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THE PION FORM FACTOR WITH GENERALIZED ρ-DOMINANCE

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1. Introduction

It is well known [1] that the pion form factor $F_\pi(t)$ in the vicinity of $t \approx m_\rho^2$ (m_ρ is the mass of the ρ-meson) is well described by a simple Breit-Wigner resonance form or by a more refined Gounaris-Sakurai [2] formula. However, for $|t| \geq 1\,(\mathrm{GeV})^2$ the predictions of these formulas differ from the existing experimental data [3]. To resolve the discrepancy the existence of higher vector mesons has been proposed [4] and more complicated models [3, 5] have been developed where by means of analyticity and unitarity the effects of the energy dependence of the width and strong inelasticity on ρ-meson propagator were investigated.

In this paper we would like to show that it is possible to explain all existing experimental data covering the interval $-2(\mathrm{GeV})^2 \leq t \leq 4.4(\mathrm{GeV})^2$ (except for the latest Frascati data [6] up to $9(\mathrm{GeV})^2$) using a simple formula obtained by the dispersion relation method.

The question of obtaining a behavoiur of $F_\pi(t)$ by means of the dispersion relation is about 15 years old. First attempts in this direction were made in connection with the investigations of nucleon structure [7], where $F_\pi(t)$ is explicitly appearing in the imaginary part of the isovector part of the nucleon form factors. Some later developments can be found in [8]. But at that time physicists could make only qualitative theoretical estimates for $F_\pi(t)$ due to the lack of experimental data. At present we have at our disposal around 70 experimental values of it [1,6,9] measured for different time-like and space-like momentum transfer t. Therefore, as R.P.Feynman mentioned in his lecture notes [10] some time ago, it should be interesting to reinvestigate the pion form factor by means of the dispersion relation method and to compare it quantitatively with the existing experimental data.

2. Dispersion Relation and Basic Assumptions

The analytic properties of $F_\pi(t)$ in the complex t-plane (see fig. 1) can be proved by means of the standard methods /11/. The discontinuity on the cut from $t=4\mu^2$ (μ is the mass of the pion) to ∞ is determined by the unitarity condition.

Further going out from the assumption

$$\lim_{|t|\to\infty} |F_\pi(t)| = \text{const}\ ^* \tag{1}$$

and using the Cauchy's theorem one gets the once-subtracted dispersion relation

$$F_\pi(t) = 1 + \frac{t}{\pi}\int_4^\infty \frac{\operatorname{Im} F_\pi(t')}{t'(t'-t)}\,dt', \tag{2}$$

where the normalization condition $F_\pi(0)=1$ is automatically taken into account and the units $\hbar=c=\mu=1$ are used.

The unitarity at the least massive state approximation gives /7/

$$\operatorname{Im} F_\pi(t) = F_\pi(t)\, e^{-i\delta_1^1(t)} . \sin\delta_1^1(t), \tag{3}$$

where $\delta_1^1(t)$ is $J=1$ $I=1$ $\pi\pi$ scattering phase shift. From this relation (due to the reality of the left-hand side) it follows that the phase of $F_\pi(t)$ is just identical to $\delta_1^1(t)$. Strictly speaking this is true only in the region $4 \leq t \leq 16$, where the approximation (3) is valid exactly.

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* It appears /12/ that the pion form factor behaves asymptotically as $\sim 1/t$. However we suppose that the effect of the difference between this behaviour and our assumption (1) will be negligible in a finite energy region.

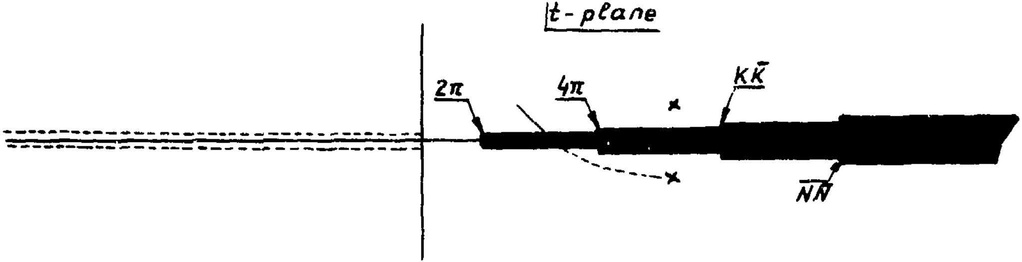

Fig. 1. The analytic properties of the pion form factor. Rho-meson poles on the second sheet are denoted by (x) .

Now substituting $\mathrm{Im}\, F_\pi(t)$ from the relation (3) into (2) one gets the so-called Muskhelishvili-Omnes integral equation

$$F_\pi(t) = 1 + \frac{t}{\pi} \int_4^\infty \frac{F_\pi(t') e^{-i\delta_1^1(t')} \sin\delta_1^1(t')}{t'(t'-t)} dt' \tag{4}$$

the general solution [13,14] of which is

$$F_\pi(t) = P_n(t) \exp\{ \frac{t}{\pi} \int_4^\infty \frac{\delta_1^1(t')}{t'(t'-t)} dt' \}, \tag{5}$$

where $P_n(t)$ is an arbitrary polynomial. The only restrictions which we may impose on the polynomial $P_n(t)$ are: $P_n(0)=1$ and that the degree of it must not be higher than $\delta(\infty)/\pi$.

To obtain the explicit formula for the pion form factor from the general expression (5) we propose to represent the energy dependence of the phase shift in the form

$$\mathrm{tg}\, \delta_1^1(t) = \frac{a q^3}{(1+q^2)(q_\rho^2 - q^2)}, \tag{6}$$

where q (the c.m. momentum) and t are connected through relation

$$t = 4(q^2 + 1). \tag{7}$$

Here $q_\rho = \frac{1}{2}\sqrt{m_\rho^2 - 4}$ and a is a parameter which can be expressed by means of the mass and the width Γ_ρ of the ρ-meson. To find this connection we require that

$$\lim_{t \to m_\rho^2} \frac{\frac{m_\rho \Gamma_\rho}{m_\rho^2 - t}}{\mathrm{tg}\, \delta_1^1(t)} = 1 \tag{8}$$

which is not in a contradiction with the first statement in the introduction. From the relation (8) we obtain

$$a = \frac{1}{2} \Gamma_\rho (1 + \frac{1}{q_\rho^2})^{3/2}. \tag{9}$$

In the next section we shall see that the factor $(1+q^2)$ in the denominator of Eq. (6) is generating a new pole of the pion form factor below the thershold on the real axis of the second Riemann sheet. A sitisfactory physical interpretation will be found for this pole.

3. Explicit Form of Pion Form Factor

To find the explicit form of the pion form factor we have to calculate the integral in Eq. (5). For the sake of simplicity of calculations we will work with the variable q using Eq. (7).

Further one can see immediately that the relation (6) can be rewritten in the following equivalent form

$$\delta_1^1(t) = \frac{1}{2i} \ln \frac{(1+q'^2)(q_\rho^2 - q'^2) + iaq'^3}{(1+q'^2)(q_\rho^2 - q'^2) - iaq'^3}. \tag{10}$$

Then the integral in Eq. (5) (taking into account the fact that the intergand in even) takes the form

$$I = \frac{(1+q^2)}{2\pi i} \int_{-\infty}^{\infty} \phi(q', q)\, dq', \tag{11}$$

where

$$\phi(q',q) = \frac{q' \ln \dfrac{(1+q'^2)(q_\rho^2 - q'^2) + iaq'^3}{(1+q'^2)(q_\rho^2 - q'^2) - iaq'^3}}{(1+q'^2)(q'^2 - q^2)}. \tag{12}$$

This form of the integral is suitable in the calculation by means of the theory of residues. It can be used in every case when the energy dependence of the phase shift is supposed to belong to the class of the general form [15]

$$\operatorname{tg} \delta_\ell(t) = (t-4)^{\frac{2\ell+1}{2}} R(t), \tag{13}$$

where $R(t)$ is a rational function. In our case this method gives

$$
I = \ln \frac{(q-q_1)}{(q+q_2)(q+q_3)(q+q_4)} \; \frac{(i+q_2)(i+q_3)(i+q_4)}{(i-q_1)}\,, \qquad (14)
$$

where q_i ($i = 1\ldots4$) are the positions of the branch points of the integrand (12) in q-plane (see fig. 2). Their explicit forms (expressed through the ρ-meson parameters) and calculation of Eq. (14) are given in Appendix.

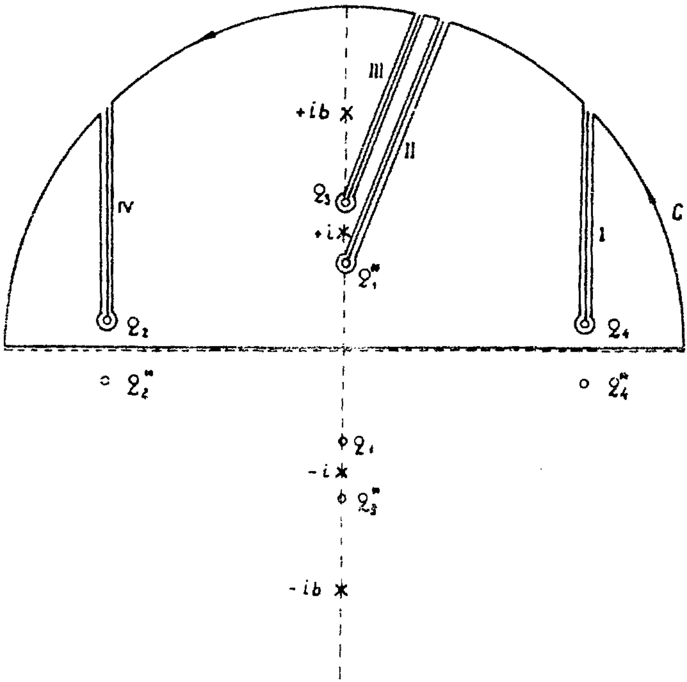

Fig. 2. The poles (x) and the branch points (o) of the integrand $\phi(q',q)$ with the contour of integration C.

The expression (14) should now be inserted into (5) and one finds then the explicit form of the pion form factor

$$F_\pi(t) = P_n(t)\,\frac{(q-q_1)}{(q+q_2)(q+q_3)(q+q_4)}\;\frac{(i+q_2)(i+q_3)(i+q_4)}{(i-q_1)} \qquad (15)$$

with a freedom of the choice of $P_n(t)$ discussed in section 2 and with $q = +\sqrt{\frac{t-4}{4}}$.

As soon as q_i ($i = 1 \ldots 4$) in (15) are expressed through ρ-meson parameters in a reasonable approximation, Eq. (15) takes the following simplified form

$$F_\pi(t) = P_n(t)\left[y_1^* + \frac{a}{q_\rho^2 y_1^*}\right]\frac{1}{\{1-(\alpha^2+\beta^2)q^2 - 2i\beta q\}}\;\frac{1-iy_1^* q}{1+iy_3^* q},$$

where

$$\alpha = \frac{1}{q_\rho}\left[1 - \frac{q_\rho^2(q_\rho^2-3)}{\{2(q_\rho^2+1)\}^3}a^2 + O(a^4)\right];$$

$$\beta = \frac{a}{2(q_\rho^2+1)}\left[1 + \frac{16\,q_\rho^2(q_\rho^2-1)}{\{2(q_\rho^2+1)\}^4}a^2 + O(a^4)\right]$$

$$y_3^* = -1 + \beta + \frac{3q_\rho^2-1}{\{2(q_\rho^2+1)\}^3}a^2 + O(a^4);$$

$$y_1^* = 1 + \beta - \frac{3q_\rho^2-1}{\{2(q_\rho^2+1)\}^3}a^2 + O(a^4); \qquad (15a)$$

The singular structure of $F_\pi(t)$ given by (15) is shown in fig. 3. It is straightforward to see that the $(-q_2)$ and $(-q_4)$ poles correspond to the complex conjugate pair of the ρ-meson poles (see fig. 1). The additional pole for $q=-q_3$ (which is also appearing on

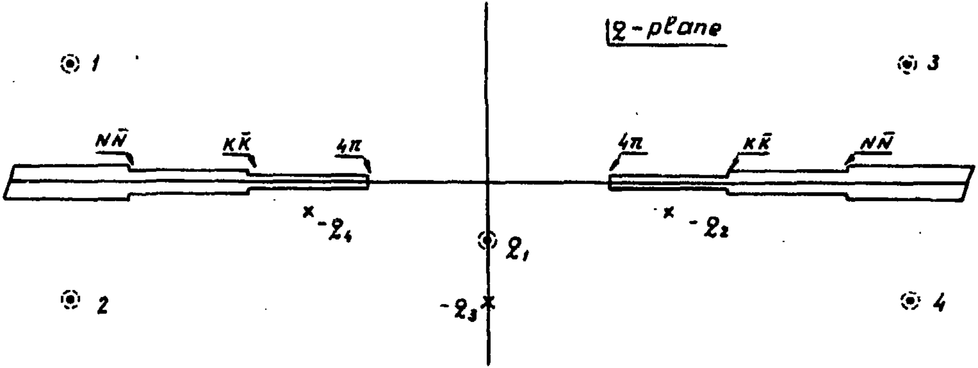

Fig. 3. The singular structure of our finite form of the pion form factor. The cuts do not follow from our formula and they are shown only for completeness. The zeros are denoted by ⊙ and the cross (x) is used for the poles.

the II sheet of t) can be explained in the following way. Using the unitarity condition of the pion form factor in the two pion approximation one can write for it the expression [14]

$$F_\pi^{II}(t) = \frac{F_\pi^{I}(t)}{1+2i\,M_{\pi\pi}^{I}(t)}, \tag{16}$$

where $M_{\pi\pi}^{I}(t)$ is the $\pi\pi$ scattering partial-wave amplitude in the $J=I=1$ state. From Eq. (16) it follows that $F_\pi^{II}(t)$, i.e. $F_\pi(t)$ on its second sheet, has all singularities of $F_\pi^{I}(t)$ and in addition all branch points of $M_{\pi\pi}^{I}(t)$ as well. The singularities of $M_{\pi\pi}^{I}(t)$ are confined to branch cuts along the real axis in the range $t \lesssim 0$ and $t \geq 4$. So $F_\pi^{II}(t)$ has an additional left-hand cut (marked in fig. 1 by the dashed lines) the contribution of which to $F_\pi(t)$ is approximated by the $(-q_3)$-pole in our considerations.

The form factor has also the zero at $q=q_1$ which in the t-variable is appearing on the second Riemann sheet. Zeros generated by the polynomial $P_n(t)$ for concrete cases will be discussed in the next section.

4. Results of the Fit and Pion's Charge Radius

Before the comparison of our formula (15) with the experimental data we have to choose the concrete form for the polynomial $P_n(t)$. Taking into account the restrictions described in section 2 and the fact that the $\delta_1^1(\infty)=\pi$ (this follows from Eq. (6)) one gets

$$P_1(t) = 1 + At\ , \tag{17}$$

where A is unknown constant which can be found only carrying out the fit to the experimental data.

Further we shall leave also the mass and the width of the ρ -meson as free parameters.

The result of the minimization [16] of our formula to the 60 experimental points with the $\chi^2 \approx 88$ and 3 free parameters with the values

$$m_\rho = 778 \pm 4 \text{ MeV}$$
$$\Gamma_\rho = 152 \pm 4 \text{ MeV}$$
$$A = 0.0027 \pm 0.0003 \frac{1}{[\mu^2]} \qquad (18)$$

is represented by dashed lines in figs. 5 and 6 (the shape of the dashed line in the resonant region would be nearly the same as that of the full line in fig. 4).

In this case the $P_1(t)$ generates one zero on the physical sheet and one on the second Riemann sheet in the same place

$$t = -\frac{1}{A} \approx -7.39 (\text{GeV})^2 . \qquad (19)$$

Ignoring the slightly higher value of m_ρ in comparison with that given in Review of Particle Properties /17/ we see from figs. 5 and 6 that the prediction of our formula differs from the experimental data for higher values of $|t|$. Concretely the formula (15) with $P_1(t)$ and parameters (18) give smaller values of $F_\pi(t)$ as they were measured experimentally.

One gets excellent description of the same experimental points using in Eq. (15) the polynomial

$$P_2(t) = 1 + B \cdot t + C \cdot t^2 \qquad (20)$$

with two unknown constants. The best fit with $\chi^2 \approx 46$ and 4 free parameters with the values

$$m_\rho = 770 \pm 4 \text{ MeV}$$
$$\Gamma_\rho = 157 \pm 5 \text{ MeV} \qquad (21)$$
$$B = 0.0038 \pm 0.0003 \frac{1}{[\mu^2]}$$
$$C = 0.000028 \pm 0.000004 \frac{1}{[\mu^4]}$$

is represented in figs. 4,5 and 6 by the full lines.

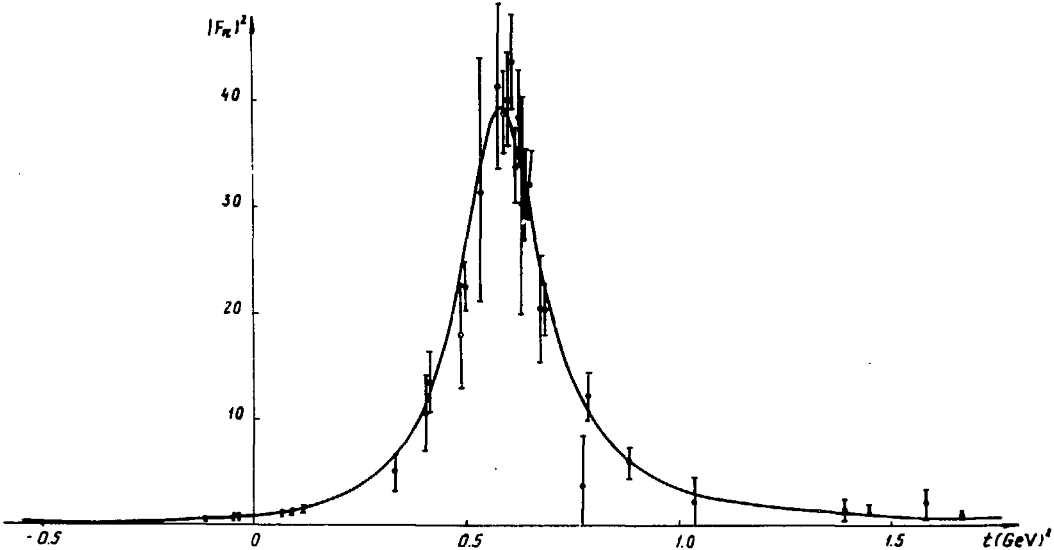

Fig. 4. Theoretical predictions of $|F_\pi(t)|^2$ by means of our formula with the values of parameters (21).

The polynomial $P_2(t)$ generates two pairs of the two complex conjugate zeros on the I and II sheets for the same values of t

$$t_1 \simeq -1.32 + i\,3.43\ (GeV)^2,$$

$$t_2 = t_1^*, \tag{22}$$

where the asterisk means complex conjugation.

The most probable interpretation of these zeros (see fig. 3) seems to be the following. As it is possible to see from Eq. (3) we have neglected all inelastic contributions or, in other words, we did not take into account all cuts in fig. 3 which (as it has been seen in confrontation of (15) and $P_1(t)$ with experimental data)can no more be neglected for higher values of $|t|$. It appears that the simplest way how to incorporate the contributions from these cuts into the pion form factor is to generate four zeros shown in fig. 3 (denoted by 1,...4) or in other words to consider Eq. (15) with the polynomial $P_2(t)$ depending on the two unknown parameters.

It should be interesting to look for the departure of $P_2(t)$ for $t = m_\rho^2$ from the value 1. The simple calculation gives $P_2(m_\rho^2) \simeq 1.14$. It means that 88% of $|F_\pi|^2_{max}$ is given by ρ -meson and for 12% the polynomial $P_2(t)$ is responsible.

The comparison of our formula with the nine values of the DESY-data [18] together with the 60 aforementioned experimental points has also been carried out. The minimization procedure gives $\chi^2 \simeq 98$ (compare it with the previous value $\chi^2 \simeq 46$) with 4 free parameters as given without principal changes by (21). So we conclude that the DESY-data (see fig. 5) are inconsistent with other experimental data.

The pion's charge radius squared is given by the following expression

$$\langle r^2 \rangle = 6\, \frac{dF_\pi(t)}{dt}\Big|_{t=0} . \tag{23}$$

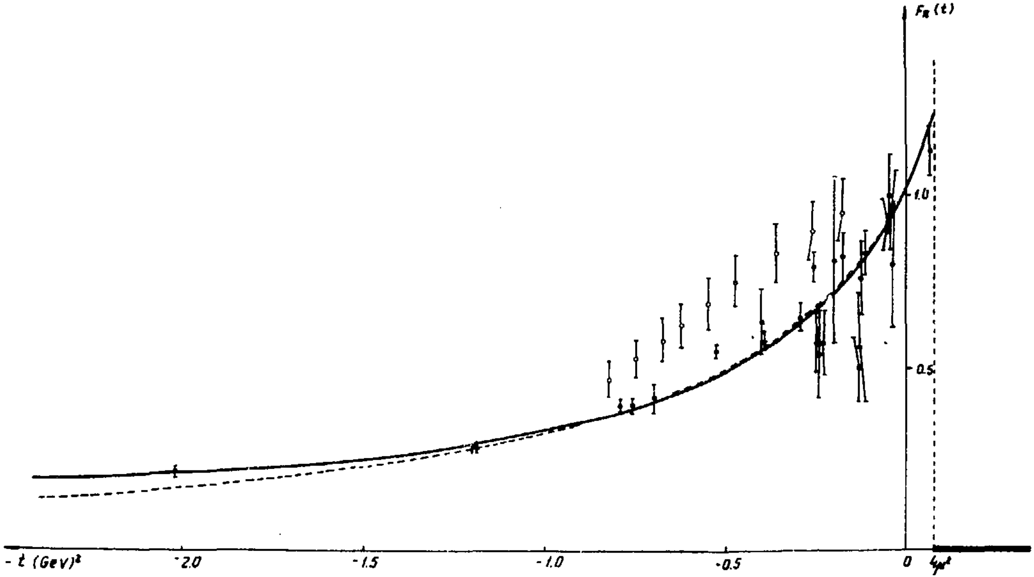

Fig. 5. The comparison of our formulas with experimental data in the space-like region. denote the DESY-data /17/.

Taking into account the explicit form of the pion form factor with the values of the parameters given by (21) one gets from (23)

$$\langle r^2 \rangle^{1/2} = 0.70 \pm 0.01\,\mathrm{F} \tag{24}$$

which is slightly higher than the ρ -dominance value [19]

$$\langle r^2 \rangle^{1/2}_{\rho\ \mathrm{dom}} \approx 0.62\,\mathrm{F} \tag{25}$$

but still much smaller than ~ 0.95F suggested by the Serpukhov-UCLA measurement /20/ of πe scattering.

5. Conclusions

In this paper we have tried to reinvestigate the pion form factor by the dispersion relation method. Starting with the assumption $\delta_1^1(\infty) = \pi$ and generating the two complex conjugate zeros on I and II Riemann sheets through the polynomial $P_2(t)$ we found the explicit form of the pion form factor which is in an excellent agreement (more than 90% confidence level) with experimental data in the region $-2.02\,(\mathrm{GeV})^2 \le t \le 4.4(\mathrm{GeV})^2$.

The latest Frascati data /6/ cannot be explained by our formula because due to the $P_2(t)$ the pion form factor for $t > 4.5(\mathrm{GeV})^2$ is increasing and therefore we consider our parametrization invalid for the aforementioned values of t .

Appendix

In what follows we outline the calculation of the integral (11) in section 3

$$I = \frac{(1+q^2)}{2\pi i} \int_{-\infty}^{\infty} \frac{q' \ln \dfrac{(1+q'^2)(q_\rho^2 - q'^2) + iaq'^3}{(1+q'^2)(q_\rho^2 - q'^2) - iaq'^3}}{(1+q'^2)(q'^2 - q^2)}\, dq' \tag{I}$$

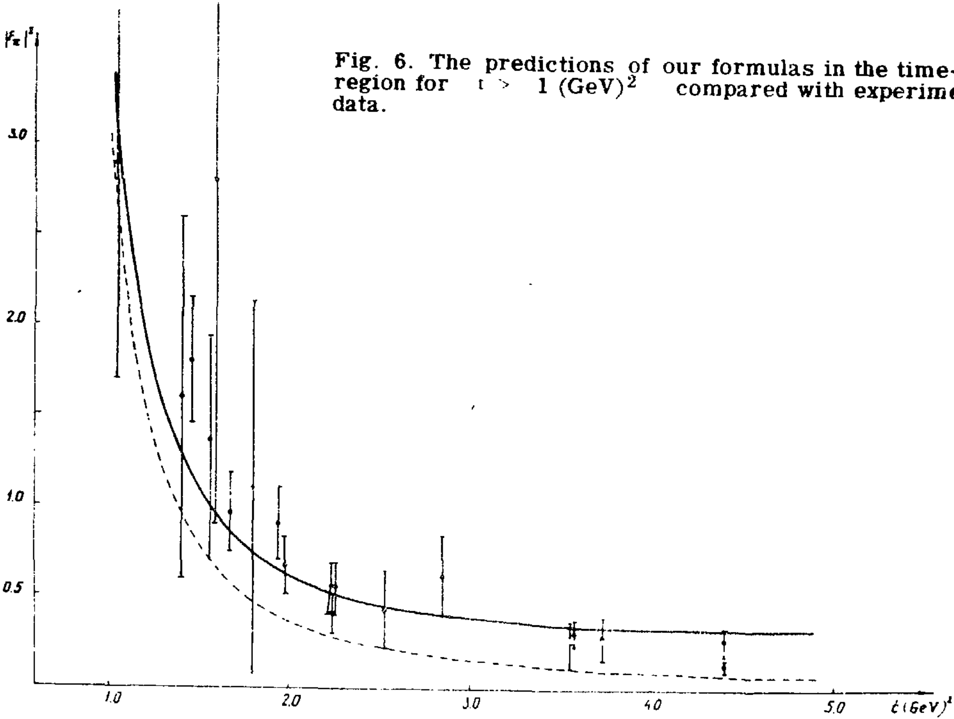

Fig. 6. The predictions of our formulas in the time-like region for $t > 1\ (\mathrm{GeV})^2$ compared with experimental data.

Due to the analyticity of $F_\pi(t)$ it has a sence for both negative and positive values of q^2. Further in the calculation of the integral (I) we shall consider the case $q^2<0$, i.e.

$$q = i\sqrt{\frac{4-t}{t}} \equiv i\,b\,. \qquad \text{(II)}$$

The result for the case $q^2>0$ can be obtained from the latter by means of the analytic continuation.

One can see immediately the singularities of the integrand in (I). It has the poles for $q' = \pm i$, $q' = \pm ib$ and to find the branch points we have to solve two complex conjugate algebraic equations of the fourth order

$$(1+q'^2)(q_\rho^2 - q'^2) \pm i a q'^3 = 0. \qquad \text{(III)}$$

The solutions of the equation with plus sign (before that the transformation $q = i/y$ is used to obtain the equation with real coefficients) can be written in the form [21]

$$q_1 = \frac{i}{-\sqrt{z_1}-\sqrt{z_2}-\sqrt{z_3}}\,; \quad q_3 = \frac{i}{-\sqrt{z_1}+\sqrt{z_2}+\sqrt{z_3}}\,;$$

$$\text{(IV)}$$

$$q_2 = \frac{i}{\sqrt{z_1}-\sqrt{z_2}+\sqrt{z_3}}\,; \quad q_4 = \frac{i}{\sqrt{z_1}+\sqrt{z_2}-\sqrt{z_3}}\,;$$

where z_ν $(\nu=1\ldots3)$ are solutions of the cubic equation

$$z^3 - \frac{(q_\rho^2-1)}{2q_\rho^2}z^2 + \frac{(q_\rho^2+1)^2}{16\,q_\rho^4}z - \frac{a^2}{64\,q_\rho^4} = 0 \qquad \text{(V)}$$

and the signs of $\sqrt{z_\nu}$ in (IV) are taken to fulfill the following condition

$$\sqrt{z_1}\sqrt{z_2}\sqrt{z_3} = -\frac{a}{8\,q_\rho^2}\,. \qquad \text{(VI)}$$

Numerically, using the values of the ρ -meson parameters given by (21), we obtain

$$
\begin{aligned}
q_1 &= -i0.95837 \ [\mu] \\
q_2 &= -2.53669 + i\,0.30054 \ [\mu] \\
q_3 &= +i1.05117 \ [\mu] \\
q_4 &= 2.53669 + i0.30054 . \ [\mu]
\end{aligned}
\qquad \text{(VII)}
$$

The solutions q_i^* of the equation (III) with the minus sign can be obtained from (VII) simply by complex conjugation. Then it is straightforward to see the relations

$$
\begin{aligned}
q_1^* &= -q_1 \\
q_2^* &= -q_4 \\
q_3^* &= -q_3 \\
q_4^* &= -q_2 .
\end{aligned}
\qquad \text{(VIII)}
$$

The singularities of the integrand in (1) with the contour of integration C in the upper half plane are shown in fig. 2. Taking into account the property of the integrand

$$
\lim_{|q'| \to \infty} \phi (q', q) = 0 \qquad \text{(IX)}
$$

and using the relations (VIII) one gets (14) in sect. 3.

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