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STRONG COUPLING DECOMPOSITION OF THE GROUND STATE FUNCTIONAL IN THE QUANTUM FIELD THEORY MODEL OF NEUTRAL SCALAR FIELD WITH THE SELFACTION $g\phi^4$ IN THE CASE OF TWO-DIMENSIONAL SPACE-TIME

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The equations for determining the coefficient functions of the ground state functional for the aforementioned model of quantum field theory (in the case of non-degenerated vacuum) can be reduced to the form

$$A^2(q,\epsilon) = q^2 + \epsilon^2$$

$$+ 6 \int [D_4(q,-q,s,-s;\epsilon) - D_4(0,0,s,-s;\epsilon)] ds$$

$$D_4(q_1,q_2,q_3,q_4;\epsilon)\ \Sigma_4 = 2 + 15 \int D_6(q_1,\ldots,q_4,s,-s,\epsilon) ds$$

$$D_6 \Sigma_6 + 4[D_4 D_4] = 28 \int D_8(q_1 \ldots q_6, s, -s, \epsilon) ds \qquad (1)$$

. . . .

where

$$\Sigma_n \equiv \sum_{i=1}^{n} A(q_i,\epsilon).$$

The first eq. of (1) shows, that

$$A^2(0,\epsilon) = \epsilon^2, \qquad (2)$$

and correspondingly

$$A(0,\epsilon) = \epsilon,\ \epsilon \geq 0 \qquad (3)$$

or

$$A(0,\epsilon) = -\epsilon \qquad (4)$$

Only the first possibility (3) defines the ground state functional.

Small values of ϵ^2 correspond to the case of strong coupling:

$$\epsilon^2 \to 0, \quad \epsilon^2 > 0 \tag{5}$$

In /1/ it is supposed, that the system (1) for zero value of ϵ has the solution $A(q,0)$, $D_4(q;0)$, $D_6(q;0)$ . . . such that

$$A(0,0)=0 \tag{3a}$$

and all the functions $D_4(q;0)$, $D_6(q;0)$... are finite for all values of their arguments q, including $q=0$, as well.

In the present paper developing this assumption we give the argumentation in favour of that a solution of the system (1) in the case of strong coupling (5) can be represented as a Taylor series in powers of ϵ, this series having non-zero convergence radius.

1. So, we shall search for a solution of the system (1) in the form

$$A(q,\epsilon)=A_0(q)+\epsilon A_1(q)+\epsilon^2 A_2(q)+\dots$$

$$\cdot$$

$$D_4(q,\epsilon)=D_{40}(q)+\epsilon D_{41}(q)+\epsilon^2 D_{42}(q)+\dots \tag{6}$$

. . . . . .

Substituting these expansions into the second and subsequent equations from (1), gives the system of integral equations, through which it is possible to express the functions D_{41}, D_{42} ... D_{61}, D_{62} via the functions A_0, D_{40}, D_{60} ..., A_1, A_2, ... In particular, there will be

$$D_{41}(q_1,q_2,q_3,q_4)=\sum_{i=1}^{4} A_1(q_i)f_i(q_1,q_2,q_3,q_4)+$$

$$+\int F(q_1,q_2,q_3,q_4;s)A_1(s)ds$$

$$D_{42}(q_1,q_2,q_3,q_4)=\sum_{i,j=1}^{4} A_1(q_i)A_1(q_j)f_{ij}(q_1,q_2,q_3,q_4)+$$

$$+\sum_{i}^{4} A_1(q_i)\int F_i(q,s)A_1(s)ds+\int F(q,s,t)A_1(s)A_1(t)ds\,dt+$$

$$+\sum_{1}^{4} A_2(q_i)f_i(q_1,q_2,q_3,q_4)+\int F(q_1,q_2,q_3,q_4;s)A_2(s)ds. \tag{7}$$

Here f_{ij}, f_i, f, F_i, F are some bounded functions, decreasing fast enough for large arguments. Substituting (6) and (7) into the first eq. (1) gives

$$2\epsilon A_0(q)A_1(q)+2\epsilon^2 A_0(q)A_2+\epsilon^2 A_1^2(q)+\ldots$$

$$=\epsilon^2-2\epsilon[A_1(q)\alpha(q)-A_1(0)\alpha(0)+\int\beta(q,s)A_1(s)ds]$$

$$-2\epsilon^2[A_2(q)\alpha(q)+\int\beta(q,s)A_2(s)ds]-\epsilon^2[A_1^2(q)\gamma(q)-A_1^2(0)\gamma(0)$$

$$+A_1(q)\int\delta(q,s)A_1(s)ds-A_1(0)\int\delta(0,s)A_1(s)ds$$

$$+\int\omega(q,s,t)A_1(s)A_1(t)ds\,dt]+\ldots \tag{8}$$

Here $\alpha, \beta; \gamma, \delta, \omega$ are some functions;

$$\beta(0,s)=0$$
$$\omega(0,s,t)=0. \tag{9}$$

It follows from (3) and (6) that

$$A_1(0)=1,\quad A_0(0)=A_2(0)=A_3(0)=\ldots=0. \tag{3b}$$

Equating to zero the coefficient for the first power of ϵ in (8) we obtain

$$[\alpha(q)+A_0(q)]A_1(q)=\alpha(0)-\int\beta(q,s)A_1(s)ds \tag{10}$$

so the function $A_1(q)$ is defined ; it follows from (9) that the solution of (10) (if it exists), satisfies the condition (3b), if

$$\alpha(0)\neq 0. \tag{11}$$

Similarly, the equation for $A_2(q)$ is

$$[\alpha(q)+A_0(q)]A_2(q)+\int\beta(q,s)A_2(s)ds=1-A_1^2(q)+\gamma(0)-$$

$$-\gamma(q)A_1^2(q)+\int\delta(0,s)A_1(s)ds-A_1(q)\int\delta\ (q,s)A_1(s)ds-$$

$$-\int\omega(q,s,t)A_1(s)A_1(t)ds\,dt\,; \qquad (12)$$

it follows from (9) and (11) that if the function $A_2(q)$ satisfies (12), it satisfies also the condition (3b).

1.1. So, apparently no obstacles are to determine the coefficient functions of the expansions (6): the solution of the system (1) in the case of strong coupling (5) can be represented as a Taylor series in powers of ε.

2. To gain any idea about the concrete form of the functions $\alpha(q)$, $\beta(q,s)$... it is useful, e.g., to consider the system

$$A^2(q,\epsilon)=q^2+\epsilon^2+6\int[D_4(q,-q,s,-s;\epsilon)-D_4(0,0,s,-s;\epsilon)]ds$$

$$D_4\Sigma_4=2+15\int D_6\ ds$$

$$D_6(\Sigma_6+6)+4[D_4D_4]=0 \qquad (13)$$

which is obtained by some truncation of (1). Concerning the system (13), one can definitely assert that its solution in the case of strong coupling (5) can be represented by the series (6), and this series has non-zero radius of convergence.

3. It is possible to illustrate the present work by simple example of unharmonic oscillator. This quantum-mechanical system defined by Hamiltonian

$$H=-\frac{1}{2}\frac{\partial^2}{\partial x^2}+\frac{1}{2}x^2+gx^4 \qquad (14)$$

is /2/ somewhat related to the considered model of quantum field theory.

After the substitution

$x=g^{-1/6}y$

Hamiltonian (14) takes the form

$$H=g^{1/3}h \qquad (15)$$

where

$$h = -\frac{1}{2}\frac{\partial^2}{\partial y^2} + y^4 + g^{-2/3} y^2 = h_0 + g^{-2/3} h_1 . \qquad (16)$$

Let us consider h_1 as a perturbation ($g \to \infty$). At large distances the perturbation potential is relatively small (in comparison with y^4), so the Taylor series in powers of z, $z = g^{-2/3}$, representing eigenfunctions and eigenvalues of operator (16), have non-zero radius of convergence whereas similar series in powers of g for the operator (14) have no circle of convergence /3/ : the perturbation potential in (14) is stronger at large distances, than x^2 .

4. For completeness we shall derive (1) /1/ from the basic Hamiltonian of the theory /2/

$$H = \frac{1}{2}\int dk \left[-\frac{\delta^2}{\delta\phi(k)\,\delta\phi(-k)} + (M^2 + k^2)\phi(k)\phi(-k)\right] +$$

$$+ g \int \prod_1^4 (\phi(k_i) dk_i)\, \delta\left(\sum_1^4 k_i\right) \qquad (17)$$

and the Schrödinger equation

$$(H - E_0)\Omega_0 = 0. \qquad (18)$$

We take the ground state functional Ω_0 in the form /2/

$$\Omega_0 = e^{-\kappa}, \qquad (19)$$

then (18) transforms into

$$\frac{1}{2}\int dk \left[-\frac{\delta\kappa}{\delta\phi(k)}\frac{\delta\kappa}{\delta\phi(-k)} + (M^2 + k^2)\phi(k)\phi(-k) + \frac{\delta^2\kappa}{\delta\phi(k)\delta\phi(-k)}\right]$$

$$+ g\int \prod_1^4 (\phi(k_i) dk_i)\delta(\Sigma k_i) = E_0 . \qquad (20)$$

After the substitutions

$$k = \sqrt{g}\, q$$

$$\phi(k) = \frac{1}{\sqrt{g}}\,\psi(q)$$

$$M^2 = m^2 + gL$$

$$\epsilon^2 = m^2/g\,, \tag{21}$$

eq. (20) and (17) get the form

$$\frac{1}{2}\int dq\,[-\frac{\delta\kappa}{\delta\psi(q)}\,\frac{\delta\kappa}{\delta\psi(-q)} + (q^2+\epsilon^2+L)\psi(q)\psi(-q) + \frac{\delta^2\kappa}{\delta\psi(q)\delta\psi(-q)}]$$

$$+\int \prod_1^4 (\psi(q_i)dq_i)\,\delta(\Sigma q_i) = \frac{E_0}{\sqrt{g}} \tag{22}$$

$$H = \sqrt{g}\,\{\int dq\,[-\frac{1}{2}\,\frac{\delta^2}{\delta\psi(q)\delta\psi(-q)} + \frac{1}{2}(q^2+\epsilon^2+L)\psi(q)\psi(-q)]$$

$$+\int \prod_1^4 (\psi(q_i)dq_i)\,\delta(\Sigma q_i). \tag{23}$$

Eq. (22) with the representation:

$$2\kappa = \int A(q,\epsilon)\,\psi(q)\psi(-q)dq$$

$$+\int D_4(q_1,q_2,q_3,q_4;\epsilon)\prod_1^4(\psi(q_i)dq_i)\,\delta(\sum_1^4 q_i)$$

$$+\ldots \tag{24}$$

gives eq. (1), if we take

$$L = -6\int D_4(0,0,s,-s\,;\epsilon)\,ds\,. \tag{25}$$

5. ϵ dependent term in (17), (23)

$$\frac{1}{2} M^2 \int \phi(k)\phi(-k)\,dk = \frac{1}{2} M^2 \int \tilde{\phi}(x)^2 dx$$

is small at large values of ϕ as compared with the last quadric term in (17), (23):

$$g \int \tilde{\phi}(x)^4 dx .$$

This is just the reason to expect that the strong coupling series (6) have non-zero radius of convergence.

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