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SEQUENCES OF PARASTATISTICS

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1. Introduction

The parastatistics, being a natural generalization of the ordinary Fermi-Dirac and Bose-Einstein statistics, are thought of as statistics whose number of particles in the symmetric state for the para-Fermi-statistics or in the antisymmetric state for the para-Bose-statistics cannot exceed a certain given number p called the parastatistics order.

The field realization of parastatistics was suggested in a classic paper of Green /1/ and was also investigated in papers of Volkov /2/ *.

Let us introduce annihilation operators a_r and hermitian conjugate operators $a^r \equiv \bar{a}_r$ for creation of a particle in a state r. Both the operators are denoted by the same symbol $a(\rho)$ whose index ρ may assume both lower and upper values. Now, the set of all the Green relations can be written in the form

$$[\, a(\rho)\, ,[\, a(\sigma)\, , a(\tau)\,]_{\epsilon}\,]_{-} = 2g(\rho\sigma)\, a(\tau) + 2\epsilon\, g(\rho\tau)\, a(\sigma)\, , \quad (1.1)$$

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* Parastatistics can also be introduced according to the axiomatic definition of the algebra of observables /3/. However, throughout the present paper we shall follow the original Green's field realization of the parastatistics.

where $[a,b]_\epsilon = ab + \epsilon ba$ and $\epsilon = +1$ for the para-Bose-statistics and $\epsilon = -1$ for the para-Fermi-statistics. The coefficients $g(\rho\sigma)$ are the following numerical values

$$g_{rs} = g^{rs} = 0\,, \quad g_r^{\ s} = -\epsilon\, g^s_{\ r} = \delta_r^s\,, \tag{1.2}$$

where δ_r^s is the Kronecker symbol.

From (1.1) it follows the relations

$$\begin{aligned}[A(\nu\mu)\,, A(\rho\lambda)]_- &= 2g(\mu\rho)\, A(\nu\lambda) + 2g(\mu\lambda)\, A(\rho\nu) + \\ &+ 2\epsilon\, g(\nu\rho)\, A(\mu\lambda) + 2\epsilon\, g(\nu\lambda)\, A(\rho\mu)\,,\end{aligned} \tag{1.3}$$

where

$$A(\nu\mu) = \epsilon\, A(\mu\nu) = [\,a(\nu)\,, a(\mu)\,]_\epsilon\,. \tag{1.4}$$

The relations (1.3) are analogous to the characteristic relations of the Lee algebras of the orthogonal ($\epsilon = -1$) and symplectic ($\epsilon = +1$) groups /4-6/.

The operator of the particle number in the state r is defined as

$$n_r = \frac{1}{2}[\,a^r, a_r\,]_\epsilon - \frac{\epsilon}{2}p\,. \tag{1.5}$$

where, as will be shown below, the constant p must coincide with the parastatistics order. Thanks to (1.1) it possesses necessary properties

$$[\,n_r\,, a_m\,]_- = -\delta_{rm}\, a_m\,, \quad [\,n_r\,, a^m\,]_- = \delta_{rm}\, a^m\,, \qquad [\,n_r\,, n_s\,] = 0\,. \tag{1.6}$$

The operator of the total particle number is obtained by summing over all the states

$$N = \sum_r n_r\,. \tag{1.7}$$

The next step must naturally be a study of the representations of the set of abstract quantities satisfying the paracommutation relations. In so doing, one usually postulates the existence of the only vacuum state /7/. Starting from this postulate and the condition of positive definiteness of the vector state norm Greenberg and Messiah have shown that an ensemble of parastatistics of all orders $p = 0,1,2,...$ really corresponds to the Green commutation relations (1.1).

It is, however, possible to discover the presence of other separable irreducible representations of the Green algebra, which correspond to the parastatistics of a given order and such in which there is a cyclic vector and the operator of the particle number can be determined. Usually such representations were rejected for the reason that in them there is a degeneration of the ground state. This was, in our opinion (see also ref. /8/), associated with an incorrect interpretation of such ground states as "vacuum" states. In reality, they are many-particle states. Below, following the authors of ref. /8/, we will refer to these states as "reservoir" states, contrary to the single vacuum state.

In the present paper we study the properties of the reservoir states and of the Green-algebra irreducible representations for the general case of para-Fermi and para-Bose-statistics of an arbitrary order generalizing the results of preceding papers /9-11/ to particular cases of para-Fermi-statistics of the second and third orders. It should be noted that in recent papers of Bracken and Green /8/ and Kraev /12/ a group classification of the Green-algebra irreducible representations has been made in extracting them from the space of the well-known Green ansatz /1/. Our consideration, which is based on the study of the properties of the reservoir state vectors being "minor" or "preceding" vectors of the irreducible representations, might be useful in a direct and consistent construction of such representations as was just the case for the mentioned particular cases /9-11/. In addition, in this paper we do not use the Green ansatz.

In section 2 we deal with the study of the characteristic features of reservoir states. We also prove a generalized version of the above-mentioned Greenberg and Messiah theorem /7/. In section 3 we establish hierarchy of the parastatistics. In Conclusion we only dwell on a possible physical interpretation of paraquantization. Our general conclusions are illustrated in Appendix 2 by the examples of the parastatistics of the first (ordinary statistics), second and third orders.

2. Irreducible Representation of the Green Algebra

As was indicated above, we reject the condition of uniqueness of the reservoir vectors and impose on them, instead of it, more weak restrictions.

i) There exists, including the single vacuum vector $|\rangle$, a complete set of linearly independent reservoir vectors $|\rangle$, $|\ell_1\rangle, |\ell_1,\ell_2\rangle$ etc. for which

$$a_r|\rangle = 0 \tag{2.1}$$

holds. Each of the indices ℓ_1, ℓ_2, etc., assumes the same values as the initial indices of the particle states r. The condition of completeness means that any vector obeying (2.1) belongs to this set.

The basis of the Green algebra (1.1) representations is obtained by the action of all possible monoms of the creation operators a^r, a^s, etc., on the reservoir vectors, We call the indices of the latter ℓ_1, ℓ_2, etc., "reservoir" while the indices of the creation operators etc., "external" indices.

ii) The particle number operator (1.5) acting on the reservoir vectors must lead to the following results

$$n_r|\rangle=0\ ,\ n_r|\ell_1,\ldots\ell_M\rangle = (\delta_r^{\ell_1}+\ldots+\delta_r^{\ell_M})\,|\ell_1,\ldots\ell_M\rangle . \tag{2.2}$$

Thus, if the unique vector $|\rangle$ describes the vacuum state then the remaining reservoir vectors describe one-, two- and more particle states.

iii) The representation space is the ordinary Hilbert space with the scalar product

$$\langle\phi|\pi\rangle = \overline{\langle\pi|\phi\rangle}\ , \tag{2.3}$$

where $|\phi\rangle$ and $|\pi\rangle$ are two arbitrary vectors which may also belong to the reservoir vectors (the line implies here complex conjugation). The vector norm is positive definite

$$\langle\phi|\phi\rangle = \|\varphi\|^2 > 0. \tag{2.4}$$

The consequence of the condition (2.2) is orthogonality of reservoir vectors with different number of indices or with different sets of the identical number of indices. The reservoir vectors differing only by permutations of identical sets of indices can be orthogonalized by forming combinations symmetrized by the Young schemes. So, we consider the reservoir vectors as orthonormalized

$$\langle\{\ell\}_M|\{\ell'\}_{M'}\rangle = \delta_{\{\ell\},\{\ell'\}}\delta_{MM'}, \tag{2.5}$$

where the symbol $\{\ell\}_M$ means a set of indices $\{\ell_1,\dots,\ell_M\}$ symmetrized according to the appropriate Young scheme.

Now let us prove two theorems being a direct generalization of the Greenberg and Messiah theorem /7/.

THEOREM I. If the conditions (2.1)-(2.3) are fulfilled, then the following equality

$$a_r a^s|\ell_1,\dots,\ell_M\rangle = p\delta^s_r|\ell_1,\dots,\ell_M\rangle + \tag{2.6}$$
$$+ 2\epsilon\,\delta_r^{\ell_1}|s,\ell_2,\dots,\ell_M\rangle + \dots + 2\epsilon\,\delta_r^{\ell_M}|\ell_1,\dots,\ell_{M-1},s\rangle$$

also holds, where p is a number independent of r, s, $\ell_1,\dots,\ell_M$.

PROOF: Let us apply to the reservoir vector $|\{\ell\}_M$ both the sides of eq. (1.1) for $a(\rho) = a_r$, $a(\sigma) = a^s$ and $a(\tau) = a_t$. Owing to (2.1) we get

$$a_r a_t a^s|\{\ell\}_M\rangle = 0. \tag{2.7}$$

Consequently, the vector $a_t a^s|\{\ell\}_M\rangle$ is reservoir and, according to condition i), may be expressed as a linear combination

$$a_t a^s|\{\ell\}_M\rangle = \sum_{\{\ell'\}_{M'}} (f^s)^{\{\ell\}_M}_{\{\ell'\}_{M'}}|\{\ell'\}_{M'}\rangle, \tag{2.8}$$

in which the unknown coefficients are of the form

$$(f_t^{\;s})^{\{\ell\}_M}_{\{\ell'\}_{M'}} = \langle \{\ell'\}_{M'} | a_t a^s | \{\ell\}_M \rangle \; . \tag{2.9}$$

It is easily seen that the latter differ from zero only for $M=M'$ (we arrive at this conclusion by inserting the total particle number operator (1.7) between the operators a_t and a^s in (2.9) and letting it act first on the right- and then on the left-hand parts, taking into account (1.6) and (2.2)).

Now we submit the same vector to the action of both sides of eq. (1.3), putting in it $a(\nu)=a^n$, $a(\mu)=a_m$, $a(\rho)=a^r$ and $a(\lambda)=a_t$. Using (2.8) and (2.5) we obtain (with the account of $M=M'$).

$$\sum_{\{\ell'\}} \left[(f_t^{\;r})^{\{\ell\}}_{\{\ell'\}} (f_m^{\;n})^{\{\ell'\}}_{\{\ell''\}} - (f_m^{\;n})^{\{\ell\}}_{\{\ell'\}} (f_t^{\;r})^{\{\ell'\}}_{\{\ell''\}} \right] = \tag{2.10}$$

$$= 2\epsilon\, \delta^r_m (f_t^{\;n})^{\{\ell\}}_{\{\ell''\}} - 2\epsilon\, \delta^n_t (f_m^{\;r})^{\{\ell\}}_{\{\ell''\}} \; .$$

Owing to (2.3) the coefficients (2.9) satisfy also the relation

$$(f_t^{\;s})^{\{\ell\}}_{\{\ell'\}} = (f_s^{\;t})^{\{\ell'\}}_{\{\ell\}} \; . \tag{2.11}$$

In the matrix form eqs. (2.10) and (2.11) look like

$$[f_t^{\;r}, f_m^{\;n}]_- = 2\epsilon\, \delta^r_m f_t^{\;n} - 2\epsilon\, \delta^n_t f_m^{\;r} \tag{2.12}$$

and

$$f_t^{\;s} = (f_s^{\;t})^+ \; , \tag{2.13}$$

where f^+ is the matrix hermitian conjugate to f.

The general solution (up to unitary transformation) of the obtained matrix algebra has the form of the sum of the unit matrix multiplied by an arbitrary number p and the matrices having a single non-zero nondiagonal element equal to 2ϵ. Omitting the nonessential here symmetrization of indices according to the Young schemes we obtain for the coefficients

$$(f_r{}^s)^{\ell_1,\dots,\ell_M}_{\ell'_1,\dots,\ell'_M} = p\,\delta_r^s\,\delta^{\ell_1}_{\ell'_1}\dots\delta^{\ell_M}_{\ell'_M} +$$

$$+\ 2\epsilon\,\delta_r^{\ell_1}\,\delta^s_{\ell'_1}\,\delta^{\ell_2}_{\ell'_2}\dots\delta^{\ell_M}_{\ell'_M} + \dots + \tag{2.14}$$

$$+\ 2\epsilon\,\delta_r^{\ell_M}\,\delta^{\ell_1}_{\ell'_1}\dots\delta^{\ell_{M-1}}_{\ell'_{M-1}}\,\delta^s_{\ell'_M}\ ,$$

from where it immediately follows (2.6). A particular case of (2.6) is the Greenberg and Messiah relation for the vacuum vector

$$a_r\, a^s > \ = p\,\delta_r^s > . \tag{2.15}$$

It follows from this relation and the condition (2.2) that the constant entering the definition for the particle number operator (1.5) must really be equal to $-\epsilon\, p/2$.

THEOREM II. Due to the condition of positive definiteness of the state vector norm (2.4) the number p entering the relation (2.6) of the preceding theorem must be non-negative whole number 0, 1, 2, etc. The number of particles being in the symmetric state for the case of parafermions and in the antisymmetric state for the case of parabosons cannot exceed this whole number p.

PROOF: We make the proof simultaneously for parafermions and parabosons by putting in brackets the expressions related to the latter case.

Let us consider a vector symmetric (antisymmetric) in all its external and reservoir indices

$$\chi_{N,M} = \sum_P \lambda_P\, a^{P\ell_1}\dots a^{P\ell_N}\,|\,P\ell_{N+1},\dots,P\ell_{N+M}> \tag{2.16}$$

The summation is carried out over all possible permutations of indices $\ell_1,\dots,\ell_{N+M}$. The indices obtained as a result of the permutation P are denoted as $P\ell_1,\dots P\ell_{N+M}$. The number λ_P is the signature of the permutation which is equal to $+1$ for parafermions and ± 1 depending on the parity of the transposition number for parabosons.

Re-application of eqs. (1.1) and (2.6) for the norm of this vector gives the following expression (see Appendix 1)

$$\| \chi_{N,M} \|^2 = (N!)^2 \prod_{j=1}^{N} (p-j+1-2M) \left(\sum_{\mu} \| \chi_{O,M}(\mu) \|^2 \right), \tag{2.17}$$

where

$$\chi_{O,M}(\mu) = \sum_{P} \lambda_P |P\ell_{\mu_1}, \ldots, P\ell_{\mu_M}> \tag{2.18}$$

is a symmetric (antisymmetric) reservoir vector. The sum over μ means the sum over all possible samples of the indices from the complete set of indices $\ell_1, \ldots, \ell_{N+M}$ according to M.

For a sufficiently large number of external indices N the norm (2.17) may turn out to be negative. In order that this might not happen the number p should be a non-negative integer: 0, 1, 2, atc. Then for $N \geq p+1-2M$ the norm (2.17) vanishes, which in the Hilbert space means vanishing of the vector (2.16) itself.

Now we fix one of the admissible values of the number p. Then all the vectors, whose number of external and reservoir indices undergoing symmetrization (antisymmetrization) satisfies the condition

$$2M + N \geq p + 1, \tag{2.19}$$

must vanish. On the contrary, only the vectors for which the requirement

$$2M + N \leq p \tag{2.20}$$

is valid, may differ from zero. Hence it follows that the number p defines really the maximum number of particles which may be in symmetric (antisymmetric) states. Thereby the theorem II is definitely proved.

COROLLARY. As a consequence of our proof, from the conditions (2.19) there arise certain restrictions imposed on the reservoir vectors. The following symmetric (antisymmetric) combinations should vanish:

For parastatistics of even orders, $p = 2k$,

$$\sum_P \lambda_P |P\ell_1, \ldots P\ell_{k+1}> = 0,$$

$$\sum_P \lambda_P a^{Pr_1} |P\ell_1, \ldots, P\ell_k> = 0,$$

$$\sum_P \lambda_P a^{Pr_1} a^{Pr_2} a^{Pr_3} |P\ell_1, \ldots, P\ell_{k-1}> = 0, \tag{2.21}$$

$$\ldots$$

$$\sum_P \lambda_P a^{Pr_1} a^{Pr_2} \ldots a^{Pr_{2k-1}} |P\ell_1> = 0.$$

For parastatistics of odd orders,

$p = 2k + 1$,

$$\sum_P \lambda_P |P\ell_1, \ldots, P\ell_{k+1}> = 0,$$

$$\sum_P \lambda_P a^{Pr_1} a^{Pr_2} |P\ell_1, \ldots, P\ell_k> = 0, \tag{2.22}$$

$$\ldots$$

$$\sum_P \lambda_P a^{Pr_1} \ldots a^{Pr_{2k}} |P\ell_1> = 0.$$

Note that the reservoir vectors may have, in addition to symmetrizable (antisymmetrizable) indices, some other indices not subjected to this operation. Their presence does not affect the above proof, therefore we always omitted them.

Thus, the reservoir vectors are defined by the conditions (2.1), (2.6), (2.21) and (2.22). These conditions should be regarded as boundary conditions for irredu-

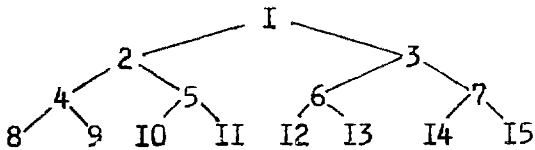
cible representations. Their further specification consists in symmetrizing the reservoir vectors according to the appropriate Young schemes.

Each irreducible representation of the parastatistics of a p -th order is thus characterized by a variable *number of* external states and by a constant number, say M, of reservoir states symmetrized according to the appropriate Young scheme. Irreducibility follows from the very method of constructing the representations. So, for example, the reservoir vectors are related to one another by the relations (2.6).

The simplest examples provide direct evidence that the above-mentioned boundary conditions for the reservoir vectors for a fixed order of p together with the general Green relations (1.1) define really the appropriate parastatistics (see Appendix 2).

3. Sequences of Parastatistics

It follows from the condition (2.20) that the reservoir vectors of the irreducible representations of a given order p have themselves the parastatistics of an order equal to the whole part $[p/2]$. We can consider the sequences of parastatistics in which the vectors being the bases of the irreducible representations of the foregoing statistics serve as reservoir vectors of the irreducible representations of a subsequent parastatistics. So, taking any basis vector of the Fock representation of the ordinary Fermi-Dirac (Bose-Einstein) statistics as a reservoir vector we can construct on it an irreducible representation of the para-Fermi (para-Bose)-statistics of the second and third order. To this end, it is necessary to operate on it by possible polynomials of new production operators requiring for the appropriate conditions to be fulfilled (Appendix 2, (A2.6)-(A2.9) and (A2.17)-(A2.20)). The basis vectors of the irreducible representations of the second and third orders obtained in this manner should be symmetrized by the Young schemes. Further, taking them as reservoir vectors, it is possible similarly to construct parastatistics of the fourth and fifth or the sixth and seventh orders, respectively. Continuing this procedure we obtain the following hierarchy of parastatistics in which each of the preceding parastatistics of an order p generates two subsequent parastatistics of an order $2p$ and $2p+1$:



It should be noted that the operators belonging to the reservoir vectors and the operators belonging to the representations themselves constructed on these vectors obey different algebras and have with one another nothing in common, except the boundary conditions.

In previous papers [9-11] when studying the parastatistics of the second and third orders we have utilized the well-known Green ansatz [1]. The representation space of the latter is a reducible representation of the Green algebra (1.1). It is possible to single out from this space the irreducible representations of the latter by finding in it reservoir vectors (see Appendix in ref. [11]). Then it is possible to make sure directly in the validity of the above considerations about the sequences of the parastatistics. It should be noted that among the irreducible representations extracted in this way for $p \geq 3$ there appear equivalent (isomorphic) representations [8,11,12]. The problem of the quantity and the classification of the equivalent representations has been studied in recent papers [8,12] by the group theory approach.

Finally we note that from the point of view of the theory of proper parafield the Green ansatz is a convenient, but not necessary, tool. The Green ansatz can, however, be considered as an extended formulation of paraquantization /8-13/. In this case the Green ansatz is thought of as a physical picture, and one should consider all the irreducible representations of the Green algebra entering it, *including* the equivalent ones. Such a situation occurs in the physical application of paraquantization to the classification of the states of the nucleon pairs in nuclear shells /14-15/. In another example the use of para-Fermi-quantization of the third order as a *field* realization of the SU(3) symmetry equivalent states are thought of as states forming isomultiplets /8,9,11/.

4. Conclusion

In the theory of paraquantization we have renounced the condition of uniqueness of the vector which plays in the theory of ordinary quantization the role of the vacuum vector. Instead we have suggested the requirement that a whole set of reservoir vectors be present and that they be the eigenvectors of the particle number operator which correspond to zero-, one-, two- and more particle states.

On the basis of theorems I and II we have established the conditions imposed on the reservoir vectors which together with the Young symmetry define the irreducible representations of paracommutation Green relations. On the basis of the study of the parastatistics hierarchy we have also suggested a way of consecutive construction of the irreducible representations of parastatistics as their orders increase.

In literature one has repeatedly noted the connection between para-Fermi-quantization and the higher spin algebra /1,2,4-6,12,16/. Our results are easily extended to this latter when the indices of the operators a^r and a_r assume the finite number of values. Then our representations are unitary representations of an orthogonal and symplectic groups which correspond to para-Fermi and para-Bose quantization. Note that the presence of the reservoir vectors is, in this case, a consequence of the algebra itself. The Fock representation corresponds to the highest spin while other representations correspond to lower spins.

In conclusion we dwell on possible applications of parastatistics. The presence of a set of the irreducible representations of the parastatistics of a given order shows a possibility of interpreting paraquantization as an implicit and field form of writing the fact that ordinary fermion and boson field have internal degrees of freedom. Then the state of the same number of particles

but related to different irreducible representations may be considered as different internal states of particles /8-13/ corresponding to the states of the SU(p) group in multiplets. It is interesting to note that once this interpretation of paraquantization is admitted then, according to the above hierarchy of parastatistics, the SU(6) symmetry must follows the SU(3) symmetry. However in this case one of the internal degrees of freedom must be the spin.

Parastatistics can also appear due to the composite structure of the objects in question /14,15/. In the case of application of parastatistics for nucleon pairs in nuclear shells to its different irreducible representations there correspond different quantum numbers of seniority or quasispin.

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Appendix 1. Norm of the Symmetric (Antisymmetric) Vector

In this appendix we calculate the norm of the symmetric for para-Fermi-statistics and antisymmetric for para-Bose-statistics vector (2.16).

First, using the induction method we prove the formula

$$a_\ell \Big(\sum_P \lambda_P a^{P\ell_1} \dots a^{P\ell_N} \,|P\ell_{N+1}, \dots, P\ell_{N+M}>\Big) = \tag{A1.1}$$

$$= N(p - N + 1 - 2M) \Big(\sum_P \lambda_P \delta_\ell^{P\ell_1} a^{P\ell_2} \dots a^{P\ell_N} |P\ell_{N+1} \dots, P\ell_{N+M}>\Big) .$$

On the basis of the fact that the vector under consideration is symmetric (antisymmetric), using (2.6) it is easy to make sure that this relation holds for $N=1$. Now we assume that it is valid for $N-1$ and come to the conclusion that it holds for N, too. Making use of the Green relations (1.1) for $a(\sigma) = a_1, a(\tau) = a^{P\ell_1}, a(\rho) = a^{P\ell_2}$ we get

$$a_\ell \Big(\sum_P a^{P\ell_1} a^{P\ell_2} \dots a^{P\ell_N} |P\ell_{N+1}, \dots, P\ell_{N+M}>\Big)$$

$$= \sum_P \lambda_P (-2\epsilon\, a^{P\ell_1} a_\ell a^{P\ell_2} - a^{P\ell_1} a^{P\ell_2} a_\ell - \tag{A1.2}$$

$$- 2\,\delta_\ell^{P\ell_1} a^{P\ell_2}) \, a^{P\ell_3} \dots a^{P\ell_N} |P\ell_{N+1}, \dots, P\ell_{N+M}> .$$

Using the induction condition as well as the symmetry (antisymmetry) of the vector in question we obtain for the r.h.s. the following expression

$$\{2(N-1)[p - (N-1) + 1 - 2M] - (N-2)[p - (N-2) + 1 - 2M] - 2\} \times$$

$$\times \left(\sum_P \lambda_P \delta_\ell^{P\ell_1} a^{P\ell_2} \dots a^{P\ell_N} \,|P\ell_{N+1}, \dots, P\ell_{N+M}>\right). \quad \text{(A1.3)}$$

After reducing such terms we are led to (A1.1).

Now we calculate the norm of the vector

$$\|\sum_P \lambda_P a^{P\ell_1} \dots a^{P\ell_N} \,|P\ell_{N+1}, \dots, P\ell_{N+M}>\|^2$$

$$= \sum_{P,P'} \lambda_P \lambda_{P'} <P'\ell_{N+M}, \dots, P'\ell_{N+1} \,|a_{P'\ell_N} \dots a_{P'\ell_1} \quad \text{(A1.4)}$$

$$a^{P\ell_1} \dots a^{P\ell_N} \,|P\ell_{N+1}, \dots, P\ell_{N+M}> .$$

By repeating application of (A1.1) for the r.h.s. of (A1.4) we obtain

$$(N!) \prod_{j=1}^{N} (p-j+1-2M) \sum_{P,P'} \lambda_P \lambda_{P'} \delta_{P'\ell_1}^{P\ell_1} \dots \delta_{P'\ell_N}^{P\ell_N}$$

$$<P'\ell_{N+M} \dots, P'\ell_{N+1} \,|P\ell_{N+1}, \dots, P\ell_{N+M}> . \quad \text{(A1.5)}$$

We conclude that the external indices N must coincide and, consequently, the reservoir indices must form identical sets which can differ from one another only by permutations. We can rewrite (A1.5) in the form

$$(N!)^2 \prod_{j=1}^{N} (p-j+1-2M) \left(\sum_\mu \|\sum_P \lambda_P |P\ell_{\mu_1}, \dots, P\ell_{\mu_M}>\|^2 \right), \quad \text{(A1.6)}$$

where the sum over μ means the sum over possible samplings of the indices with respect to M from the set of indices $\ell_1, \dots, \ell_{N+M}$. Introducing the notation (2.18) we go over to the formula (2.17).

Appendix 2. Boundary Conditions for Parastatistics of the First, Second and Third Orders

Taking the parastatistics of the first (usual) and the second (Green-Volkov ones) orders as the simplest examples we show how the boundary conditions for the reservoir vectors and the general Green relations (1.1) define the appropriate statistics. For the third order we give only the boundary conditions and do not introduce characteristic relations, because of their complexity/4/.

Case $p=1$. In this case the condition (2.20) shows that it holds only for $M=0$, otherwise there is the unique Fock representation based on the vacuum vector $>$. The boundary conditions (2.1) and (2.6) for the latter are of the form

$$a_r > = 0 , \quad a_r a^s > = \delta_r^s > . \tag{A2.1}$$

Next, from the same condition (2.20) it follows that in this case not more than one particle can be in the symmetric (antisymmetric) state. However, this means an anticommutation (commutation) of the operators

$$a^{m_1} a^{m_2} \dots a^{m_N} > - \epsilon\, a^{m_2} a^{m_1} \dots a^{m_N} > = 0 . \tag{A2.2}$$

Finally we show that

$$a_m a^{m_1} a^{m_2} \dots a^{m_N} > = \epsilon\, a^{m_1} a_m a^{m_2} \dots a^{m_N} > + \\ + \delta_m^{m_1} a^{m_2} \dots a^{m_N} > \tag{A2.3}$$

takes place. For $N=1$ the validity of this relation follows from (A2.1). For an arbitrary N the proof is performed by the induction method. First using (A2.2) we interchange in the l.h.s. of (A2.3) the operators a^{m_1} and a^{m_2}. In order to displace the operator a_m to the right we use one of the Green relations (1.1) for $a(\rho) = a^{m_1}$, $a(\sigma) = a_m$, $a(\tau) = a^{m_2}$. We get

$$a_m a^{m_1} a^{m_2} \dots a^{m_N} > = (\epsilon\, a^{m_1} a_m a^{m_2} - a^{m_2} a_m a^{m_1} +$$

$$+ a^{m_1} a^{m_2} a_m + 2\delta_m^{m_1} a^{m_2})\, a^{m_3} \dots a^{m_N} > . \tag{A2.4}$$

If now according to the induction condition for the second term in the r.h.s. of (A2.4) we make use of the relation (A2.3) for the $N-1$ operator and then using (A2.2) we interchange in it anew the operators a^{m_1} and a^{m_2} then we arrive at the relation (A2.3) for N operators.

Thus, in the Fock space under consideration which is defined by the conditions (A2.1) the usual relations

$$[\, a(\rho)\, ,\, a(\sigma)\,] = g(\rho\sigma) \tag{A2.5}$$

are valid.

Case $p=2$. For this simple generalization of the usual statistics we have a set of reservoir vectors satisfying, according to (2.1) and (2.6) the conditions

$$a_r\, |\ell_1, \dots, \ell_M > = 0\, , \tag{A2.6}$$

$$a_r a^s\, |\ell_1, \dots, \ell_M > = 2\delta_r^s\, |\ell_1, \dots, \ell_M > +$$

$$+ 2\epsilon\, \delta_r^{\ell_1}\, |s, \ell_2, \dots, \ell_M > + \dots + 2\epsilon\, \delta_r^{\ell_M} |\ell_1, \dots, \ell_{M-1}, s > . \tag{A2.7}$$

In addition from the condition (2.21) for $k = 1$ it follows that the reservoir vectors are antisymmetric (symmetric)

$$|\ell_1,\dots,\ell_i,\dots,\ell_j,\dots,\ell_M\rangle = \epsilon|\ell_1,\dots,\ell_j,\dots,\ell_i,\dots,\ell_M\rangle \qquad \textbf{(A2.8)}$$

and that the relation

$$a^r|\ell_1,\dots,\ell_M\rangle = \epsilon\, a^{\ell_1}\,|r,\ell_2,\dots,\ell_M\rangle \qquad \textbf{(A2.9)}$$

holds. We show that in this representation there take place the Green [1] Volkov [2] relations

$$a(\rho)\,a(\sigma)\,a(\tau) - \epsilon\, a(\tau)\,a(\sigma)\,a(\rho) = 2g(\rho\sigma)\,a(\tau) + 2g(\sigma\tau)\,a(\rho)\,. \qquad \textbf{(A2.10)}$$

We employ the two Green relations

$$a^r a^s a^t - \epsilon\, a^t a^s a^r = a^t a^r a^s - \epsilon\, a^s a^r a^t\,,$$
$$a^r a^s a^t - \epsilon\, a^t a^s a^r = a^s a^t a^r - \epsilon\, a^r a^t a^s\,. \qquad \textbf{(A2.11)}$$

According to theorem II, for $p=2$ the vector symmetric (antisymmetric) in the three external indices

$$(a^r a^s a^t - \epsilon\, a^r a^t a^s - \epsilon\, a^s a^r a^t + a^s a^t a^r + a^t a^r a^s + $$
$$+ a^t a^r a^s - \epsilon\, a^t a^s a^r)\, a^{\ell_1}\dots a^{\ell_N}|\ell_{N+1},\dots,\ell_{N+M}\rangle = 0$$

$$\textbf{(A2.12)}$$

must vanish. It follows from (A2.11) and (A2.12)

$$(a^r a^s a^t - \epsilon\, a^t a^s a^r)\, a^{\ell_1}\dots a^{\ell_N}|\ell_{N+1},\dots,\ell_{N+M}\rangle = 0 \qquad \textbf{(A2.13)}$$

which proves (A2.10) for the operators a^r, a^s, a^t.

Let now prove, e.g., the relation

$$(a_r a^s a^t - \epsilon\, a^t a^s a_r - 2\delta_r^s a^t)\, a^{\ell_1}\dots a^{\ell_N}|\ell_{N+1},\dots,\ell_{N+M}\rangle = 0\,. \qquad \textbf{(A2.14)}$$

For $N=0$ the validity of this relation is easily established with the aid of the Green relation

$$a_r a^s a^t - \epsilon a^t a^s a_r = a^t a_r a^s - \epsilon a^s a_r a^t + 2\epsilon \delta^t_r a^s \tag{A2.15}$$

and the conditions (A2.7) - (A2.9). For an arbitrary N the relation (A2.14) is proved by induction. Using (A2.15) we displace the operator a_r to the right and utilize the induction condition. We obtain the expression

$$\{a^t(\epsilon a^{\ell_1} a^s a_r + 2\delta^s_r a^{\ell_1}) - \epsilon a^s(\epsilon a^{\ell_1} a^t a_r + 2\delta^t_r a^{\ell_1}) + 2\epsilon \delta^t_r a^s a^{\ell_1} - 2\delta^s_r a^t a^{\ell_1}\} a^{\ell_2} \dots a^{\ell_N} |\ell_{N+1}, \dots, \ell_{N+M}\rangle . \tag{A2.16}$$

Taking into account the relation (A2.13) proved by us when reducing the similar terms we are led to (A2.14). The other relations of (A2.10) are proved in the same way.

Case $p=3$. The boundary conditions imposed on the reservoir vectors are in this case

$$a_r |\ell_1, \dots, \ell_M\rangle = 0 , \tag{A2.17}$$

$$a_r a^s |\ell_1, \dots, \ell_m\rangle = 3\delta^s_r |\ell_1, \dots, \ell_M\rangle + 2\epsilon \delta^{\ell_1}_r |s, \ell_2, \dots, \ell_M\rangle + \dots + 2\epsilon \delta^{\ell_M}_r |\ell_1, \dots, \ell_{M-1}, s\rangle , \tag{A2.18}$$

$$|\ell_1, \dots, \ell_i, \dots, \ell_j, \dots, \ell_M\rangle = \epsilon |\ell_1, \dots, \ell_j, \dots, \ell_i, \dots, \ell_M\rangle \tag{A2.19}$$

$$\sum_P \lambda_P a^{Pr_1} a^{Pr_2} |P\ell_1, \ell_2, \dots, \ell_M\rangle = 0 . \tag{A2.20}$$

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