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ON THE TIME EVOLUTION OF PHYSICAL SYSTEMS

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ON THE TIME EVOLUTION OF PHYSICAL SYSTEMS

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О временной эволюции физических систем

Определен общий класс временной эволюции физических систем, c -эволюции. Эти эволюции характеризуются тем, что каждый инвариантный вогнутый, "энтропия-подобный" функционал растет с течением времени. Показано, что различные физические процессы являются c -эволюцией.

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On the Time Evolutions of Physical Systems

In this paper a general class of time evolutions of physical systems, the c -evolutions, will be defined. These evolutions are characterized by the property that any invariant concave functional, "entropy-like" functional, increases in time. Different physical processes are shown to be c -evolutions.

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In this paper a general class of evolutions of physical systems $(\mathfrak{A}, \Sigma)$, $\mathfrak{A}$ observable algebra, Σ state-set, will be introduced. These evolutions we call c-evolutions (concave evolutions). A c-evolution is characterized by the property that in the Schrödinger picture a state ρ_t after the time $t > 0$ is ''more mixed'' than at the time $t = 0$ in the sense of Uhlmann /7/, i.e. every invariant concave function on the convex set Σ of sets increases for $t \to +\infty$.

This class of c-evolutions contains the strong determined evolutions of classical mechanics and quantum mechanics but also more general evolutions, generated by Markov processes. In this sense the class of c-evolutions is very wide. On the other hand they have the property that any ''entropy like'' functions increase in dependence of the time. We regard this class of c-evolutions also for systems with unbounded observables.

I.

The basic constituents of a physical system $(\mathfrak{A}, \Sigma)$ are its observable $*$-algebra $\mathfrak{A}$ and the set Σ of states. The symmetric elements $A = A^+$ of $\mathfrak{A}$ are the observables. A state $\rho \in \Sigma$ is a linear positive and normed functional on $\mathfrak{A}$, $\rho(A^+A) \geq 0$, $\rho(I) = 1$, where I is the identity of $\mathfrak{A}$, $\rho(A)$ is the expectation value of the observable A in the state ρ. We suppose, that Σ is a convex set, but we do not assume any normalization of $\mathfrak{A}$. Also algebras $\mathfrak{A}$ of unbounded operators will be regarded.

For a physical system $(\mathfrak{A}, \mathfrak{Z})$ "physical" topologies in $\mathfrak{A}$ and $\mathfrak{Z}$ are automatically defined, namely simply by the requirements that a sequence A_ν of observables converges to A, if for all $\rho \in \mathfrak{Z}$ the expectation values $\rho(A_\nu)$ converge to $\rho(A)$. Analogously one defines the convergence $\rho_\nu \to \rho$ in $\mathfrak{Z}$ by the requirement that $\rho_\nu(A)$ converges to $\rho(A)$ for every $A \in \mathfrak{A}$. The topologies defined in this way are the weakest "physical" topologies /5/ and we denote them in $\mathfrak{A}$ as well as in $\mathfrak{Z}$ by σ. These topologies σ are exactly the weak topologies of the dual pair $(\mathfrak{A}, L_{\mathfrak{Z}})$, where $L_{\mathfrak{Z}}$ is the linear space of linear functionals generated by $\mathfrak{Z}$.

The evolution of the physical system can be described in the Schrödinger picture by the evolution $\rho = \rho_0 \to \rho_t$, $t > 0$ of the states. If ρ_0 is a state at $t = 0$, then ρ_t is the state after the time t.

Usually one requires the

Properties E.

1. $\rho \to \rho_t = V_t \rho$ is a linear transformation, continuous with respect to the topology σ (weakly continuous).

2. $t \to V_t$ is a representation of the additive semi-group $T_+ = \{t;\ t > 0\}$, i.e. there holds

$$V_{t_1} V_{t_2} = V_{t_1 + t_2}$$

or

$$(\rho_{t_1})_{t_2} = \rho_{t_1 + t_2} .$$

We will see below that these requirements are general enough, so that irreversible processes can be described in this way, too.

Well known is the transition from the Schrödinger picture to the Heisenberg picture, by which the states are fixed and the observables $A \in \mathfrak{A}$ change,

$A \to A_t = W_t A$.

The connection between the Schrödinger picture and the Heisenberg picture is that the expectation values (ρ_t, A) and (ρ, A_t) coincide, i.e.

$$(\rho_t, A) = (V_t \rho, A) = (\rho, W_t A) = (\rho, A_t) . \qquad (1)$$

Properties E′:

1′. The operator W_t with the property (1) exists and is a weakly continuous linear operator from $\mathfrak{A}$ into itself.

2′. W_t is a representation of the semi-group T_+.

3′. W_t is commutative with the involution in $\mathfrak{A}$, i.e. we have

$$(A^+)_t = W_t A^+ = (W_t A)^+ = (A_t)^+ .$$

According to this the observables $A = A^+$ go over to observables $A_t = A_t^+$.

Proof. 1′ is valid since V_t is weakly continuous and therefore exists the adjoint operator W_t to V_t in the dual pair and is weakly continuous, too.

2′ follows immediately from 2.

Because $(\rho_t, A) = (\rho, A_t)$ is valid and ρ_t is a state, it follows that (ρ, A_t) is really for all states ρ. But from this it follows that A_t is an observable, for if this were not so, we could write it in the form $A_t = B_1 + i B_2$, with B_1, B_2 observables and $B_2 \neq 0$.

Then there exists a state ρ with $(\rho, B_2) \neq 0$ and, therefore, $(\rho, A_t) = (\rho, B_1) + i(\rho, B_2)$ is not real.

II.

The first well-known example for an evolution of physical system in the form discussed in the previous section is the quantum mechanical system.

Let $\mathfrak{A}$ be a $*$-algebra of operators in a unitary space (unbounded operators are not excluded) and $\mathfrak{Z}$ a convex set positive nuclear operators with $\operatorname{tr} \rho = 1$ and $\operatorname{tr} A\rho < \infty$ for every $A \in \mathfrak{A}$. The evolution is then described in the Schrödinger picture by

$$\rho \to \rho_t = V_t \rho = e^{-iHt} \rho\, e^{iHt} , \quad \rho \in \mathfrak{Z} \qquad (2)$$

and in the Heisenberg picture by

$$A \to A_t = W_t A = e^{iHt} A\, e^{-iHt} . \qquad (3)$$

Of course (2) and (3) are well defined if and only if $\rho_t \in \mathcal{Z}$ and $A_t \in \mathfrak{A}$ for any $t \geq 0$. This is the case if $\mathfrak{A} = \mathcal{B}(\mathcal{H})$, the $*$-algebra of all bounded operators, and $\mathcal{Z}$ the set of all trace-states. But we can also take examples with unbounded operators. For example let $\mathcal{S} = \mathcal{S}(R^1)$ be the Schwartz space and $\mathfrak{A} = \mathcal{L}^+(\mathcal{S})$ the Op^*-algebra of all (unbounded) operators in $L^2(R^1)$ for which $A\mathcal{S} \subset \mathcal{S}$ and $A^+\mathcal{S} \subset \mathcal{S}$ hold and $\mathcal{Z}$ the convex set of all positive and normed (to $\rho = 1$) nuclear operators mapping L_2 into $\mathcal{S}$. Then $\rho(A) = \mathrm{tr}\, A\rho$ is well defined /6/. The operators $Q = x$ and $P = 1/i \cdot d/dx$ are elements of $\mathfrak{A}$ and hence observables for this system $(\mathfrak{A}, \mathcal{Z})$. The evolution (2), (3) is well defined for example if H is the Hamiltonian of the harmonic oscillator, $H = 1/2m\,P^2 + Q^2$, since $e^{iHt}\mathcal{S} = \mathcal{S}$.

Now we regard a classical mechanical system. Let the phase space Ω be a locally compact group. The group operation we write additive. As the set $\mathcal{Z}$ of states we take a convex set of positive normed Borel measures μ, $\int d\mu = 1$ on Ω which contains the point measures $\int_\Omega d\mu = \delta(x - x'_0)\,dx$. $\delta(x) = \delta_0$ is the point-measure concentrated at the point 0.

For the observable algebra $\mathfrak{A}$ we take an $*$-algebra of locally bounded Borel functions on Ω which contains all continuous functions with compact support, such that $\mu(A) = \int |A(x)|\,d\mu < \infty$ for any $\mu \in \mathcal{Z}$, $A \in \mathfrak{A}$. There are two important special cases

$\mathfrak{A}_I$ = algebra of all bounded Borel functions;
$\mathcal{Z}_1$ = set of all positive normed measures. (4)

In this case $\mathfrak{A}_I$ is a C^*-algebra.

$\mathfrak{A}_2$ = algebra of all locally bounded Borel functions;
$\mathcal{Z}_2$ = set of all positive normed measures with compact support. (5)

$\mathfrak{A}_2$ can not be equipped with an algebra-norm.

The classical strong determined dynamic is described by the one-parametric groups (trajectories) $\phi(t, x_0)$ of transformations of Ω, $\phi(t_1 + t_2, x_0) = \phi(t_1, \phi(t_2, x_0))$, $\phi(0, x_0) = x_0$, satisfying the equations of motion (Hamilton equations).

In the language of the physical system $(\mathfrak{A}, \mathcal{Z})$ the classical dynamic is given by the Schrödinger picture

$$\mu \to \mu_t = V_t\,\mu \qquad \mu \in \mathcal{Z}$$

where

$$\mu_t(M) = \mu(\phi(-t) \cdot M) \tag{6}$$

for any Borel subset M of Ω, or in the Heisenberg picture $A \to A_t = W_t A$ with

$$A_t(x) = A(\phi(t, x)). \tag{7}$$

This is in accordance with the Properties E and E′ and the relation (1).

Up to now we have not used the group-properties of Ω. These we need only for the next examples of general evolution of a physical system. Let $\{P_t, 0 \leq t \leq \infty\}$ be a homogeneous stochastic process on the locally compact group Ω^3, i.e. a family of probability measures P_t with the Markov-property

$$P_{t+s} = P_t * P_s \tag{8}$$

and the continuity condition $P_t \to \delta_e$ (weak) for $t \downarrow 0$, where the evolution is defined by

$$P_t * P_s(B) = \int_\Omega P_t(B - x)\, P_s(dx). \tag{9}$$

The evolution of the physical system generated by the stochastic process P_t in the Schrödinger picture is then given by

$$\mu \to \mu_t = V_t\,\mu = P_t * \mu \tag{10}$$

and we suppose that from $\mu \in \mathcal{Z}$ it follows $\mu_t \in \mathcal{Z}$. It is clear that the properties E are satisfied. If $\Omega = R^N$ an example for such a stochastic process is the Brown motion with

$$P_t(dx) = \frac{1}{\sqrt{2\pi t}^{\,3N}}\, e^{-\frac{x^2}{2t}}\, dx \tag{11}$$

Example 1. The strongly determined evolution of a quantum mechanical system $(\mathfrak{A}, \mathfrak{Z})$ is of course a c -evolution, since ρ and ρ_t are equivalent (Def.1).

Example 2. The processes $\rho \to \rho_t$ regarded in /2/ and given by

$$\rho(t+dt) = (1-\lambda dt)\, e^{-iHdt} \rho(t)\, e^{iHdt} + \lambda dt \sum_n P_n \rho(t) \quad (17)$$

are c -evolutions. $\Sigma\, \omega_n P_n = \Omega$ is the spectral resolution of an observable Ω which is repeatedly measured at an unpredictable times, randomly distributed with a given mean frequency λ.

That (17) described a c -evolution follows from the fact, that $\mu = \Sigma P_n \rho P_n$ is more mixed than ρ. For the case of finitely many P_n, $\sum_{n=1} P_n = I$ this can be directly seen if we put

$$U_i = \sum_{n=1} \lambda_i^n P_n, \quad \lambda_i^n = \pm 1; \quad i = 1,\dots,M$$

with the orthogonal $\sum_i \lambda_i^n \lambda_i^m = M\, \delta_{nm}$.
Then it holds

$$\frac{1}{M} \sum_{i=1}^{M} U_i \rho U_i^* = \frac{1}{M} \sum_{n,m} \sum_i \lambda_i^n \lambda_i^m P_n \rho P_m = \sum_n P_n \rho P_n = \mu .$$

The λ_i^i can be chosen for N=3 by

$$\begin{array}{lrrrr} \lambda_i^1 : & 1 & 1 & 1 & 1 \\ \lambda_i^2 : & 1 & 1 & -1 & -1 \\ \lambda_i^3 : & 1 & -1 & 1 & -1 \end{array}$$

and analogously in the general case. For infinitely many P_n, the more mixedness of $\Sigma P_n \rho P_n$ can be proved with the help of Lemma 1 ii).

Example 3. Kossakowski in /4/ regarded some classes of dynamical semi-groups V_t in $(\mathfrak{B}(\mathcal{H}), \mathcal{P}(\mathcal{H}))$ induced by Markov processes in topological groups G. $\mathfrak{B}(\mathcal{H})$ is the $*$ -algebra of all bounded operators of a Hilbert space $\mathcal{H}$ and the state-set $\mathcal{P}(\mathcal{H})$ - the set of all density operators ρ ($\rho > 0$, $\mathrm{tr}\,\rho = 1$). Let $\mathfrak{M}(G)$ be the space of all probability measures on G. $\mathfrak{M}(G)$ is a semi-group with respect to the convolution $\mu * \nu$. Let $g \to U(g)$ be a unitary representation of G in $\mathcal{H}$ Then

$$V_\mu \rho = \int_G U(g)\, \rho\, U(g^{-1})\, \mu(dg) \quad (18)$$

is a representation of $\mathfrak{M}(G)$ in the semi-group of "linear" endomorphisms of $\mathcal{P}(\mathcal{H})$. If μ_t is a one parameter convolution semi-group in $\mathfrak{M}(G)$ then $\rho \to \rho_t = \int_G U(g)\, \rho\, U(g^{-1})\, \mu_t(dg)$ is a dynamical semi-group, which Kossakowski called "quantum Poisson process"

$$\text{if } \mu_t = e^{-at} \sum_{n=0}^{\infty} \frac{(at)^n}{n!} \mu^{*n} . \quad (19)$$

It follows immediately from the definitions that (18) describes a c -evolution of the physical system $(\mathfrak{B}(\mathcal{H}), \mathcal{P}(\mathcal{H}))$.

IV.

In the last section let us yet regard the evolution of the mechanical system. For the sake of definiteness of the motion we take the physical system $(\mathfrak{A}_2, \mathfrak{Z}_2)$ (5), which contains also unbounded observables for non-compact phase spaces Ω. We will see that the evolutions (6) or (14) are c -evolutions. For this let G be the group of all homoemorphisms of Ω, for $g \in G$ we define

$$A_g(x) = A(g(x)) . \quad (20)$$

In this sense G is also a group of σ -continuous $*$ - automorphism of $\mathfrak{A}_2$.

For any $\mu \in \mathfrak{Z}_2$ μ_g is then defined by

$$\mu_g(A) = \mu(A_g) = \int A(g(x))\, d\mu = \int A(x)\, \mu(g^{-1}(dx)) \cdot (21)$$

Hence
$$\mu_g(M) = \mu(g^{-1}(M)) .$$

Example 4. For the strongly determined classical dynamic (6) one has

$$\mu \to \mu_t = \mu_{\phi(t,\cdot)} \qquad \phi(t,\cdot) \in G \tag{22}$$

and therefore it is a c -evolution. More precisely the states μ_t and μ are even $(G-)$ equivalent (Def. 1).

Example 5. Now we regard the Markov process (10) in $(\mathfrak{A}_2, \mathcal{Z}_2)$ supp P_t compact. It holds

$$\mu_t(B) = (P_t * \mu)(B) = \int_\Omega P_t(B-x)\, \mu(dx)$$

$$= \int_\Omega \mu(-y+B)\, P_t(dy) = \int_\Omega \mu_y(B)\, P_t(dy) \tag{23}$$

with the denotion $\mu_z(B) = \mu(-z+B)$ for $z \in G$.
If we denote by $g \in G$ the homoemorphism $x \to z + x$ of Ω, then in consistence with (21) we have

$$\mu_z = \mu_{g_z} . \tag{24}$$

Therefore μ_t is more mixed than μ (with respect to G), since from (23) $\mu_t \in M_\mu$ follows (Def. 1). Hence (10) $\mu \to P_t * \mu$ is a c-evolution.

Let T be the set of all translation - automorphisms

$$A(x) \to A_{g_z}(x) = A(z+x) \tag{25}$$

of $\mathfrak{A}_2$. T is a subgroup of G. From (23) we see that

$$\mu_t \overset{T}{\longleftarrow} \mu \qquad \text{for} \quad t > 0 , \tag{26}$$

i.e. μ_t is already with respect to T more mixed than μ.

Finally let us summarize another two properties of the semiorder $\rho \overset{T}{\longleftarrow} \mu$ in $\mathcal{Z}_2$.

Lemma 2

1) All point measures δ_x are equivalent and any state $\rho \in \mathcal{Z}_2$ is more mixed than every point measure δ_x.

ii) $\mathcal{Z}_2$ contains a maximal mixed state if and only if Ω is a compact group. This maximal mixed state is uniquely determined and equal to the normed Haar measure.

Proof

i) $\delta_x \overset{T}{\approx} \delta_y$ since $\delta_x = (\delta_y)_{g_{x-y}}$. Let μ be any measure then

$$\mu(M) = \int (\delta_0)_{g_x}(M)\, \mu(dx) \tag{27}$$

(the integral converges with respect to the topology σ).

Hence $\mu \overset{T}{\longleftarrow} \delta_0 \overset{T}{\approx} \delta_x$.

ii) Let Ω be a compact group and μ, $\int d\mu = 1$ the Haar measure of Ω. Let ρ be an arbitrary measure on Ω. Then it follows that

$$\mu(M) = \int_\Omega \mu(M-x)\, \rho(dx) = \int_\Omega \rho(-x+M)\, \mu(dx)$$

$$= \int_\Omega \rho_x(M)\, \mu(dx)$$

and therefore $\mu \overset{T}{\longleftarrow} \rho$. On the other side M_μ contains only μ. Hence the maximal state μ is uniquely determined. Now let Ω be not compact and μ an arbitrary state of $\mathcal{Z}_2$; supp $\mu = K$ is a compact set. $K-K$ is also a compact set and we take $x \in \Omega$, $x \notin K-K$. Then $-x+K \cap K = \emptyset$.

We put $\rho(B) = \frac{1}{2}\mu(B) + \frac{1}{2}\mu_x(B) = \frac{1}{2}(\mu(B) + \mu(-x+B))$.

Then we have $\rho \overset{T}{\longleftarrow} \mu$ and $\rho(K) = \frac{1}{2}$. Further for any $\chi \in M_\rho$ it is $\chi(K) \leq \frac{1}{2}$. Therefore μ is not an element of M_ρ, i.e. μ is not a maximal mixed state. Since μ was arbitrary $\mathcal{Z}_2$ does not contain any maximal state.

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