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ULTRAVIOLET ASYMPTOTICS IN THE PRESENCE OF NONABELIAN GAUGE FIELDS

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I. Introduction

The regular method of investigating ultraviolet asymptotics in quantum field theory is based on the use of the renormalization group[1,2] (RG). The calculations performed by this method utilize as a starting point perturbation expansion and allow to effectively improve its approximation properties. However it is well known that such calculations lead, as a rule, to the difficulty of principle connected with the departure from the framework of weak coupling. The lowest approximation of the RG equation which is equivalent to the summation of the main logarithmic terms leads in quantum electrodynamics and two-charge meson-nucleon theory to the ghost-pole trouble(the so-called "zero-charge" problem[3]). The analysis of the next approximations[4] (see also 43.2 in[5]) shows that in the vicinity of the ghost pole the lower logarithmic terms become important. This does not permit making any solid statement about the existence of the ghost pole on the basis of the first approximation. In quantum electrodynamics the lower logarithmic terms are calculated up to the sixth order[6]. All these terms appear to be positive. Therefore they merely shift the position of the ghost. As a result, it became a general belief that the account of the lower logarithmic terms does not change the situation qualitatively.

Recently it was discovered[7,8] that massless Yang-Mills theory possesses a remarkable ultraviolet behaviour, quite different from the one mentioned above. In this theory the invariant coupling constant (ICC) tends asymptotically to zero (see below eq.(9)). Such a theory, considered as a model for strong interactions, leads to the scale invariant behaviour without anomalous dimensions. However massless Yang-Mills theory cannot evidently pretend to describing strongly interacting vector mesons. Moreover, it possesses serious troubles due to infrared divergences.

At present the only known method of ascribing mass to vector fields in a gauge invariant way is based on the Higgs mechanism of spontaneous symmetry breaking[9]. All such models include inevitably the interaction of massive vector fields with scalar fields and also quartic self-interaction of scalar fields. Thus, the second coupling constant arises in the theory. The solution of the renormalization group equations for this ICC corresponds in the main logaritamic approximation to going outside the framework of weak coupling (see below eq. (IO)). In paper[7] the wide class of gauge theories including scalar fields was considered. Unfortunately the authors have not succeeded in finding any physically acceptable model which is asymptotically free with respect to both coupling constants. Therefore they were forced to hope on some hypothetical "dynamical" mechanism of symmetry breaking.

In the present paper we investigate the role of lower logarithmic terms in the theory of Yang-Mills field interacting with scalar and spinor particles, including spontaneously broken models. We also consider the effect of additional Yukawa-type coupling which arises in some models of strong and weak interactions.[IO]

2. The Role of Scalar Particles

Consider the simplest nonabelian spontaneously broken gauge model[II]. The original Lagrangian looks as follows

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} + |\partial_\mu \chi - ig\frac{\vec{\tau}}{2}\vec{B}_\mu \chi|^2 + \frac{m^2}{2}|\chi|^2 - \lambda(\bar{\chi}\chi)^2, \tag{I}$$

where $F^a_{\mu\nu} = \partial_\mu B^a_\nu - \partial_\nu B^a_\mu + g\epsilon^{abc} B^b_\mu B^c_\nu$

and χ is a nonhermitian isospinor, $\chi = \begin{pmatrix} \chi^0 \\ \chi^- \end{pmatrix}$.

Due to spontaneous symmetry breaking the vacuum expectation of the field χ is different from zero. This provides vector mesons with nonzero masses. With the help of the canonical transformation one can go over from the fields χ to the fields φ^a, σ having zero vacuum expectation. In terms of these fields the Lagrangian (I) has the form

$$\begin{aligned} \mathcal{L} = & -\frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} + \frac{M^2}{2}\vec{B}_\mu^2 + \frac{1}{2}\partial_\mu \vec{\varphi}\, \partial_\mu \vec{\varphi} + \\ & + \frac{1}{2}\partial_\mu \sigma \partial_\mu \sigma - \frac{m^2}{2}\sigma^2 + M \vec{B}_\mu \partial_\mu \vec{\varphi} - \\ & - \frac{g}{2} B^a_\mu \left(\sigma \partial_\mu \varphi^a - \varphi^a \partial_\mu \sigma - \epsilon^{abc} \varphi^b \partial_\mu \varphi^c \right) - \\ & - \frac{Mg}{2}\sigma \vec{B}_\mu^2 + \frac{g^2}{8}(\sigma^2 + \vec{\varphi}^2)\vec{B}_\mu^2 + \\ & + \frac{gm^2}{4M}\sigma(\sigma^2 + \vec{\varphi}^2) - \frac{g^2 m^2}{32M^2}(\sigma^2 + \vec{\varphi}^2)^2. \end{aligned} \tag{2}$$

Gauge invariance of the Lagrangian (2) makes it possible to develop a renormalizable perturbation theory. The transversal gauge $\partial_\mu \vec{B}_\mu = 0$ is particularly convenient. In this gauge the free vector meson propagator looks as follows.

$$D^{(0)}_{\mu\nu}(k) = \frac{1}{M^2 - k^2}\left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} \right) . \qquad (3)$$

The interaction Lagrangian includes, besides the vertices written explicitly in eq.(2), the additional term describing the interaction with the so-called Faddeev-Popov ghosts[12]

$$i\int \tilde{\mathcal{L}}(x)\,dx = -\sum_{n=2}^{\infty} \frac{g^n}{n} \int \epsilon^{abc} \ldots \epsilon^{\ldots f a} \times$$

$$\times B^b_{\mu_1}(x_1) \ldots B^f_{\mu_n}(x_n)\, \partial_{\mu_1} D_0(x_1 - x_2) \ldots \partial_{\mu_n} D_0(x_n - x_1)\, dx_1 \ldots dx_n , \qquad (4)$$

where

$$D_0(x) = \frac{1}{(2\pi)^4} \int \frac{e^{ikx}\,dk}{k^2 + i\epsilon} . \qquad (5)$$

Asymptotics of all Green functions are known to be expressed in terms of asymptotical ICC's which we choose in the form

$$\bar{g}^2(L) = g^2 \Gamma^{-2}_{3B} D^3_B \;\; ; \quad \bar{g}^2(0) = g^2$$

$$\bar{h}(L) = h \Gamma_{4\sigma} D^2_\sigma \;\; ; \qquad (6)$$

$$\bar{h}(0) = h \equiv \left(\frac{3}{4\pi^2}\right) \frac{g^2 m^2}{32 M^2} .$$

Here Γ_{3B} and $\Gamma_{4\sigma}$ are normalized symmetric vertex fu[illegible], D_σ and D_s are dimensionless Green functions, e.g.,

$$D_{\mu\nu}(k) = D^{(c)}_{\mu\nu}(k) D_B(k)$$

and $L = \ln(-k^2/\lambda^2)$, where λ^2 is the point of subtraction. The factor $3/4\pi^2$ is introduced to simplify ensuring expressions. The ICC's $\bar{h}$ and $\bar{g}^2$ satisfy the system of differential Lie equations [2,13]

$$\frac{d\bar{h}(L)}{dL} = \varphi(\bar{g}^2, \bar{h}), \tag{7a}$$

$$\frac{d\bar{g}^2(L)}{dL} = \psi(\bar{g}^2, \bar{h}), \tag{7b}$$

The functions φ and ψ are defined by the conditions

$$\varphi(g^2, h) = \left.\frac{d\bar{h}}{dL}\right|_{L=0} \; ; \; \psi(g^2, h) = \left.\frac{d\bar{g}^2}{dL}\right|_{L=0}$$

and can be obtained from the perturbation calculations as the series in g^2 and h ;

$$\varphi(g^2, h) = \alpha h^2 - \beta h^3 - \gamma g^2 h + \delta (g^2)^2 + \dots$$

$$\psi(g^2, h) = -c(g^2)^2 + d(g^2)^2 h^2 + e(g^2)^3 + \dots$$

In the lowest order with respect to $h \sim \bar{g}^2 \ll 1$ equations (7) become independent

$$\frac{d\bar{h}}{dL} = a\,\bar{h}^2 , \qquad (8a)$$

$$\frac{d\bar{g}}{dL} = -c\,\bar{g}^3 \qquad (8b)$$

and may be solved easily

$$\bar{g}^2(L) = \frac{g^2}{1 + c\,g^2 L} , \qquad (9)$$

$$\bar{h}(L) = \frac{h}{1 - a\,h\,L} . \qquad (10)$$

To determine the coefficients a and c it is sufficient to obtain the lowest order corrections to the Green and vertex functions and extract from them the logarithmic asymptotics. The corresponding diagrams are shown on Fig.1. In the model under consideration

$$\begin{aligned} &\Gamma_{B}\;[\text{illegible}] = [\text{illegible}]\ \frac{[\text{illegible}]}{16\pi^2} \\ &[\text{illegible}]\;(L) = 1 + O(h^2) \\ &[\text{illegible}] = 1 + \frac{\lambda}{2}\,\frac{[\text{illegible}]}{16\pi^2} \\ &\Gamma_{4\varphi}\;(L) = 1 + 4hL . \end{aligned} \qquad (11)$$

Hence both coefficients $c = \frac{[\text{illegible}]}{16\pi^2}$, $a = 4$ are positive. Therefore ICC $\bar{g}^2(L)$ goes to zero as $L \to \infty$ and we get the asymptotical freedom with respect to g^2 (this fact was first established in papers [7,8]).

At the same time due to positiveness of h the ICC $\bar{h}(L)$ grows as $L \to \infty$ and violates the condition $\bar{h} \ll 1$. We go outside the domain of weak coupling.

Note that if we do not limit ourselves by Kibble model and consider the case $h<0$ (see below Sect. III), we do not obtain the asymptotical freedom with respect to $\bar{h}$. Indeed, now $\bar{h}(L)$ decreases in absolute value as $L\to\infty$ and enters the region $-\bar{h}\sim\bar{g}^2$, where the terms $\sim\gamma$ and δ in the r.h.s. of (7a) become essential. We get instead of Eq. (8a):

$$\frac{d\bar{h}}{dL} = \alpha\bar{h}^2 - \gamma\bar{g}^2\bar{h} + \delta(\bar{g}^2)^2 \; . \tag{8c}$$

The system (8b),(8c) was studied in [7] . For all cases of physical interest the r.h.s. of Eq.(8c) is positive. Therefore as L grows the ICC $\bar{h}$ increases, pass through zero, becomes positive, enters the region $\bar{h}\sim\bar{g}^2$, then the region $\bar{h}\sim\bar{g}$ and goes out the domain of weak coupling.

Hence the next terms in the perturbation expansion of φ in h should be considered. Eq. (10) corresponds to the summation of the main logarithmic terms $(hL)^n$. Taking into account of the next correction to the r.h.s. of (7a) is equivalent to summing of lower logarithms $h(hL)^n$.

To determine this correction it is necessary to calculate lower logarithmic asymptotics of the diagrams shown on Fig.2. Diagram 2(c) is the product of two lower order diagram, and therefore does not give terms h^2L we are interested in. Indeed, after subtraction asymptotic of the simple closed loop (Fig.1 (f)) is proportional to hL . Therefore the diagram 2(c) gives a contribution $\sim h^2L^2$. So, it is sufficient to consider only diagrams 2(a) and 2(b). The calculation method is described in the Appendix.

The diagrams 2(a) and 2(b) give contributions $\frac{2}{3}h^2L$ and $-\frac{56}{3}h^2L$ correspondingly. Consequently, the coefficient β in the function $\varphi(\bar{g}^2,\bar{h})$ is equal to $-2\cdot\frac{2}{3}+\frac{56}{3}=\frac{52}{3}$

and the Lie equation for $\bar h(L)$ in this approximation looks as follows

$$\frac{d\bar h}{dL} = 4\bar h^2 - \frac{52}{3}\bar h^3 = \varphi_2(\bar h)\,. \tag{I2}$$

One can see that account of the next approximation changes the solution drastically. Previously the function $\bar h(L)$ was singular at some finite value of L. Contrary to that, the solution of eq. (I2) has no singularities in the asymptotical region and tends to the constant $H = \frac{3}{13}$ as $L \to \infty$ [14].

The simplest way to see that is to rewrite the equation

$$\frac{d\bar h}{dL} = \varphi(\bar h(L))$$

in the Gell-Mann-Low form

$$L = \int_h^{\bar h(L)} \frac{dh'}{\varphi(h')}\,. \tag{I3}$$

The solution (I0) corresponds to the approximation $\varphi_1(h) = 4h^2$ for which the r.h.s. of eq. (I3) is finite for an arbitrary upper limit $\bar h(L)$. Hence eq.(I3) cannot be valid for sufficiently large L. In other words, the solution (I0) is contradictory. In the case (I2) the function $\varphi_2(h)$ has a zero at some finite value of h and the integral (I3) diverges as $\bar h \to 3/13$. It follows that when $L \to \infty$ $\bar h(L)$ tends to the constant value $H = \frac{3}{13}$.
The solution of eq.(I2) can be represented as follows

$$\bar h(L) = \frac{h}{1 - 4hL + \frac{h}{H}\ln\left\{\frac{H-h}{H-\bar h(L)}\cdot\frac{\bar h(L)}{h}\right\}}\,. \tag{I4}$$

This result means that charge renormalization in the theory of self-interacting scalar field described by the Lagrangian $h(\varphi_i\varphi_i)^2$ is finite in the second logarithmic approximation. If this fact remained valid after taking into account the next terms in the expansion of $\varphi(\bar{h})$, $\sim \bar{h}^n$ and we would have a gauge invariant renormalizable theory of massive vector fields in which both ICC exhibit physically acceptable behaviour: $\bar{g}^2 \to 0$, $\bar{h} \to H < \infty$. Indeed, in that case all terms in eqs.(7 a) and (7 b) which are of higher degree in $\bar{g}$ may be omitted, and the equations are actually independent (at least for sufficiently small H).

Such a theory might be considered as a realistic model of strong interactions for the finiteness of asymptotical ICC provides a scale invariant behaviour of the Green functions (with anomalous dimensions different from zero, generally speaking).

So, the question of relative contribution of the next terms in the function $\varphi(h)$ is very crucial. Below we investigate it for the model of self-interacting scalar field

$$\mathcal{L}_{int} = -\frac{4\pi^2}{3} h(\varphi_i\varphi_i)^2, \quad i = 1,2,\ldots n \tag{15}$$

All conclusions may be extended in a clear way to the case when the scalar field interacts also with a gauge vector field. It appears that the function $\varphi(h)$ calculated up to the h^4 looks as follows *)

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* An analogous result was obtained by Belavin A.A. and Avdeeva G.M.

$$\varphi_3(h) = \frac{N+8}{3}h^2 - \frac{6N+28}{3}h^3 + [1.05N^2 + 33.4N + 119.2]h^4 . \qquad (I6)$$

Contributions of separate diagrams to eq.(I6) are written in the table.

The equation $\varphi_3(h) = 0$ has no real roots for any natural N. This means that we have come back to the situation qualitatively equivalent to the main logarithmic approximation $\varphi_1(h)$. By inspection of formula (I6) one can see that the order of magnitude of the first, second and third coefficients in the series

$$\varphi(h) = \sum_{n \geq 2} \varphi_n h^n \qquad (I7)$$

is determined by purely combinatorial factors (the second column of the table). Once this property is supposed to remain valid in higher orders as well one can show that $\varphi_n \sim (-1)^n \cdot n!$ This means that the series (I7) is asymptotical and it is impossible to draw any reliable conclusion concerning the existence of zeroes of the function $\varphi(h)$ by examining any finite number of its terms. In this disappointing situation one fact may be considered as a source of some expectations. Contrary to the electrodynamics, the signs of the terms in the perturbation series for the Gell-Mann-Low function alternate. So, one may suppose the existence of some "compensation mechanism" leading to zero

$$\varphi(H) = 0 .$$

Accepting such a hypothesis one can analyse the theoretical and experimental consequences. It follows from eqs.(2) describing spontaneously broken gauge theory that the point
will be a stable knot for sufficiently small h .

3. The Influence of Yukawa Interaction

The models of weak and electromagnetic interactions[10] include in addition to the minimal gauge-invariant interaction the Yukawa type interaction $\bar{\psi}\Gamma\psi\varphi$ which is necessary for the description of lepton mass spectrum. It is important to know the effect of this interaction on the results of the preceding section. It can also be useful for possible application of nonabelian gauge fields to the theory of strong interactions because such theories contain usually a direct meson-nucleon interaction.

As can be seen from section 2, the "additional" vertices with dimensional coupling constants (from the last line of eq.(2)) as well as the mass terms originated from the canonical transformation are nonessential for an asymptotic analysis. Thus, in analysing simultaneously the Yang- Mills and Yukawa interactions we use for brevity the Lagrangian (I)*

$$\mathcal{L} = \mathcal{L}_{YM}(B) + \bar{\psi}\left(i\hat{\partial} + gB^a_\mu T^a_+ \gamma^\mu\right)\psi + \\ + \frac{1}{2}\left(\partial_\mu\varphi^a - igB^b_\mu (T^b_\varphi)^{ac}\varphi^c\right)^2 + 2\bar{\psi}\gamma^5 T^a_+ \varphi^a \psi - h(\varphi^a\varphi^a)^2 . \qquad (18)$$

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*) Here ユ is the Japanese character from the katakana alphabet pronounced as "ju".

Here B is the gauge field connected with the group SU(n), the fields ψ and φ realize some representations of this group. For simplicity we consider only one type of the Yukawa interaction.

In this case besides $\bar g^2$ and $\bar h$ we have the third ICC $\bar f^2$

$$\bar f^2 = f^2 D_\psi^2 \Gamma_{\psi\psi\varphi}^2 D_\varphi \tag{19}$$

and the system of the Lee equations is of the form

$$\frac{d\bar g^2}{dL} = -c(\bar g^2)^2, \tag{20a}$$

$$\frac{d\bar f^2}{dL} = a(\bar f^2)^2 - b\bar f^2 \bar g^2, \tag{20b}$$

$$\frac{d\bar h}{dL} = \alpha \bar h^2 - \gamma \bar h \bar g^2 + \varkappa \bar h \bar f^2 + \delta(\bar g^2)^2 - \rho(\bar f^2)^2. \tag{20c}$$

The r.h.s. of these equations are written in the lowest approximations under the assumption

$$g^2 \sim f^2 \sim h. \tag{21}$$

The numerical parameters a, b, and c are

$$\begin{aligned} 16\pi^2 c &= \frac{11}{3} n - \frac{4}{3} t_\psi - \frac{1}{6} t_\varphi \\ 16\pi^2 a &= 3T_\psi^2 + 2t_\psi - n \\ 16\pi^2 b &= 6\left(T_\psi^2 - \frac{n}{2}\right) + 3T_\varphi^2 \end{aligned} \tag{22}$$

with $[T^a, T^b] = i f^{abc} T^c$;

$$T^2 \equiv T^a T^a \; ; \quad Sp(T^a T^b) = t\, \delta^{ab} .$$

We do not write down explicitly the coefficients of the last equation. Note only that all of them are positive. Once the general solution of the first equation (20a) is known, we consider the second one (20b). Its general solution can be expressed in terms of $\bar{g}^2(L)$ (see eq.(9)) as follows

$$\bar{\mathcal{I}}^2(L) = \bar{g}^2(L) \frac{\sigma_c}{1 + k\left(\frac{g^2}{\bar{g}^2(L)}\right)^{\mu}} , \tag{23}$$

where

$$\sigma_c = \frac{b-c}{a} \; ; \; k = \frac{g^2}{\mathcal{I}^2}\sigma_c - 1 \; ; \; \mu = \frac{b-c}{c} . \tag{24}$$

If

$$\sigma_c > 0 \tag{25}$$

and

$$k > 0 \qquad \text{i.e.} \quad \frac{\mathcal{I}^2}{g^2} < \sigma_c \tag{26}$$

we have

$$\bar{\mathcal{I}}^2(L) \to 0 \qquad \text{as} \qquad L \to \infty \; .. \tag{27}$$

In the opposite case $\sigma_c < \frac{\mathcal{I}^2}{g^2}$ the ICC $\bar{\mathcal{I}}^2(L)$ tends to infinity in a "ghost-type" manner.

The whole picture looks very transparent on the phase plane (see Fig.3). The condition (25) means that the singular solution

$$\bar{\lambda}^2 = \sigma_c \bar{g}^2 \tag{28}$$

lies in the first quadrant, in other words, eq.(25) is the condition of the existence of the region

$$0 < \frac{\lambda^2}{g^2} < \sigma_c$$

in which the phase curves tend to the origin. Eq.(26) imposes a restriction on the low-energy values

$$\lambda^2 = \bar{\lambda}^2(0) \quad ; \quad g^2 = \bar{g}^2(0) \, .$$

The fulfilment of eq.(25) would mean that in the presence of the Yang-Mills field the Yukawa interaction can asymptotically be free. For the simplest possibility when the group is SU(2) and ψ is an isodublet and φ a triplet

$$16\pi^2 a = \frac{5}{4} \ , \ 16\pi^2 b = \frac{9}{2} \ , \ 16\pi^2 c = \frac{19}{3}$$

and eq.(25) is not satisfied. However for the octet representation of SU(3) group

$$16\pi^2 a = 12 \ , \ 16\pi^2 b = 18 \ , \ 16\pi^2 c = \frac{13}{2}$$

Consider now the last equation (20c). As follows from eq.(28), in the limit $L \to \infty$ the ICC $\bar{\lambda}^2$ tends to zero more quickly than $\bar{g}^2$. (The singular solution $\bar{\lambda}^2 \sim \bar{g}^2$ is unstable). This means that in the asymptotic analysis of eq. (20c) one can neglect $\bar{\lambda}^2$

with respect to $\bar{g}^2$. So, we get an equation without the Yukawa interaction considered in ref.[7].

As this equation has no asymptotically free solutions for physically interesting cases the ICC $\bar{h}$ becomes $\gg \bar{g}^2 \cdot \bar{\imath}^2$ when $L \to \infty$, and the initial assumption, eq.(21), is violated. Hence the terms depending on $\bar{h}$ become essential in the r.h.s. of eqs. (20a) and (20b). The account of such terms can change the behaviour of $\bar{g}^2$ and $\bar{\imath}^2$ completely. In the r.h.s. of eq.(20b) for any finite H the term $\bar{\imath}^2 \bar{h}^2$ becomes asymptotically dominant. This term, being positive, destabilize the origin. In other words, the effective Yukawa coupling goes outside the weak coupling domain. In eq.(20a) the corresponding term is proportional to $\bar{g}^4 \bar{h}^2$. For H large enough it can compete with $-c\bar{g}^4$.

Note that contrary to the "pure" scalar field theory (15) the absence of asymptotical freedom does not depend on the sign of the low-energy value $h = \bar{h}(0)$.

4. Discussion

Thus, a reliable quantitative ultraviolet asymptotics can be obtained for asymptotically free theories only. These models can describe only massless Yang-Mills field interacting solely with fermions. Any interactions with scalar fields (besides some high unitary groups discovered in ref.[7]) inevitably destroy asymptotical freedom. This excludes practically all the models of weak and strong interactions except for gauge invariant interaction of massless vector fields with quarks. Even if we take into account that the mass of Yang-Mills field can appear as a result of spontaneous symmetry breaking mechanism different from the Higgs one [15], we still face the problem of transition from the quark model to the description of real hadrons.

Our analysis of lower logarithmic terms (Section 2) shows that in principle there exists another possibility which is quite acceptable from the physical point of view: the Yang-Mills ICC $\bar{g}^2$ tends to zero and the quartic ICC $\bar{h}$ tends to some finite value H. Although the perturbative approach cannot give any reliable estimate of this value, we can check the experimental consequences of such hypothesis. The most important of them is the existence of non-zero anomalous dimensions.

The popular belief is that the data on deep inelastic scattering prove the absence of anomalous dimensions. However, the careful analysis performed in ref.[16] shows that the existing world data do not exclude small anomalous dimensions. Note here that in the model (15) the anomalous dimension of the scalar field in the second logarithmic approximation (12) turns out to be very small ($6/169 \simeq 0.04$). This means that even for rather large asymptotical H values the anomalous dimensions can be small.

Due to these facts the models of such a type should not be rejected ad hoc. Unlike the existing asymptotically free theories such models provide the possibility of making computations in the high energy region as well as near the mass shell and can be considered as a basis for realistic description of strong and weak interactions.

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Appendix

I. Consider, for example, the calculation of the integral corresponding to the diagram of Fig.4

$$R_\lambda \int \frac{dq\, dl}{q^2 (q-k)^2 l^2 (l-(q-p))^2}\,, \tag{A.I}$$

where the symbol R_λ means R-operation with subtraction at λ^2. This diagram contains one divergent subgraph, Fig.I(f). The divergent integral corresponding to this subgraph is

$$\int \frac{dl}{l^2 (l-(q-p))^2} = i\pi^2 \left[\ln \frac{\Lambda^2}{(q-p)^2} + 1 \right], \tag{A.2}$$

where Λ^2 is the usual Feynman cut-off.
After subtraction we have

$$-i\pi^2 \ln \frac{(q-p)^2}{\lambda^2}\,. \tag{A.3}$$

Substituting (A.3) into (A.I) we find

$$-i\pi^2 \int dq \frac{\ln \frac{(q-p)^2}{\lambda^2}}{q^2 (k-q)^2} = -i\pi^2 \int dq \frac{\left[\ln \frac{q^2}{\lambda^2} + \ln \frac{(q-p)^2}{q^2} \right]}{q^2\, (k-q)^2}\,. \tag{A.4}$$

To calculate the integral

$$\int \frac{dq \ln \frac{q^2}{\lambda^2}}{q^2 (k-q)^2} = i\pi^2 \int_0^\infty \frac{dq^2 \cdot q^2 \ln \frac{q^2}{\lambda^2}}{q^2} \int \frac{d\Omega_q}{2\pi^2} \frac{1}{(k-q)^2}\,, \tag{A.5}$$

where in the r.h.s. all the momenta are euclidean, we use the method proposed in [17].

$$\frac{1}{(k-q)^2} = \ldots \frac{1}{\ldots} \sum_n \left(\frac{q}{k}\right)^{n+1} C_n(\hat{k}\hat{q}) \ldots \qquad (A.6)$$

where $k = \sqrt{k^2}$, $\hat{k}\hat{q} \equiv \cos\theta_{kq} = \frac{k q}{k \cdot q}$ ... and C_n are the Chebyshev polynomials ... which have the following properties:

$$\int \frac{d\Omega_q}{2\pi^2} C_n(\hat{k}\hat{q}) C_m(\hat{k}\hat{q}) = \delta_{nm}, \qquad (A.7)$$

$$\int \frac{d\Omega_q}{2\pi^2} C_n(\hat{k}\hat{q}) C_m \ldots = \frac{\delta_{mn}}{n+1} C_n(\ldots). \qquad (A.8)$$

The substitution of (A.6) into (A.5) gives

$$\pi^4 \int_0^{\Lambda^2} \frac{dq^2 \ln q^2/\lambda^2}{k \cdot q} \sum_n \left(\frac{q}{k}\right)^{n+1} \int \frac{d\Omega_q}{2\pi^2} C_n(\hat{q}\hat{k}) \qquad (A.9)$$

Using (A.7) and performing an angular integration in (A.9) we have

$$\pi^4 \int_0^{\Lambda^2} \frac{dq^2 \ln q^2/\lambda^2}{k \cdot q} \left\langle \frac{q}{k} \right\rangle = \pi^4 \left[\int_0^{k^2} \frac{dq^2 \ln q^2/\lambda^2}{k^2} + \int_{k^2}^{\Lambda^2} \frac{dq^2}{q^2} \ln \frac{q^2}{\lambda^2} \right]$$

$$= \pi^4 \left[\ln \frac{k^2}{\lambda^2} - 1 + \frac{\ln^2 \ldots}{2} - \frac{\ln^2 k^2}{2} - \ln \frac{\Lambda^2}{k^2} \ln \lambda^2 \right]. \qquad (A.10)$$

Subtracting (A.10) at λ^2 we find for (A.5)

$$-\pi^4 \left[\frac{L^2}{2} - L \right], \tag{A.11}$$

where $L = \ln \frac{\kappa^2}{\lambda^2}$.

Finally for (A.1)

$$-\pi^4 \left[\frac{L^2}{2} - L - \frac{1}{i\pi^2} R_\lambda \int \frac{dq \ln \frac{(q-p)^2}{q^2}}{q^2 (\kappa - q)^2} \right]. \tag{A.12}$$

The last integral is nonessential for symmetrical logarithmic asymptotics ($p^2 \sim \kappa^2 \to \infty$) of (A.1). It contributes to the terms $h^4 \ln \frac{\kappa^2}{\lambda^2}$ which are important only for higher logarithmic asymptotics.

II. Here we describe the method allowing calculation of the lowest logarithmic terms for the arbitrary order diagrams in scalar theory. Consider one-particle-irreducible diagram G. The corresponding integral is

$$\frac{\pi^{2\ell}}{i^{\ell+c}} T_G(\ldots \kappa_i \ldots) = \int \frac{dp_1 \ldots dp_c}{\ldots (\kappa_i - p_j)^2 \ldots},$$

where κ_i are external momenta, ℓ is a number of internal lines. In the symmetrical case all κ_i are proportional to one momentum κ^2, and T_G becomes a function of one variable.

To remove the divergences we are to perform an R-operation with subtraction point $\kappa^2 = \lambda^2$

$$T_G(\kappa^2) \to R_\lambda T_G(\kappa^2).$$

In the α-representation

$$T_G(\kappa^2) = \int \frac{\prod d\alpha \; e^{i A_G(\alpha, \kappa^2)}}{D_G^2(\alpha)}. \tag{A.13}$$

Here we use the notation of ref.[5], [16].

The R - operation may be written in the form

$$R = (1 - M_G)(1 - M_{\gamma_1}) \ldots (1 - M_{\gamma_k}) \ldots , \qquad (A.14)$$

where γ_i is a divergent subgraph of G, and

$$M_{\gamma} T_G(\kappa^2) = \int \prod d\alpha \, \frac{e^{iA_\gamma(\alpha,\lambda^2)} \, e^{iA_{G/\gamma}(\alpha,\kappa^2)}}{D^{2}_{\gamma}(\alpha) \, D^{2}_{G/\gamma}(\alpha)} . \qquad (A.15)$$

The asymptotic form of $R_\lambda T_G(\kappa^2)$ is

$$R_\lambda T_G(\kappa^2) \to a_m \left(\ln \frac{\kappa^2}{\lambda^2}\right)^m + \ldots + a_1 \ln \frac{\kappa^2}{\lambda^2} \qquad \kappa^2 \to \infty .$$

To calculate the Gell-Mann-Low function one needs the coefficient a_1,

$$a_1 = \kappa^2 \frac{\partial}{\partial \kappa^2} R_\lambda T_G(\kappa^2) \Big|_{\kappa^2 = \lambda^2} . \qquad (A.16)$$

Having in mind that $A(\alpha,\kappa^2) = \kappa^2 A(\alpha)$ and performing the change of variables $\alpha_i \to \mu \alpha_i'$, $\sum_i \alpha_i' = 1$, we get

$$\int \prod^{\ell} d\alpha \to \int_0^\infty \mu^{\ell-1} d\mu \int_0^1 \prod^{\ell} d\alpha \, \delta\left(1 - \sum^{\ell} \alpha\right), \quad A(\alpha,\kappa^2) \to \kappa^2 \mu A(\alpha)$$

All diagrams G, $\gamma_1, \ldots \gamma_k$ are assumed to diverge logarithmically. Then

$$M_\gamma T_G(\kappa^2) \to \int_0^\infty \frac{d\mu}{\mu} \int_0^1 \prod d\alpha \, \delta\left(1 - \sum \alpha\right) \frac{e^{iA_\gamma(\alpha)\mu\lambda^2} \, e^{iA_{G/\gamma}(\alpha)\mu\kappa^2}}{D^{2}_{\gamma}(\alpha) \, D^{2}_{G/\gamma}(\alpha)} . \qquad (A.17)$$

The integration over μ gives

$$\kappa^2 \frac{\partial}{\partial \kappa^2} M_\gamma T_G(\kappa^2)\Big|_{\kappa^2 = \lambda^2} = -\int \prod d\alpha \, \delta(1 - \sum \alpha) \frac{A_{G/\gamma}(\alpha)}{D_\gamma^2(\alpha) D_{G/\gamma}^2(\alpha) \left[A_\gamma(\alpha) + A_{G/\gamma}(\alpha) \right]} \quad \text{(A.I8)}$$

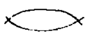
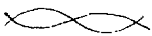
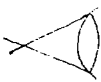
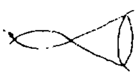
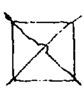
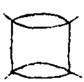
All other contributions to (A.I6) are calculated similarly. Substituting (A.I8) and (A.I4) to (A.I6) we obtain the coefficient a_1. Every integral of the type (A.I8) diverges, but in a_1 all divergences are cancelled.

The quadratically divergent diagrams are treated by the same method, but the resulting formulas are slightly different.

This method was used to calculate the coefficients at h^4 in eq. (13) with the help of a computer. The results are the same as the ones given in the table.

T A B L E

$$\mathcal{L} = -\frac{4\pi^2 h}{3}(\varphi^a\varphi^a)^2 \qquad a = 1, 2, \ldots N \qquad L = [illegible]$$

Diagram	Combinatorial factor	The integral value
	$-\frac{8+N}{3}h$	$-L$
	$\frac{N^2+6N+20}{9}h^2$	L^2
	$4/9\,(5N+22)\,h^2$	$\frac{L^2}{2}-L$
	$2\,\frac{2+N}{9}h^2$	$\frac{L}{2}$
	$-\frac{N^3+8N^2+24N+48}{27}h^3$	$-L^3$
	$-4\,\frac{3N^2+22N+56}{27}h^3$	$-L\,(\frac{L^2}{2}-L)$
	$-8\,\frac{5N+22}{27}h^3$	$-6\,\zeta(3)\,L$
	$-4\,\frac{N^2+20N+60}{27}h^3$	$-\frac{L^3}{3}+L^2-2L+2LJ$

(to be continued)

	$-4\,\frac{3N^2+22N+56}{27}\,h^3$	$-\frac{L^3}{3}+L^2-2L+2LJ$
	$-16\,\frac{N^2+20N+60}{27}\,h^3$	$-\frac{L^3}{6}+L^2-2L$
	$-2\,\frac{3N^2+22N+56}{27}\,h^3$	$-\frac{L^3}{3}+L^2+2L-2JL$
	$-4\,\frac{N^2+10N+16}{27}\,h^3$	$-\frac{L^2}{4}+\frac{L}{2}-\frac{\ln(3/4)}{2}\,L$
	$-2\,\frac{N^2+10N+16}{27}\,h^3$	$-\frac{L^2}{2}+\frac{3}{2}\,L$

In the table the corrections $\Delta\Gamma$ and ΔD to the vertex function and the propagator $\Gamma = 1+\Delta\Gamma$, $D = 1+\Delta D$ are presented. The third column contains the values of the integrals of the type

$$\frac{i}{\pi^4}\int \frac{dp_1\,dp_2\,dp_3}{\dots(\kappa-p_1)^2\dots}$$

corresponding to the diagrams of the first column. The subtractions are performed at the point $s = t = u = \frac{4}{3}\kappa_1^2 = \lambda^2$.

The second column contains the combinatorial factors.

$$J \equiv J\left(\tfrac{3}{4}\right) \approx 0.75 \;;\quad J\left(\tfrac{p^2}{\kappa^2}\right) = \frac{1}{i\pi^2}\int \frac{dq\,\ln\frac{(q-p)^2}{\lambda^2}}{q^2(\kappa-q)^2}\Bigg|_{(p-\kappa)^2=p^2}$$

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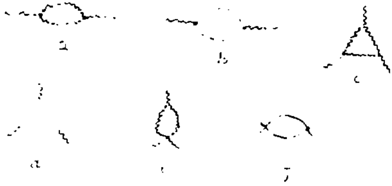

Fig. 1

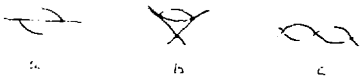

Fig. 2

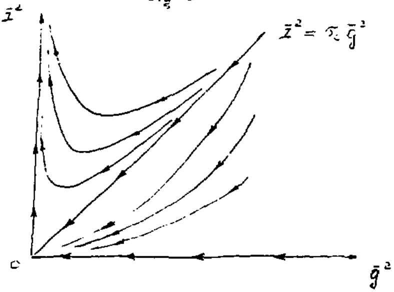

Fig. 3

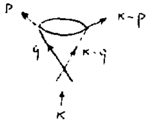

Fig. 4