}y\left[-i\pi + \ln\frac{1+y}{1-y}\right] & ;\ x > 1 \\ \frac{1}{2}y\ \ln\frac{1+y}{y-1} & ;\ x < 0 \end{cases}\Bigg|_{y=(1-1/x)^{1/2}} \tag{2.9}$$

Our further task is to determine four indefinite parameters A,B,C and D of the expression obtained. We will be able to fix them using, on the one hand, the SP method for calculating the pion loop contribution and requiring, on the other hand, that in the limit $m=0$ our expression be the same as the well known expression found by Lehmann [3]. The latter requirement allows us, in particular, to employ the expressions found in refs.[3,6] for the contribution to the $\pi\pi$ scattering amplitude from the nucleon or all baryon loops in the $1/F^4$ approximation.

3. Pion-loop contribution

In order to employ the SP method for finding the parameters A_π, B_π, C_π and D_π we should consider the whole set of two-vertex diagrams with pion loops where to every vertex two external pion fields are joined (Fig.1). The corresponding matrix element $S_x(\pi)$ in the conventional "naive" perturbation theory is of the form

$$S_x(\pi) = \frac{i^2}{2} \langle 0 | T_r^* \left[: \int d^4x \, \mathcal{L}^{(2)}(\pi/r) : \right]^2 | 0 \rangle , \tag{3.1}$$

where $\mathcal{L}^{(2)}(\pi/r)$ means that the Lagrangian $\mathcal{L}_{(\pi+r)}(x)$ is decomposed into the external asymptotical field $\pi(x)$ and in (3.1) the second order of this expansion is taken ; $r(x)$ is the internal quantum field over which T^*-ordering is made. The final results achieved here will depend on the form of chiral Lagrangian, even in the limit $m=0$. The covariant calculation method of (3.1) proposed by Honerkamp [5,10,11] and extended to the Lagrangian formalism in paper [14] , corresponds to dicriminating all two-vertex diagrams from those with an arbitrary set of vertices (see Introduction) and consists in replacing the Lagrangian $\mathcal{L}^{(2)}(\pi/r)$ in (3.1) by the one $\mathcal{L}^{(2)}_{cov}(\pi/r)$ found by using the procedure of differentiation covariant with respect to the group SU(2) x SU(2) (see Appendix and ref. [14]).

The results obtained by the covariant method are independent of a choice of pion coordinates in the limit $m=0$. The output shape of the Lagrangian $\mathcal{L}^{(2)}_{cov}(\pi/r)$ will be almost the same as that found by using Gursey coordinates:

$$\mathcal{L}^{(2)}_{cov}(\pi/r) = \mathcal{L}_I(\pi/r) + \mathcal{L}_{II}(\pi/r) - \frac{m^2}{2} \varphi^{(2)}(\pi/r) , \tag{3.2}$$

$$\mathcal{L}_I(\pi/r) = \frac{1}{2F_\pi^2} (\pi^a \partial_\mu \pi^b - \pi^b \partial_\mu \pi^a) r^b \partial_\mu r^a f(\vec{r}^{\,2}) , \tag{3.3}$$

$$\mathcal{L}_{II}(\pi/r) = \frac{1}{2F_\pi^2} (\partial_\mu \pi^a \partial_\mu \pi^b)(r^a r^b - \vec{r}^{\,2} \delta_{ab}) f(\vec{r}^{\,2}) , \tag{3.4}$$

$$\varphi^{(2)}(\pi/r) = \frac{1}{F_\pi^2} \left[(\vec{\pi}\vec{r})^2 - \vec{r}^{\,2} \vec{\pi}^{\,2} \right] \sum_{n=0}^{\infty} C_\varphi(n) \vec{r}^{\,2n} . \tag{3.5}$$

Here $f(\vec{r}^{\,2})$ is the same function as in (2.3), the coefficients $C_\varphi(n)$ are dependent on the choice of Lagrangian, ($m \neq 0$). For Gursey coordinates this coefficient, up to the third term of the sum, is roughly equal to [x]

$$C_\varphi(n) = \frac{2^{2n}(2 - n + n^2)}{F_\pi^{2n}(2n+2)!(2n+3)!!} \; ; \quad C_\varphi(0) = \frac{1}{3} . \tag{3.6}$$

All calculations in what follows will be carried out by making use of the covariant expansion of $\mathcal{L}$. Performing T^*-ordering for $S_x'(\pi)$ we get the expression

$$S_x' = S_I' + S_{II}' , \tag{3.7}$$

$$i S_I' = \frac{1}{2} \int d^4x_1 \, d^4x_2 \, (\pi_1^a \partial_\mu \pi_1^b - \pi_1^b \partial_\mu \pi_1^a)(\pi_2^a \partial_\nu \pi_2^b - \pi_2^b \partial_\nu \pi_2^a) \, \delta^I_{\mu\nu}(x_1 - x_2) ; \tag{3.8}$$

$$\begin{aligned} i S_{II}' = \frac{1}{2} \int d^4x_1 \, d^4x_2 \Big\{ & (\partial_\mu \vec{\pi}_1)^2 (\partial_\nu \vec{\pi}_2)^2 \, \delta_1^{II}(x_1 - x_2) + (\partial_\mu \vec{\pi}_1 \, \partial_\nu \vec{\pi}_2)^2 \, \delta_2^{II}(x_1 - x_2) + \\ & + m^4 (\vec{\pi}_1)^2 (\vec{\pi}_2)^2 \, \delta_3^{II}(x_1 - x_2) + m^4 (\vec{\pi}_1 \cdot \vec{\pi}_2)^2 \, \delta_4^{II}(x_1 - x_2) - \\ & - m^2 \left[(\partial_\mu \vec{\pi}_1)^2 (\vec{\pi}_2)^2 + (\partial_\mu \vec{\pi}_2)^2 (\vec{\pi}_1)^2 \right] \delta_5^{II}(x_1 - x_2) - \\ & - m^2 \left[(\partial_\mu \vec{\pi}_1 \vec{\pi}_2)^2 + (\partial_\mu \vec{\pi}_2 \vec{\pi}_1)^2 \right] \delta_6^{II}(x_1 - x_2) \Big\} \end{aligned} \tag{3.9}$$

Here $\pi_i^a = \pi^a(x_i)$

$$\delta^I_{\mu\nu}(x) = -\frac{i}{4F_\pi^4} \sum_{n=0}^{\infty} \left(\partial_\mu \Delta^c(x) \partial_\nu \Delta^c(x) - \Delta^c(x) \partial_\mu \partial_\nu \Delta^c(x) \right) \left[-i \Delta^c(x) \right]^{2n} a^I(n) , \tag{3.10}$$

$$\delta_j^{II} = -\frac{i}{2F_\pi^4} \sum_{n=0}^{\infty} \left(\Delta^c(x) \right)^2 \left[-i \Delta^c(x) \right]^{2n} a_j^{II}(n) \tag{3.11}$$

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[x] How to compute the function $\varphi^{(2)}(\pi/r)$ see Appendix.

and $\Delta^c(x)$ is the propagator of free scalar field. The coefficients $a(n)$ are expressed in terms of $C_f(n)$, C_φ (see eq. (2.3)) in the following way:

$$a^I(n)=R(n)C_f^2(n)\ ;\ a_4^{I}=\frac{6n+5}{5}a^I(n)\ ;\ a_2^{II}=\frac{2n+5}{5}a^I(n)\ ;$$
$$a_3^{II}=\frac{6n+5}{5}R(n)C_\varphi^2(n)\ ;\ a_4^{II}=\frac{2n+5}{5}R(n)C_\varphi^2(n)\ ; \qquad (3.12)$$
$$a_5^{II}=\frac{6n+5}{5}R(n)C_\varphi(n)C_f(n)\ ;\ a_6^{II}=\frac{2n+5}{5}R(n)C_\varphi(n)C_f(n)\ ;$$
$$R(n)=\frac{(2n+3)!(2n+1)!}{3}\ ;\quad C_f(n)=\frac{2^{2n+1}(-1)^n}{F^{2n}(2n+2)!}\ .$$

Further, since we are interested only in the one-loop approximation, $n=0$, then in passing to the momentum space in (3.10), and (3.11) we will employ the massive form of free propagators just for expressions enclosed in parentheses. The remaining propagators may be taken, for the sake of simplicity, in the massless form, as it was made in our papers [8,15]. Then for pion scattering amplitude we get the final expression

$$A^{\pi}(s,t,u)=A^I+A^{II}\ , \qquad (3.13)$$
$$A^I=2[(t-s)\tilde{\delta}^I(\bar{u})+(u-s)\tilde{\delta}^I(\bar{t})]\ , \qquad (3.14)$$
$$A^{II}=(s-2m^2)^2\tilde{\delta}_1^{II}(\bar{s})+\frac{1}{2}(t-2m^2)^2\tilde{\delta}_2^{II}(\bar{t})+\frac{1}{2}(u-2m^2)^2\tilde{\delta}_2^{II}(\bar{u})+$$
$$+4m^4\tilde{\delta}_3^{II}(\bar{s})+2m^4[\tilde{\delta}_4^{II}(\bar{t})+\tilde{\delta}_4^{II}(\bar{u})]+4m^2(s-2m^2)\tilde{\delta}_5^{II}(\bar{s})$$
$$+2m^2[(t-2m^2)\tilde{\delta}_6(\bar{t})+(u-2m^2)\tilde{\delta}_6(\bar{u})]\ , \qquad (3.15)$$

where

$$\tilde{\delta}\left(\frac{q^2}{4m^2}\right)=\int d^4x\, e^{iqx}\delta(x)\ ;\quad \tilde{\delta}^I_{\mu\nu}=g_{\mu\nu}\tilde{\delta}^I \qquad (3.16)$$

and

$$\left.\begin{aligned}\tilde{\delta}^I(x)&=-\frac{m^2a^I(0)}{3(4\pi F_\pi)^2F_\pi^2}\left[x\gamma^I-1+(1-x)J(x)\right],\\ \tilde{\delta}_j^I(x)&=\frac{a_j^{II}(0)}{(4\pi F_\pi)^2F_\pi^2}\left[-J(x)+\gamma_j^{II}\right].\end{aligned}\right\} \qquad (3.17)$$

$$\left.\begin{aligned}\gamma^I&=\ln\frac{4\pi}{m}-\frac{1}{4}\left(\ln a^I(z)\right)'\Big|_{z=0}-c+1\ ,\\ \gamma_j^{II}&=\ln\frac{4\pi}{m}-\frac{1}{4}\left(\ln a_j^{II}(z)\right)'\Big|_{z=0}-c+\frac{1}{2}\ .\end{aligned}\right\} \qquad (3.18)$$

and c is the Euler constant. Hence for the parameters A_π, B_π, C_π and D_π we obtain

$$A_\pi=6\beta_0+\frac{32}{10}\simeq 6{,}63\ ;\ B_\pi=6\beta_0+3\frac{57}{20}\simeq 11{,}8\ ;\ C_\pi=3\ ;\ D_\pi=-\frac{3}{2}\ ; \qquad (3.19)$$
$$\beta_0=\ln\frac{2\pi F_\pi}{m}-\frac{3}{2}c=0{,}451\ .$$

For the noncovariant approach one has

$$A_\pi'=B_\pi'=6\beta_0+14-\frac{7}{20}\simeq 17\ ;\ C_\pi=0\ ;\ D_\pi=-6\ . \qquad (3.20)$$

4. Baryon-loop contribution [x]

First of all we consider the interaction of pions with nucleons only, following Lehmann [3]. Then, to the lowest orders in F_π^{-1} the pion interaction Lagrangian is splitted into three parts:

[x] This section is a brief sketch of the results of works [3,6].

$$\mathcal{L}_1(\pi,N) = \frac{M_N g_A}{F_\pi} :\bar{\psi}_N(x)\gamma_5 \vec{\tau}\vec{\pi}(x)\psi_N(x): \tag{4.1}$$

$$\mathcal{L}_2(\pi,N) = \frac{M_N g_A^2}{2F_\pi^2} :\bar{\psi}_N(x)\psi_N(x)\vec{\pi}^2(x): \tag{4.2}$$

$$\mathcal{L}_3(\pi,N) = \frac{(g_A^2-1)}{4F_\pi^2} :\bar{\psi}_N(x)\gamma_\mu \vec{\tau}(\vec{\pi}(x)\times\partial_\mu\vec{\pi}(x))\psi_N(x) \tag{4.3}$$

Here M_N is the nucleon mass, $g_A = 1{,}25$ is the normalization constant. In the $1/F_\pi^4$ approximation to the $\pi\pi$ scattering amplitude the five one-loop diagrams will contribute (Fig.2) . Two of them, b and d, contain the vertices where the connection with derivative of ($\mathcal{L}_3$) enters. We are interested only in terms with q^4 since the subsequent terms of the expansion in powers of q^2 include the small factor q^2/M_N^2 . In this approximation all diagrams provide the finite contribution to the pion scattering amplitude, except for the diagrams b having two vertices with derivatives. In calculating this diagram an arbitrary parameter appears.

As the vertices related to $\mathcal{L}_3$ contain the small factor (g_A^2-1) Lehmann does not consider the diagrams b and d; and other diagrams give the finite contribution to the pion scattering amplitude:

$$\frac{A^N(s,t,u)}{32\pi} = \alpha_0^2\,\frac{\pi}{2}\,3g_A^4(-\bar{s}^2+\bar{t}^2+\bar{u}^2) \tag{4.4}$$

In order to allow for the contribution to the amplitude from other members of baryon octet it is convenient to write the interaction Lagrangian in the form invariant with respect to SU(3) $\times$ SU(3). Similar calculations have been recently performed by Ecker and Honerkamp [6] . Without discussing in detail

their calculations we only mention that the part of the Lagrangian responsible for interaction with the derivative of $\mathcal{L}_3(\pi,B)$ will not now include the small factor ($g_A^2 - 1$) in front of the whole expression, therefore no reason exists here, generally speaking, to neglect the diagrams b and d.

However, the estimates of the work [6] have shown that if one fixes an arbitrary constant of the expression corresponding to the diagram b, by fitting the correct position of the p-wave ρ resonance, then the contribution to the amplitude from this diagrams appears to be rather small as compared to those from all other diagrams. Therefore in the following calculations we will take into account only the contributions from the diagrams a,c,d and e, where all the coefficients for the q^4 terms are strictly determined. As a result, instead of (4.4) we get the following expression

$$\frac{A^B_{(1)}(s,t,u)}{32\pi} = \frac{2g_A^4 M_N^4}{3\pi^2 F_\pi^4}\Big\{-\bar{s}^2\Big[\alpha^4+\frac{34}{3}\alpha^2(1-\alpha)^2+(1-\alpha)^4\Big] + (\bar{t}^2+\bar{u}^2)\Big[\frac{13}{9}\alpha^4+\frac{26}{3}\alpha^2(1-\alpha)^2+5(1-\alpha)^4\Big]\Big\}, \tag{4.5}$$

$$\frac{A^B_{(2)}(s,t,u)}{32\pi} = \frac{4g_A^2 M_N^4}{3\pi^2 F_\pi^4}(2\bar{s}^2-\bar{t}^2-\bar{u}^2)\Big\{g_A^2\Big[\frac{11}{9}\alpha^4 + 10\alpha^2(1-\alpha)^2+3(1-\alpha)^4\Big]-\Big[\frac{5}{3}\alpha^2+3(1-\alpha)^2\Big]\Big\} \tag{4.6}$$

Here α is the mixing SU(3) parameter , $\alpha = \frac{D}{F+D}$. As in paper [15] we will put it to be equal to about 0.67. $A^B_{(1)}$ is the contribution to the amplitude from the diagrams without derivatives in the vertices a, c, e . $A^B_{(2)}$ is the one from the diagram d where one vertex includes the derivative. Summing up these expressions we obtain

$$\frac{A^{B}(s,t,u)}{32\pi} = \alpha_0^2 \frac{\pi}{2} \left\{ -6\bar s^2 + 8{,}7(\bar t^2 + \bar u^2) \right\} . \tag{4.7}$$

Just this expression will be used in the subsequent works.

5. Scattering phases and lengths

The expressions found in the preceding sections (3.19), (3.20) (4.4) and (4.7) make it possible to fix completely arbitrary parameters A,B,C and D in (3.8) for four different cases. These cases correspond, firstly, to two different methods of calculations of contributions to the pion scattering amplitude from the pion loops, and secondly, to account of an additional contribution from only the nucleon loops or from all baryon loops. To choose the most effective variant we will carry out the phase shift analysis of the amplitude for all the four cases.

When calculating the partial wave amplitudes we shall use the formula

$$J_n(a) = \int_0^a dx\, J(-x)\, x^n = \frac{a^{n+1}}{2(n+1)}\left(\Gamma - \frac{1}{n+1}\right) + \frac{(-1)^n (2n-1)!!}{2^{n+2}(n+1)!}\Big[\frac{L}{2} + \sum_{\ell=1}^{n} (-2a)^{\ell} \frac{(\ell-1)!}{(2\ell-1)!!}\left(\Gamma - \frac{1}{\ell}\right)\Big], \tag{5.1}$$

$$\Gamma = y \ln \frac{y+1}{y-1}, \quad L = \left(\ln \frac{y+1}{y-1}\right)^2; \quad y = \left(1 + \frac{1}{a}\right)^{1/2}; \quad a = \bar s - 1$$

As a result, we get the following expression

$$t_\ell^I(s) = \frac{\pi}{2}\alpha_0 B_\ell^I(\bar s) + i\left(\frac{\pi}{2}\alpha_0 B_\ell^I(\bar s)\right)^2 \left(1 - \frac{1}{\bar s}\right)^{1/2} + \frac{\pi}{2}\alpha_0^2 \Pi_\ell^I(\bar s) . \tag{5.2}$$

Here B_ℓ^I is the Born contribution, Π_ℓ^I is the real part of contribution from the one-loop diagrams:

$$B_0^0 = 2(3\bar s - 1); \quad B_0^2 = -(3\bar s - 1); \quad B_1^1 = (\bar s - 1); \tag{5.3}$$

$$\Pi_0^0 = -\mathrm{Re}\,J(\bar s)\,4(3\bar s - 1)^2 - 12\Big[\frac{5 J_2(a)}{a} + \left(\frac{3}{a} - 1\right) J_1(a) - J_0(a)\Big] + \frac{2}{3} a^2 (4B + A) + \bar s^2 (2B + 3A) + C(2\bar s + 1) + 5D; \tag{5.4}$$

$$\Pi_0^2 = -\mathrm{Re}\,J(\bar s)(3\bar s - 1)^2 - 6\Big[4\frac{J_2(a)}{a} + \frac{a+3}{a} J_1(a) + \frac{a+1}{a} J_2(a)\Big] + \frac{2}{3} a^2 (A + B) + 2B\bar s^2 - aC + 2D; \tag{5.5}$$

$$\Pi_1^1 = -\mathrm{Re}\,J(\bar s)\, a^2 + \frac{2}{a}\Big[6\left(\frac{2J_3(a)}{a} - J_2(a)\right) - 3(a-1)\left(\frac{2J_2(a)}{a} - J_1(a)\right) - (3a+1)\left(\frac{2J_1(a)}{a} - J_0(a)\right)\Big] + \frac{a^2(B-A)}{3} + \frac{aC}{3} . \tag{5.6}$$

The scattering phases will be computed by the well-known formula

$$\left(\mathrm{ctg}\, \delta_\ell^I - i\right)^{-1} = \left(1 - \frac{1}{\bar s}\right)^{1/2} t_\ell^I . \tag{5.7}$$

Expanding in powers of small parameter α_0 we arrive at the expression

$$\mathrm{ctg}\, \delta_\ell^I = \frac{B_\ell^I(\bar s) - \alpha_0 \Pi_\ell^I(\bar s)}{\frac{\pi}{2}\alpha_0 \left(1 - \frac{1}{\bar s}\right)^{1/2} [B_\ell^I(\bar s)]^2} \tag{5.8}$$

For $m=0$ this expression coincides with that found by Lehmann expanding $ctg\,\delta_P^I$ for small q^2 in the effective range (see ref. [2]).

A behaviour of phases δ_0^0, δ_0^2, δ_1^1 is shown in Figs.3 and 4. From these plots it is easy to see that to the experimental values those curves are the most close which are calculated by using the covariant method for the pion loops and by allowing for the baryon contributions.

The scattering lengths obtained by the formulas

$$a_0^0 = t_0^0(\bar{s}=1)\;;\quad a_0^2 = t_0^2(\bar{s}=1)\;;\quad a_1^1 = t_1^1/(\bar{s}-1)\Big|_{\bar{s}=1} \qquad (5.9)$$

in this variant have the following values

$$a_0^0 = 0{,}16\,m_\pi^{-1}\;;\; a_0^2=-0{,}05\,m^{-1}\;;\; a_1^1 = 0{,}033\,m^{-1}\,. \qquad (5.10)$$

At the same time, in the Born approximation these are

$$a_0^0 = 0{,}12\,m^{-1}\;;\; a_0^2 = -0{,}06\,m^{-1}\;;\; a_1^1 = 0{,}03\,m^{-1}\,. \qquad (5.11)$$

Hence it follows that the loop diagrams give the most contribution to a_0^0 .

6. Conclusion

The calculations performed made it possible to obtain the expressions for the massive pion scattering amplitude with the required behaviour in the threshold region. The formulae for amplitudes contain the explicit information on the pion scattering length. At the same time the plots evidence that at energies ~ 800 MeV the pion masses are not essential at all. Note should be made also that the behaviour of δ_0^0 in the near threshold region depends rather much on a choice of the pion coordinates due to the broken chiral symmetry for $m\neq 0$. For instance, in the

Weinberg parametrization (see eq. (A.8)) the slope angle for phase δ_0^0 in the threshold region, and consequently the scattering length, increase by 30% (starting from threshold the corresponding curve for this phase lies just in the middle of curves I with $m=0$ and $m\neq 0$, Fig.3).

We consider Gursey coordinates to be much more preferable as their choice along the geodesics is naturally connected with geometry of the group SU(2)×SU(2).

Analysing various methods of allowing for the loop diagram contribution to the $\pi\pi$ scattering amplitude exhibits the following: Of rather strong influence on the scattering phase behaviour is the choice of calculation technique for the pion loop contribution. The covariant one is not only independent of the choice of the form of chiral Lagrangian in the massless case, but also results in a behaviour of scattering phases the most close to the experimental values. At the same time an account of the baryon, or only nucleon, loops gives almost no difference in the behaviour of phases δ_1^1 and δ_0^2 and is important only for correct description of the phase δ_0^0 where account of the baryon loops improves explicitly the phase behaviour calculated by the covariant method. Thus, we consider the covariant method of calculations of the pion contributions to the $\pi\pi$ scattering amplitude, with allowing for the baryon loops, to be most justified both from the purely theoretical point of view and from that of agreement with experiment. At the same time the kaon loops contribute negligibly in the energy region of interest. These conclusions agree with the results of Ecker and Honerkamp [6] .

APPENDIX

The usual second-order expansion of the Lagrangian (2.2) $\mathcal{L}^{(x)}_{(\pi+r)}$ in powers of asymptotic fields π may be represented in the form (rearranging the derivatives and allowing for the equation $\partial_\mu^2 \pi = -m^2 \pi$):

$$\mathcal{L}^{(2)}_{ex}(\pi|r) = \mathcal{L}^{I}_{ex} + \mathcal{L}^{II}_{ex} + \Delta\mathcal{L} - \frac{m^2}{2}\Phi_{ex}(\pi|r); \qquad (A.1)$$

$$\mathcal{L}^{I}_{ex} = \frac{1}{2F_\pi^2}\left(\pi^a \partial_\mu \pi^b - \pi^b \partial_\mu \pi^a\right)\partial_\mu r^a r^b \sum_{n=0}^{\infty} C_5(n)\frac{4}{2n+4}\vec{r}^{\,2n};$$

$$\mathcal{L}^{II}_{ex} = \frac{1}{2F_\pi^2}\partial_\mu \pi^a \partial_\mu \pi^b \left(r^a r^b - r^2\delta_{ab}\right)\sum_{0}^{\infty} C_5(n)\frac{4(4n+3)}{(2n+4)(2n+3)}\vec{r}^{\,2n};$$

$$\Phi_{ex}(\pi|r) = \frac{1}{F_\pi^2}\left((\vec{r}\vec{\pi})^2 - \vec{r}^{\,2}\vec{\pi}^{\,2}\right)\sum_{n=0}^{\infty} C_5(n)\frac{4}{2n+4}\,\frac{4n+1}{2n+3}\vec{r}^{\,2n}; \qquad (A.2)$$

$$\Delta\mathcal{L} = \sum_{n=1}^{\infty} n\, C_5(n+1)\Big\{\frac{r^{2n}}{2}\left[\pi^2(\partial_\mu r)^2 + 2(\partial_\mu r\pi)^2\right] - \frac{3r^{2(n-1)}}{2}\pi^2(r\partial_\mu r)^2 +$$

$$+ \frac{3+2n}{2}r^{2(n-1)}(r\pi)^2(\partial_\mu r)^2 + 2n\, r^{2(n-1)}(\partial_\mu r\pi)(\pi r)(r\partial_\mu r) - (4n+3)r^{2(n-1)} \times$$

$$\times(\pi r)(r\partial_\mu r)(r\partial_\mu \pi) - \frac{3}{2}r^{2n}m^2(r\pi)^2 - \frac{2n+3}{n+1}r^{2(n+1)}\left[(\partial_\mu \pi)^2 - m^2\pi^2\right].$$

The formulae (3.1) and (A.1) give the superpropagator (i.e. sums of all the two-vertex diagrams) in the conventional perturbation theory neglecting contractions. Note should be made that the main contribution to the scattering amplitude in the one-loop approximation comes from the first three terms of Lagrangian (A.1).

To allow for the contractions the following is necessary[14]:

1. To expand the Lagrangian $\mathcal{L}_{(r+L)}$ by making use of the covariant differentiation with respect to the variable L

2. To change variables with the help of drei-bein fields $e^{ai}_{(x)}$

$$L^i = r^a e^{ai}(x)\ ;\ \left(e^{ai}(x)\, e^a_j(x) = \delta^i_j\ ;\ e^{ai}(x)\, e^{aj}(x) = g_{ij}(x)\right) \qquad (A.3)$$

3. To expand the expression found in this way for the Lagrangian in powers of the asymptotic fields π . The covariant part of Lagrangian (2.2) has been computed in the paper [14] . To recall we illustrate briefly the covariant method by expanding the non-covariant mass additive term to Lagrangian written in the Gursey coordinates.

Let $\varphi(x)$ be an arbitrary function of π then the general expression for expanding in the covariant derivatives is of the form [x]

$$\varphi_{cov}(\pi(+)L) = \varphi(\pi) + \partial_\cdot\varphi L^\cdot + \sum_{n=2}^{\infty}\frac{1}{n!}\varphi^{(n)}_{\underbrace{;\cdot;\cdot;\ldots;\cdot}_{n}}(\pi)\,\overbrace{L^\cdot L^\cdot \ldots L^\cdot}^{n}; \qquad (A.4)$$

$$\varphi^{(2)}_{;\cdot;\cdot} = \partial_\cdot(\partial_\cdot\varphi) - \Gamma^{\ m}_{\cdot\cdot}(\pi)\,\partial_m\varphi(\pi);$$

$$\varphi^{(n)}_{\underbrace{;\cdot;\cdot;\ldots;\cdot}_{n}} = \partial_\cdot\varphi^{(n-1)}_{\underbrace{;\cdot;\cdot;\ldots;\cdot}_{n-1}} - \sum_{i=0}^{n-2}\Gamma^{\ m}_{\cdot\cdot}(\pi)\,\varphi^{(n-1)}_{\underbrace{;\cdot;\cdot;\ldots}_{i};m,\underbrace{\cdot;\ldots;\cdot}_{n-2-i}}; \qquad (A.5)$$

where $\Gamma^{\ n}_{\cdot\cdot}$ are the Christoffel symbols.

[x] We use notation of the paper by D.V.Volkov 12 .
$A.B = \sum_{i=1}^{3} A_i B^i \quad , \quad \partial_\cdot\varphi = \frac{\partial}{\partial x^\cdot}\varphi$

Changing variables of eq. (A.3), we get

$$\varphi_{cov}(\pi(+)L) = \varphi(\pi) + (\partial_{\cdot}\varphi\, e^{\cdot a})\Gamma^{a} + \Phi(\pi|\Gamma) \;,$$

$$\Phi(\pi|\Gamma) = \sum_{n=0}^{\infty} \frac{1}{(n+2)!}\Big[\varphi^{(n+2)}_{(\pi);\underbrace{\cdot;\cdot;\ldots;\cdot}_{n+2}}\; \underbrace{e^{\cdot a}_{(\pi)}\, e^{\cdot a}_{(\pi)} \ldots e^{\cdot a}_{(\pi)}}_{n+2}\Big]\underbrace{\Gamma^{a}\Gamma^{a}\ldots\Gamma^{a}}_{n+2} \;. \quad \text{(A.6)}$$

For the Gursey coordinates we have

$$e^{a\cdot} = \delta_{\cdot a} + P_{\cdot a}\,\pi^{2}\left(\frac{z}{\sin z} - 1\right) \;;\quad z = \sqrt{\frac{\vec{\pi}^{2}}{F_{\pi}^{2}}} \;;\quad P_{\cdot a} = \delta_{\cdot a}\vec{\pi}^{2} - \pi_{\cdot}\pi_{a} \;;$$

$$J_{\cdot\cdot}^{\;m} = 2C_{f}(1)[\delta_{\cdot m}\pi_{\cdot} - \delta_{\cdot\cdot}\pi_{m}] + \frac{1}{\pi^{4}}\Big[2\pi_{\cdot}P_{\cdot m}\Big(z\,\mathrm{ctg}\,z - 1 + \frac{z}{3}\Big) -$$

$$- \pi_{m}P_{\cdot\cdot}\Big(\frac{\sin 2z}{2z} - 1 + \frac{2z}{3}\Big)\Big] \;.$$

To exemplify we calculate the first term of the expansion (A.6):

$$\varphi(\pi) = \pi^{2} \;,$$

$$\partial_{\cdot}\varphi = 2\pi_{\cdot} \;,$$

$$\varphi^{(2)}_{;\cdot;\cdot;} = 2\delta_{\cdot\cdot} - 2J_{\cdot\cdot}^{\;m}\pi_{m} = 2\delta_{\cdot\cdot} + 4C_{f}(1)P_{\cdot\cdot} + O(\pi^{4}) \;,$$

$$[\varphi^{(2)}_{;\cdot;\cdot}e^{a\cdot}e^{a\cdot}]\Gamma^{a}\Gamma^{a} = 2\vec{\Gamma}^{2} + 2C_{f}(1)\big(\vec{\Gamma}^{2}\vec{\pi}^{2} - (\vec{\Gamma}\vec{\pi})^{2}\big) + O(\pi^{4}) \;.$$

Calculating the subsequent expansion terms in an analogous way we arrive at the expression (3.5). We have found the first six expansion terms the even ones being described by the function $C_{\varphi}(n)$ in (3.5)

$$C_{\varphi}(n) \simeq \frac{2^{2n}(2 - n + n^{2})}{(2n+2)!\,(2n+3)!!\,F_{\pi}^{2n}} \;, \qquad \text{(A.7)}$$

The equivalence of conventional perturbation theory allowing for contractions and the covariant method for the massless case has been proved in a paper of Honerkamp et al. [17].

Now, in addition, we will demonstrate how the coefficients $C_{\varphi}(n)$ and $\varphi^{(2)}(\pi/\Gamma)$ change if one uses instead of the Gursey coordinates the Weinberg parametrization

$$\varphi_{W}^{(2)}(\pi/\Gamma) = \Big((\vec{\Gamma}\vec{\pi})^{2} - \frac{\vec{\Gamma}^{2}\vec{\pi}^{2}}{2}\Big)\frac{1}{F_{\pi}^{2}}\sum_{n=0}^{\infty} C_{\varphi}^{W}(n)\,\vec{\Gamma}^{2n} \;,$$

$$C_{\varphi}^{W}(n) \simeq \frac{4 - 7n + 11n^{2}}{4(2n+2)!}\,\frac{1}{F_{\pi}^{2n}} \;. \qquad \text{(A.8)}$$

Note that the use of this parametrization of Lagrangian results in the 30% increase of the scattering length a_{0}^{0} and changes slightly the behaviour of phases δ_{1}^{1} and δ_{0}^{2} .

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Fig.1. Superpropagator diagrams for $\pi\pi$ scattering.

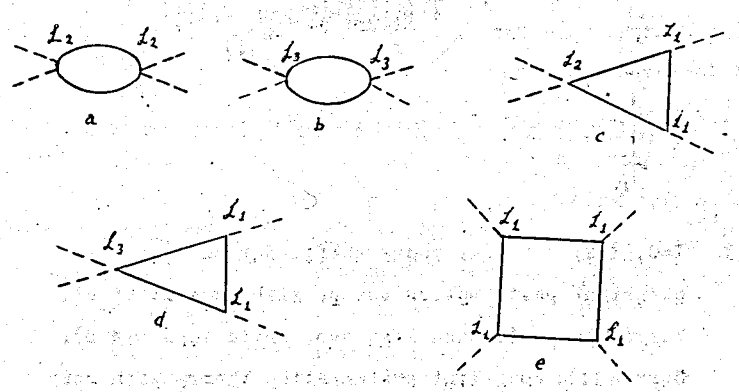

Fig.2. Baryon one-loop diagrams for $\pi\pi$ scattering from Lagrangians (4.1)-(4.3) .

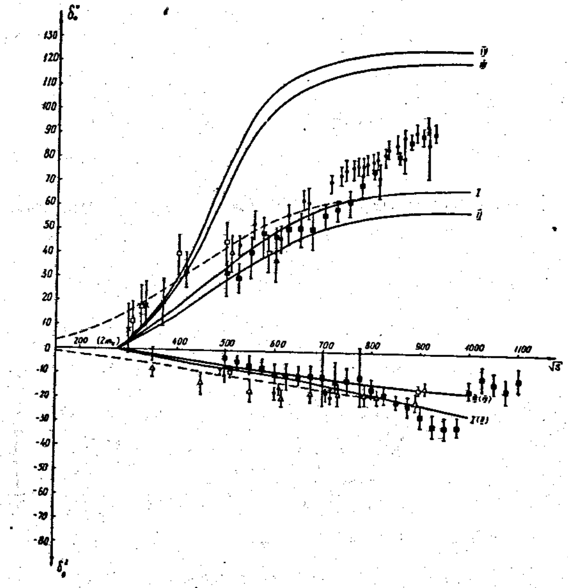

Fig.3. I=0, I=2; S wave phase shift. Curves 1): covariant perturbation theory with account of all baryon loops (dashed line m=o, solid line $m \neq 0$). Curves II): covaraint perturbation theory with only nucleon loops, $m \neq o$. Curves III) and VI): "naive" perturbation theory with account of all baryon loops (IV) and with only nucleon loops (III), Data points from ref. [6] .

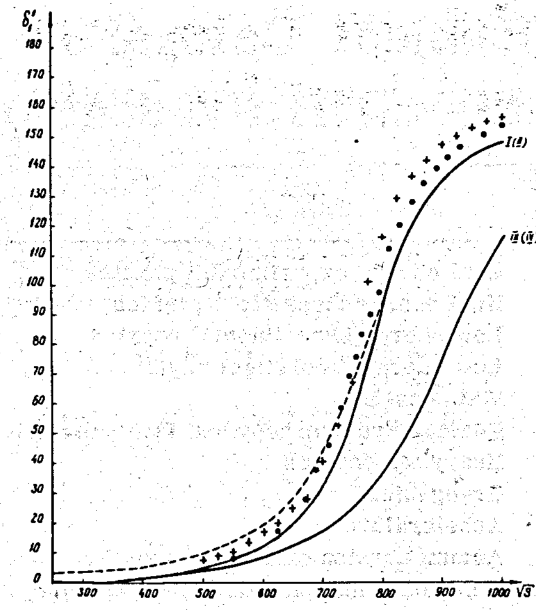

Fig.4. I=1; p wave phase shift. Curves 1(II): covariant theory (dashed line m=0, solid line $m \neq 0$); Curves III(IV): "naive" perturbation theory. Data points are taken from ref. [5] .