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# EXISTENCE PROOFS FOR WIGHTMAN TYPE FUNCTIONALS

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## 1. Introduction

In the algebraic description of a quantum field, first given in [1,8], the Wightmann functions $W_n(x_1,\ldots,x_n)$ are collected to one functional $W(f)$ on the algebra $\mathfrak{A} = \mathfrak{A}_0 + i\,\mathfrak{A}_0$ of test-functions. Hence the Wightmann functional is uniquely determined by the hyperplane $E_w = \ker W$, the kernel of $W$. The positivity of the Wightmann functional $W$ leads to the property that the cone $K$ of positive elements of $\mathfrak{A}_0$ lies on one side of $E_w$. The other properties of $W$, the locality, the spectrality and the invariance lead to the fact that $E_w$ contains a certain linear space L, the so-called Wightmann kernel.

In this way the existence problem for quantum fields is transformed to the problem of finding all hyperplanes $E_w$ in $\mathfrak{A}_0$ with the properties mentioned before. Therefore to prove the existence of Wightman (type) functionals with certain properties one can apply the well-known seperation theorems for convex sets in locally convex spaces. This method was outlined already in [6,7], where it was proved the existence of certain functionals with the property of positivity only. Further results in this direction were obtained in [2,3], where the existence of certain Lorentz invariant positive functionals could be proved. Recently in [9] a general extension theorem for linear functionals on a subspace $M$ of $\mathfrak{A}_0$ to a Wightman (type) functional on $\mathfrak{A}_0$ was proved. In this paper we describe a method to prove the existence of fields with certain properties by applying this general extension theorem. We give a simple proof on the existence of a local invariant quantum field (without spectrality) different from a generalized free field. We are convinced of the possibility to prove in analogous way also the existence of quantum fields with more difficult properties (non-trivial fields).

## 2. Geometric Interpretation of the Wightman Functionals

A quantum field $\phi(x)$ (neutral, scalar) is uniquelly determined by the Wightman functional

$$W(f) = \sum_{n\geq 0} \int W_n(x_1,\dots,x_n) f_n(x_1,\dots,x_n)\, dx_1 \dots dx_n \tag{1}$$

$$W_n(x_1,\dots,x_n) = \langle 0 \,|\, \phi(x_1)\dots\phi(x_n) \,|\, 0\rangle,$$

$$f_n(x_1,\dots,x_n) \in \mathcal{S}_n = \mathcal{S}(R^{4n}),$$

where

$$f = \{ f_0, f_1(x_1), \dots, f_n(x_1,\dots,x_n), 0, 0 \dots \}$$

is an element of the field algebra

$$\mathfrak{A} = \mathcal{S}_0 \oplus \mathcal{S}_1 \oplus \mathcal{S}_2 \oplus \dots, \quad \mathcal{S}_0 = C. \tag{2}$$

$\mathfrak{A}$ is the tensor algebra over the Schwartz space $\mathcal{S}_1 = \mathcal{S}(R^4)$. $\mathfrak{A}$ is a locally convex topological $*$-algebra. The involution is defined by

$$(f^*_n)(x_1,\dots,x_n) = \overline{f_n(x_n,\dots,x_1)}, \tag{3}$$

the multiplication by

$$(f\cdot g)_n (x_1,\dots,x_n) = \sum_{\ell+k=n} f_\ell(x_1,\dots,x_\ell)\, g_k(x_{\ell+1},\dots,x_n) \tag{4}$$

and the topology $\tau$ by the system of seminorms

$$\tau: \; \| f \|_{(\gamma_n)(\nu_n)} = \sum_{n\geq 0} \gamma_n \| f_n \|_{\nu_n}, \tag{5}$$



where $(\gamma_n)$ is an arbitrary sequence of positive numbers, $(\nu_n)$ an arbitrary sequence of nonnegative integers and $\| f_n \|_\nu$, $\nu = 0,1,2,\dots$, the denumerable system of seminorms defining the topology of $\mathcal{S}_n$. Some properties of the topology $\tau$ and of another important topology $\tau_\infty$ in $\mathfrak{A}$ are investigated in [4,5].

The hermitean part $\mathfrak{A}_0$ of the field algebra $\mathfrak{A}$ is given by

$$\mathfrak{A}_0 = \{ f \in \mathfrak{A} \;;\; f^* = f \}. \tag{6}$$

By $K_0$ we denote the cone of positive elements generated by $f^*\cdot f$, $f \in \mathfrak{A}$, and by $K = \bar{K}_0$ its topological closure. K is also a cone [6].

The Wightman functional $W(f)$ is characterised by the following properties [1,8].

1. <u>Positivity</u>

$$W(f) \geq 0 \quad \text{for } f \in K \tag{7}$$

and $W(1)=1$, where $1=\{1,0,0,\dots\}$ is the unity element of $\mathfrak{A}$. In [9] it is proved that the continuity of $W$ is a consequence of the positivity.

2. <u>Locality</u>

$$W(f) = 0 \quad \text{for } f \in I_{loc}, \tag{8}$$

where $I_{loc}$ is the two-sided ideal generated by the elements $f_n(x_1,\dots,x_n) = g_n(\dots,x_j,x_{j+1},\dots) - g_n(\dots,x_{j+1},x_j,\dots)$

with $g_n(x_1,\dots,x_n) = 0$ for $(x_{j+1}-x_j)^2 \geq 0$.

3. <u>Spectrality</u>

$$W(f) = 0 \quad \text{for } f \in I_{sp}, \tag{9}$$

where $I_{sp}$ is the linear space generated by $f_n(x_1\dots,x_n)$

with $\tilde{f}_n(p_1,\dots,p_n) = \frac{1}{(2\pi)^{2n}} \int e^{i\sum p_i x_i} f_n(x_1,\dots,x_n)\, dx_1 \dots dx_n = 0$

for $\sum_{\ell=1}^{k} p_\ell \in \overline{V^+}$ (the closed forward cone), $\sum_{\ell=1}^{n} p_\ell = 0$ ,

$k = 1,2,\dots, n-1$ ,

### 4. Poincare invariance

$$W(f) = 0 \quad \text{for } f \in I_{inv} , \qquad (10)$$

where $I_{inv}$ is the linear space generated by the elements

$$f_n(x_1,\dots,x_n) = g_n(x_1,\dots,x_n) - g_n(\Lambda^{-1}(x_1-a),\dots,\Lambda^{-1}(x_n-a))$$

$$= g_n - (\Lambda, a)\, g_n \; .$$

Since $W(f)$ is a positive functional, it is hermitean and therefore a real positive functional on the semiordered space $\mathfrak{A}_0$ with the closed cone $K$ .

Let $I = \Sigma(I_{\ell oc}, I_{sp}, I_{inv})$ be the closed linear subspace generated by these three linear spaces. Since $f^* \in I$ if $f \in I$, the linear space $I$ has the decomposition

$$I = L + iL , \qquad (11)$$

where $L$ is a real subspace of $\mathfrak{A}_0$ . We call $I$ respectively $L$ the Wightman kernel. Collecting all mentioned properties we can say that a Wightman functional $W(f)$ is a (real) linear functional on the real linear space $\mathfrak{A}_0$, satisfying the following properties.

**Wightman functional**

$$\begin{aligned} &a)\quad W(f) \geq 0 \qquad \text{for } f \in K \\ &b)\quad W(f) = 0 \qquad \text{for } f \in L \qquad\qquad (12) \\ &c)\quad W(1) = 1 \end{aligned}$$



Let be $E_W = \ker W = \{ f \in \mathfrak{A}_0 ;\ W(f) = 0 \}$.

Then $E_W$ is a hyperplane in $\mathfrak{A}_0$. Since we assume the normality $W(1)=1$ the correspondence between $E_W$ and $W$ is unique. So we have proved

**Lemma 1**

There is a one-to-one correspondence between a quantum field $\phi(x)$ with the Wightman functional $W$ and a hyperplane $E_W$ of $\mathfrak{A}_0$ with the properties (Fig. 1)

a) the cone $K$ lies on one side of $E_W$ ,

(13)

b) $L \subset E_W$ .

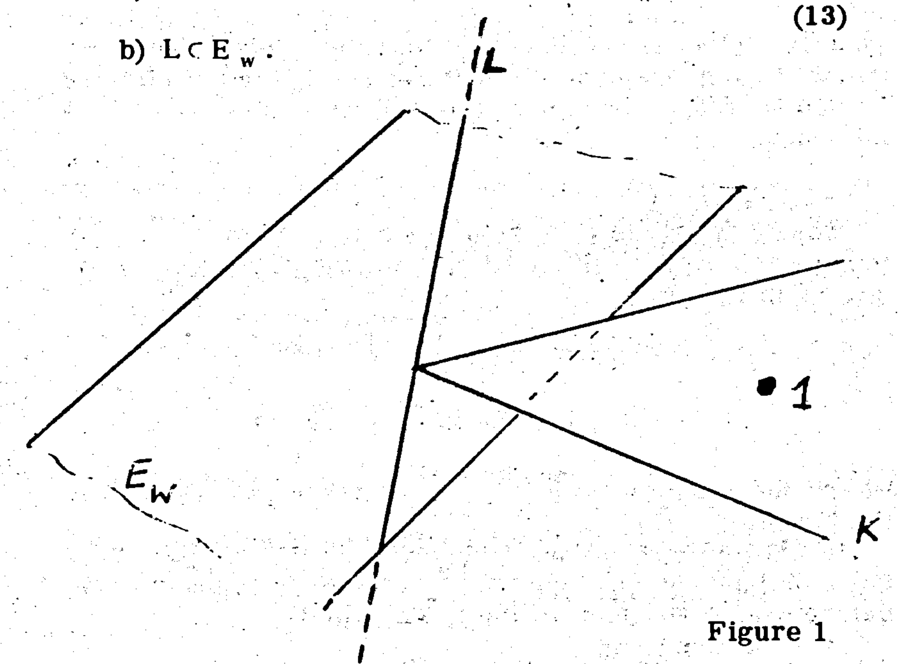


Figure 1

Thus the problem to determine all quantum fields $\phi(x)$ (neutral, scalar) is equivalent to the problem to determine all hyperplanes $E$ of $\mathfrak{A}_0$ with the properties (13). This conception of determining of the Wightman fields we had already formulated some years ago in /7/. Some results in this direction have been obtained in /2,3/, where properties of positive functionals of the Wightman - type (satisfying not all conditions) are proved. Recently Wyss [9] proved some theorems about the extension of linear functionals in $\mathfrak{A}_0$ to positive functionals. Let M be a subspace of $\mathfrak{A}_0$ containing the identity 1 and T(f) be a linear real functional on M normed by T(1)=1. T is then uniquely determined by its kernel

$$\ker T = \{ f \in M ; \; T(f) = 0 \}.$$

Question: Under which conditions does there exist a Wightman functional W(f) being an extension of T, i.e. W(f) = T(f) for $f \in M$?

That is equivalent to the question under which conditions there is a hyperplane $E_w$ satisfying (13) and containing L and ker T. The answer to this question is given by the following

Theorem 1 [9]

There exists an extension of T to a Wightman functional W if and only if one of the following equivalent conditions is satisfied:

$$\text{i)} \; \mathfrak{A}_0 \neq \overline{K + L + \ker T} ,$$

$$\text{ii)} \; -1 \notin \overline{K + L + \ker T} , \qquad (14)$$

where the closure in $\overline{K+L+\ker T}$ is taken with respect to the topology $\tau$ (5).

In what follows we shall demonstrate how one can prove the existence of quantum fields with certain properties, but we cannot yet exclude the trivial fields.

### 3. Existence of Wightman Type Functionals

The next lemmas are important for the method proving the existence of certain Wightman functionals which we shall demonstrate in what follows.

Lemma 2

i) If $f_n \in I$ then

$$\int f_n(x_1, \ldots, x_n)\, dx_1 \ldots dx_n = 0 \quad \text{or equivalent}$$

$$\tilde{f}_n(0,\ldots,0) = 0. \qquad (15)$$

ii) If $f_n \in I' = \bar{\mathfrak{L}}(I_{\ell oc}, I_{inv})$ then

$$\int f_n(x, \ldots, x)\, dx = 0 \qquad (16)$$

Proof:

i) For $f \in I_{\ell oc}$ or $I_{sp}$ or $I_{inv}$ it follows $\tilde{f}_n(0,\ldots,0) = 0$. For instance, if $f \in I_{\ell oc}$ has the form

$$f = f_n(x_1, \ldots, x_n) = g_n(\ldots x_j, x_{j+1} \ldots) - g_n(\ldots x_{j+1}, x_j \ldots)$$

then it is $\tilde{f}_n(0,\ldots 0) = \frac{1}{(2\pi)^{2n}} \int f_n(x_1, \ldots, x_n)\, dx_1 \ldots dx_n = 0$.

Analogous for $f \in I_{inv}$. If $f \in I_{sp}$, $\tilde{f}(0,\ldots,0) = 0$ holds by definition. Since $f \to \tilde{f}(0,\ldots,0)$ is a linear continuous functional on $\mathfrak{A}$, $\tilde{f}_n(0,\ldots,0) = 0$ holds for any $f \in I = \bar{\mathfrak{L}}(I_{\ell oc}, I_{sp}, I_{inv})$.

ii) If

$$f = f_n(x_1, \ldots, x_n) = g_n(\ldots x_j, x_{j+1} \ldots) - g_n(\ldots x_{j+1}, x_j \ldots) \in I_{\ell oc}$$

then $\int f_n(x, \ldots, x)\, dx = 0$

and analogous for $f = f_n \in I_{inv}$.

In consequence of the continuity and linearity of the functional $f \to \int f_n(x, \ldots, x)\, dx$

we have proved (16) for any $f \in I' = \bar{\mathfrak{L}}(I_{\ell oc}, I_{inv})$.

Let $\mathcal{P}$ be the algebra of all real polynomials $p = \sum_{\nu=0}^{\infty} a_\nu t^\nu$, $a_\nu \in R$ in one real variable, equipped with the strongest locally convex topology $\tau_{\mathcal{P}}$. The set $K_{\mathcal{P}}$ of all positive polynomials, $p(t) \geq 0$ for all real t, is a cone in $\mathcal{P}$. With this cone $\mathcal{P}$ becomes a semiordered space.

Lemma 3:

Both the linear mappings of $\mathfrak{A}_0$ into $\mathcal{P}$ defined by

$$f = \{ f_0, f_1(x_1), f_2(x_1,x_2), \dots \} \in \mathfrak{A}_0$$

$$f \to \hat{f} = \hat{f}(t) = \sum_{n\geq 0} \frac{t^n}{(2\pi)^{2n}} \int f_n(x_1,\dots,x_n)\,dx_1 \dots dx_n$$

$$= \sum_{n\geq 0} \tilde{f}_n(0,\dots,0)\,t^n \tag{17}$$

and

$$f \to \check{f} = \check{f}(t) = \sum_{n\geq 0} t^n \int f_n(x,\dots,x)\,dx \tag{18}$$

are continuous and positive, i.e. if $f \in K$ then $\hat{f}, \check{f} \in K_{\mathcal{P}}$ .

**Proof:**

The mappings $f_n \to \hat{f}_n$ and $f_n \to \check{f}_n$ are continuous on the subspace $\mathcal{S}_{n0} = \mathfrak{A}_0 \cap \mathcal{S}_n$ of the homogeneous elements of degree n. Since $\mathfrak{A}_0[\tau]$ is the topological direct sum of the $\mathcal{S}_{n0}$, the continuity of these mappings on $\mathfrak{A}_0$ is proved. Further, since the cones K and $K_{\mathcal{P}}$ are closed, the positivity of the mappings (17) and (18) is proved, if we show $\widehat{(f^+ \cdot f)} \in K_{\mathcal{P}}$ and $\widecheck{(f^+ \cdot f)} \in K_{\mathcal{P}}$ for every $f \in \mathfrak{A}$ . Now it is $(f^+_n \cdot f)(0,\dots,0) = \bar{\tilde{f}}_n(0,\dots 0)\,\tilde{f}_n(0,\dots 0)$

and therefore

$$\widehat{f^+ \cdot f} = \sum_{n\geq 0} t^n \sum_{k+\ell=n} \bar{\tilde{f}}_k(0,\dots,0)\,\tilde{f}_\ell(0,\dots,0) = \Big| \sum_m t^m \tilde{f}_m(0,\dots,0) \Big|^2 \geq 0 \tag{19}$$

for all t, i.e. $\widehat{f^+ \cdot f} \in K$.

For $\widecheck{(f^+ \cdot f)}$ we can conclude on the following way

$$\widecheck{(f^+ \cdot f)} = \sum_{n\geq 0} t^n \sum_{k+\ell=n} \int \bar{f}_k(x,\dots,x)\,f_\ell(x,\dots x)\,dx$$

$$= \int \sum_{n\geq 0} \sum_{k+\ell=n} t^k \bar{f}_k(x,\dots x)\,t^\ell f_\ell(x,\dots x)\,dx$$



$$= \int \Big| \sum_{m\geq 0} t^m \bar{f}_m(x,\dots,x) \Big|^2 dx \geq 0 \tag{20}$$

for all t.

Now we can state and prove

**Theorem 2**

Let

$M = \{ g = \lambda 1 \oplus \mu \mathring{f};\ \lambda, \mu \in R \}$, $\mathring{f} = (0,0,f_2,0,\dots) \in K$

with $\tilde{f}_2(0,0) \geqslant 0$ and T the linear functional on M defined by $T(\mathring{f}) = \beta > 0$, $T(1) = 1$.

Then there exists a Wightman functional W(g) (local, convex, invariant) which is an extension of T(g).

**Proof:**

In consequence of Theorem 1 we have only to prove

$$-1 \notin \overline{K + L + \ker T}\,.$$

Now $\ker T = \{ \lambda(-\beta 1 \oplus f_2), \lambda \in R \}$.

If $-1 \in \overline{K+L+\ker T}$, then it would exist a sequence

$$k^\alpha + \ell^\alpha + v^\alpha \in K + L + \ker T,\ k^\alpha \in K,\ \ell^\alpha \in L,$$

$$v^\alpha = \lambda^\alpha(-\beta 1 \oplus f_2) \in \ker T \quad \text{with } k^\alpha + \ell^\alpha + v^\alpha \longrightarrow -1\,.$$

$\{\alpha\}$ is a directed set of indices. By applying the mapping (17) we obtain

$$\hat{k}^\alpha + \hat{\ell}^\alpha + \hat{v}^\alpha \longrightarrow -1 \tag{21}$$

in $\mathcal{P}$. From Lemma 2 (15) we get $\hat{\ell}^\alpha = 0$ and therefore it follows

$$\hat{k}^\alpha(t) + \lambda^\alpha \tilde{f}_2(0)\,t^2 - \lambda^\alpha \beta \longrightarrow -1 \tag{22}$$

with respect to the topology $\tau_{\mathcal{P}}$ of $\mathcal{P}$.

In consequence of Lemma 3 the zero-component of

$\hat{k}^{\alpha}$ is positive and therefore $\lambda^{\alpha} \geq 0$ for $\alpha \geq \alpha_0$.
Further we take a $t_0$ such that $\tilde{f}_2(0,0)\, t_0^2 - \beta \geq 0$.
Then $\hat{k}^{\alpha}(t\ ) + \lambda^{\alpha} (\tilde{f}_2(0,0)\, t_0^2 - \beta) \geq 0$ for $\alpha \geq \alpha_0$.
This is a contradiction to (22), since from the convergence with respect to $\tau_{\varphi}$ it follows the convergence for any $t$.
Thus the Theorem is completely proved.

Theorem 2 states the existence of a Wightman functional and consequently also of a field. Of course this Theorem says nothing about the existence of nontrivial fields, since we can find a free field $\phi_0(x)$ the Wightman functional $W_0$ of which is an extension of $T$, i.e. $W_0(f) = T(f)$.
The proof of Theorem 2 is only the simplest example to demonstrate, how one can prove the existence of fields with the help of Theorem 1.
A more general result we will obtain with the next Theorem. We start with a test function $g_4(x_1, x_2, x_3, x_4) \in \mathfrak{A}_0$ of the form

$$g_4(x_1, x_2, x_3, x_4) = h(x_1)\, u(x_2)\, u(x_3)\, h(x_4) \qquad (23)$$

with the properties

$$\tilde{h}(p) = \tilde{h}(-p) \geq 0, \quad \tilde{u}(p) = \tilde{u}(-p) \geq 0 \qquad (24)$$

Further we take $h$ and $u$ in such a way that supp $\tilde{h}$ and supp $\tilde{u}$ are compact and

$$\text{supp}\, \tilde{h} \subset \{ p; p^2 > 0 \}, \quad \text{supp}\, \tilde{u} \subset \{ p; p^2 < 0 \} \qquad (25)$$

We can choose $g_4$ in such a way that

$$\int g_4(x,x,x,x)\, dx = (2\pi)^2 \int \delta(p_1 + p_2 + p_3 + p_4)\, \tilde{h}(p_1)\, \tilde{u}(p_2) \times$$

$$\times\, \tilde{u}(p_3)\, \tilde{h}(p_4)\, dp_1 \ldots dp_4 > 0, \qquad (26)$$

since for the point $(p_1, p_2, p_3, p_4)$ in Fig. 2 the integrand on the right-hand side is positive.



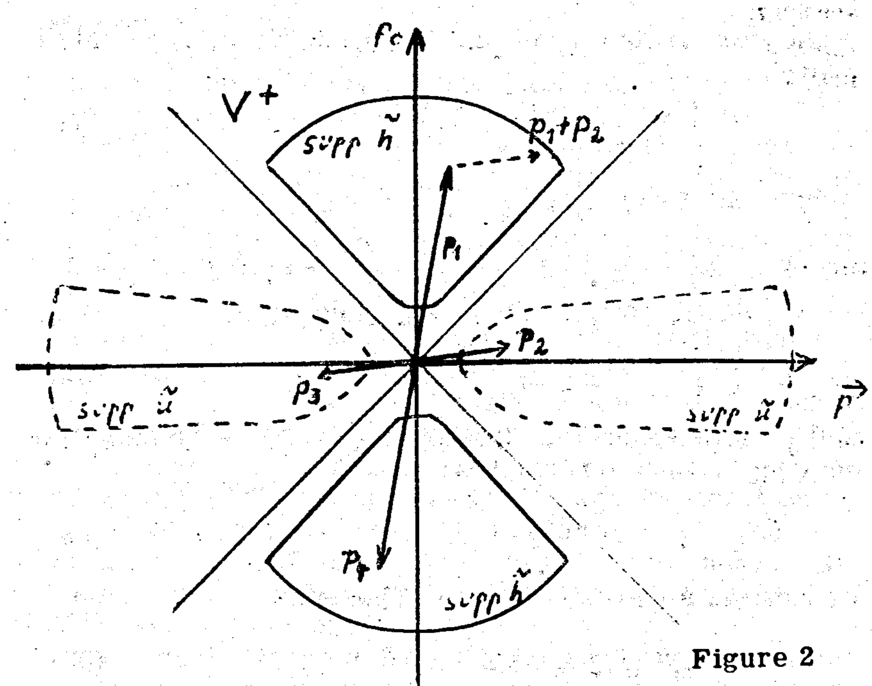


Figure 2

**Theorem 3**

Let $M = \{ f = \lambda 1 \oplus \mu g_4 ; \lambda, \mu \in R \}$ be the linear subspace of $\mathfrak{A}_0$ generated by $1$ and $g_4$ and $T(f) = \lambda + \mu\beta$, $T(g_4) = \beta > 0$ a linear functional on $M$.
Then there exists positive functional $W(f)$ on $\mathfrak{A}_0$, which is a continuation of $T$ and satisfies $W(f) = 0$ for

$$f \in L', \quad I' = L' + iL' = \mathfrak{L}(I_{loc}, I_{inv}).$$

$W(f)$ is then a Wightman-like functional. It is local and invariant. By $W$ we can construct a field $\phi(x)$ which satisfied all assumptions of the Wightman axioms except the spectrality. The field $\phi(x)$ obtained on this way is different from the generalized free field, since

$W(g_4) = \langle 0 | \phi(h)\phi(u)\phi(u)\phi(h) | 0 \rangle > 0,$

but for a generalized free field $\phi_{gen}$ it holds $\phi_{gen}(u) = 0$.

**Proof:**
Analogous to the proof of Theorem 2 we have only to prove

$$-1 \notin \overline{K + L' + \ker T}, \tag{27}$$

where $\ker T = \{\lambda(-\beta 1 \oplus g_4); \lambda \in R\}$.

Let $k^a + \ell^a + v^a \in K + L' + \ker T$, $k^a \in K$, $\ell^a \in L'$ and $v^a = \lambda^a(-\beta 1 \oplus g_4) \in \ker T$ be a sequence with

$$k^a + \ell^a + v^a \longrightarrow -1 \tag{28}$$

in $\mathfrak{A}_0$ with respect to the topology $\tau$. Now we apply the mapping $f \longrightarrow \check{f}$ (18) and get

$$\check{k}^a + \check{\ell}^a + \check{v}^a \longrightarrow -1 \tag{29}$$

By Lemma 2 ii) it is $\check{\ell}^a = 0$. Therefore

$$\check{k}^a(t) + \lambda^a (c t^4 - \beta) \longrightarrow -1 . \tag{30}$$

with $c = \int g_4(x,x,x,x)dx > 0$. By the same conclusions as in the end of the proof of Theorem 2 (30) leads to a contradiction. Thus (28) is impossible and the Theorem is completely proved.

Since for the points $p_1, p_2, p_3, p_4$ of Fig. 2 $\tilde{g}_4(p_1,p_2,p_3,p_4) > 0$ and $p_1, p_1 + p_2, p_1 + p_2 + p_3 \in V^+$, $p_1 + p_2 + p_3 + p_4 = 0$, $g_4$ is not an element of $I_{sp}$. Therefore one can hope with methods analogous to those demonstrated before also to prove the existence of an extension $W(f)$ of $T(f)$, which satisfies $W(f) = 0$ for $f \in I_{sp}$. Then $W(f)$ would be a full Wightman functional of a field different from the generalized free fields.



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