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FIVE-PARTICLE AMPLITUDES NEAR PHYSICAL REGIONS

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Within the framework of local field theory, very little is known about the analyticity properties of scattering amplitudes involving five or more particles. In particular the following problem is not completely solved: given the hadronic process $a_{\nu+1}+\ldots+a_n \to a_1+\ldots+a_\nu (\nu \geq 2, \quad n \geq 4)$, find the region of all the physical points where the scattering amplitude is the boundary value of a single analytic function on the mass shell. If $\nu = n-\nu = 2$, the whole physical region consists of such points [1]. The hope that the last property persists for $n > 4$ is dashed by several counter-examples in both local field and perturbation theories [2,3]. In this note we shall be concerned with the study of the preceding problem for the case $\nu = 3$ and $n = 5$.

Consider the hadronic reaction

$$a_4 + a_5 \to a_1 + a_2 + a_3 \tag{1}$$

and denote by m_j the mass of particle $a_j (j=1,\ldots,5)$. Let C^{16} be the space of 5-ples $k=(k_1,\ldots,k_5)$ of complex four-momenta with $k_1+\ldots+k_5=0$. The complex mass shell M_C is the space of all points $k \in C^{16}$ with $k_j^2 = m_j^2$, $j=1,\ldots,5$ *. The physical region Q for reaction (1) consists of all points $p=(p_1,\ldots,p_5) \subset M_C$ such that each four-momentum p_j is real, $p_j^{(0)} > 0$ for $j=1,2,3$, and $p_j^{(0)} < 0$ for $j=4,5$.

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*The scalar product of two four-vectors $k=(k^{(0)},\vec{k})$ and $k'=(k'^{(0)},\vec{k}')$ is defined as $kk'=k^{(0)}k'^{(0)}-\vec{k}\vec{k}'$ in both the real and complex Minkowski spaces.

We start with the following result of Bros, Epstein and Glaser [2] : if the physical point $p \in Q$ belongs to the boundary of a domain in M_C which intersects the tubes *

$$T_r = \{k \mid k \in C^{16}, \mathrm{Im}(k_4 + k_5) \in V_-, \mathrm{Im}(k_4 + k_5 + k_r) \in V_-\}, r = 1,2,3, \quad (2)$$

then there exists a complex neighbourhood N of p in M_C such that the scattering amplitude F for reaction (1) can be represented in $Q \cap N$ as

$$F = \sum_{r=1}^{3} F_r , \quad (3)$$

where each F_r is the boundary value (in the sense of distributions [2,4]) of a function analytic in the localized tube $T_r \cap N$. The conditions of this statement are fulfilled if no pair of incoming and no pair of outgoing four-momenta are mutually parallel (i.e. $p_j p_{j'} \neq m_j m_{j'}$, $j < j'$, where either $j > 3$ or $j' < 4$) [2] . The scattering amplitude is deduced from the general principles of local field theory involving only massive stable particles ($m_j > 0$ for $j = 1, ..., 5$) and its spin dependence is not explicit.

The amplitude F is the limiting value of a single analytic function whenever the intersection of the localized tubes $T_r \cap N$ is not empty. This condition holds if and only if [2]**

$$G(p) > m_1^2 m_2^2 m_3^2 , \quad (4)$$

where $G(p) = \det(p_r p_{r'})$, $1 \le r , r' \le 3$, is the Gram determinant of the scalar products of the outgoing four-momenta.

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* The Minkowski cone V_- consists of all real four-vectors such that $q^2 > 0$ and $q^{(0)} < 0$.

** Notice that the restriction of Mueller's optical theorem to the mass shell for the inclusive reaction $a_1 + a_2 \to \bar{a}_3 +$ anything holds in the region given by (4) with p_3 replaced by $-p_3$ (see ref. [3] , p. 1017).

The domain in Q given by (4) is empty if $\sqrt{s}-m_1-m_2-m_3$ is small enough. Here $\sqrt{s}$ denotes the total centre-of-mass energy for reaction (1). In the equal mass case $m = m_r$, $r = 1, 2, 3$, this domain is nonvoid if and only if $s > 12m^2$. In the general mass case, we choose the point $p \in Q$ such that $p_r p_{r'} / m_r m_{r'} = \omega$ for $1 \leq r < r' \leq 3$. Then

$$G(p) - m_1^2 m_2^2 m_3^2 = m_1^2 m_2^2 m_3^2 \omega^2 (2\omega - 3),$$

$$s = \sum_{1 \leq r \leq 3} m_r^2 + 2\omega \sum_{1 \leq r < r' \leq 3} m_r m_{r'} . \tag{5}$$

It follows from (5) with $\omega > 3/2$ that there exist some points $p \in Q$ satisfying (4) for

$$\sqrt{s} > \frac{2}{\sqrt{3}} (m_1 + m_2 + m_3). \tag{6}$$

It is convenient to introduce the following kinematical variables [2]:

$$x_r = p_r (p_1 + p_2 + p_3)/s , \quad \mu_r = m_r / \sqrt{s} , \quad r = 1,2,3, \tag{7}$$

where $x_r \sqrt{s}$ is the c.m. energy of particle a_r. Then the physical region for reaction (1) is given by

$$\Gamma(x, s) = s^{-3} G(p) \geq 0, \quad x_1 + x_2 + x_3 = 1, \quad x_r \geq \mu_r , \; r = 1,2,3, \tag{8}$$

where $x = (x_1, x_2, x_3)$ and

$$\Gamma(x, s) = \frac{1}{2} \sum_{1 \leq r < r' \leq 3} (x_r^2 - \mu_r^2)(x_{r'}^2 - \mu_{r'}^2) - \frac{1}{4} \sum_{r=1}^{3} (x_r^2 - \mu_r^2)^2. \tag{9}$$

The (x_1, x_2)-projection of Q at a given value of s (the Dalitz plot) is defined by (8) and denoted by $D(s)$. It follows from (4) and (8) that the subdomain $D_1(s)$ of $D(s)$, where the scattering amplitude is the boundary value of a single analytic function, can be characterized as

$$\Gamma_1(x,s) = \Gamma(x,s) - \mu_1^2\mu_2^2\mu_3^2 > 0, \quad x_1 + x_2 + x_3 = 1. \qquad (10)$$

The domain $D_1(s)$ is not empty if and only if $s > s_1$, where s_1 is the unique solution of the equation $\max\Gamma_1(x,s_1) = 0$ on $D(s_1)$. To show this, it suffices to remark that the restriction of $\max\Gamma_1(x,s)$ to $D(s)$ is strictly increasing with respect to s, negative at the threshold $x_r = \mu_r$, $r = 1,2,3$, and positive for some x whenever (6) holds. Notice that the closure of $D_1(s_1)$ consists of the only barycentre of the Dalitz plot $D(s_1)$. If s ($s > s_1$) increases, it follows from (8)-(10) that the convex domain $D_1(s)$ exists and its closure $\bar{D}_1(s)$ admits always the same barycentre as $D(s)$.

It is obvious that the distance between the boundaries of $D(s)$ and $D_1(s)$ goes to zero when s tends to infinity. Moreover, an inequality of Hörmander [5,6] shows that there exist two strictly positive constants c and ν such that $\Gamma(x,s) > c[d(x,s)]^{\nu}$ for any $(x_1,x_2) \in D(s)$, $x_3 = 1 - x_1 - x_2$, $s > s_1$, where $d(x,s)$ denotes the distance from (x_1, x_2) to the boundary of $D(s)$. In view of (10), if the point (x_1, x_2) belongs to the boundary of $D_1(s)$ (i.e. the level line $\Gamma(x,s) = m_1^2 m_2^2 m_3^2 s^{-3}$), then the distance $d(x,s)$ is decreasing with s more rapidly than a strictly positive power of $1/s$. The common limit of both $\bar{D}(s)$ and $\bar{D}_1(s)$ is the triangle defined by $x_1 \le 1/2$, $x_2 \le 1/2$, and $x_1 + x_2 \ge 1/2$.

The above-mentioned behaviour of the domain $D_1(s)$ is illustrated in Fig. 1 for the reaction $p + p \to p + p + \pi^0$ ($a_1 = a_2 = p$, $a_3 = \pi^0$ with $m_1 = 0.93825$ GeV and $m_3 = 0.13497$ GeV given in ref. [7]). The boundary of $D(s)$ is represented by the thin full curve inscribed in the Dalitz triangle $x_1 \ge \mu_1$, $x_2 \ge \mu_2$, $x_1 + x_2 \le 1 - \mu_3$. The Dalitz plot $D(s)$ is the convex hull of this curve. $D_1(s)$ is the convex domain bounded by the dashed line. In this particular case, we have $\sqrt{s_1} \simeq 2.202$ GeV, where s_1 is the real solution of the following equation:

$$s_1^3 - 2(6m_1^2 + m_3^2)s_1^2 + (48m_1^4 - 20m_1^2 m_3^2 + m_3^4)s_1 - 4m_1^4(16m_1^2 - m_3^2) = 0. \qquad (11)$$

Figures 1a), b) and c) are represented at $\sqrt{s} = 2.053$ GeV 2.205GeV, 3.000 GeV, respectively. The domain $D_1(s)$ is empty in Fig. 1a) and small in Fig. 1b). In Fig. 1b) the common barycentre of $D(s)$ and $\bar{D}_1(s)$ is denoted by B . Fig. 1c) shows that the domain $D_1(s)$ extends rather quickly when s increases.

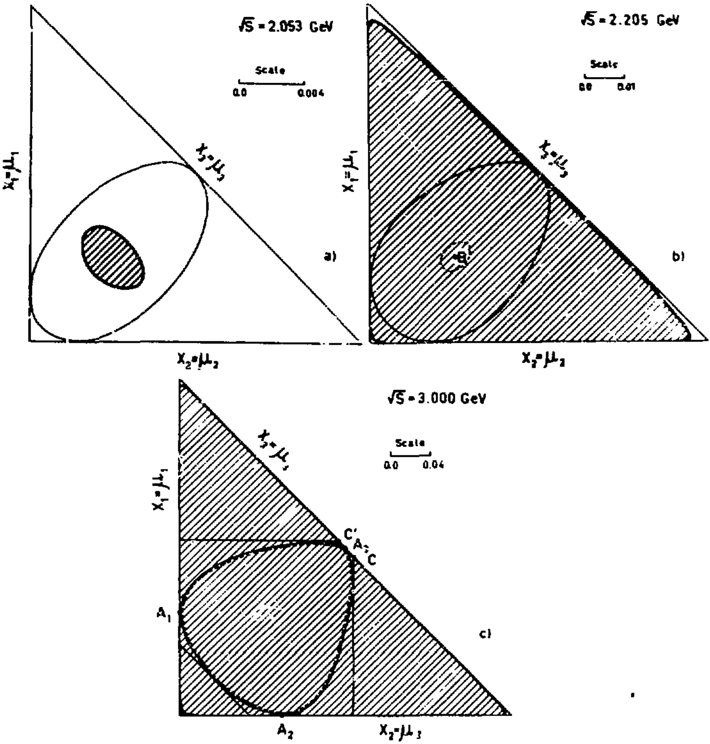

Fig. 1. Some physical regions where the scattering amplitude for the reaction $p+p \to p+p+\pi^0$ is the boundary value of a single analytic function on the mass shell.

We now recall that the decomposition of the scattering amplitude given by (3) persists if the tubes T_r, $r=1,2,3$, are replaced by the extended tubes $T'_r = L_+T_r$, where L_+ is the connected complex Lorentz group. This result has been pointed by Epstein and Glaser [8]. By virtue of both the global and linearized descriptions of the three-point extended tube [2,9], a necessary and sufficient condition for a point $p \in Q$ to lie on the boundary of the intersection $T'_1 \cap T'_2 \cap T'_3 \cap M_C$ is that there exist some $\rho_r = \mathrm{Im}(k_4 + k_5 + k_r)^2 / \mathrm{Im}(k_4 + k_5)^2$ such that

$$(\frac{2x_r}{\mu_r^2}-1)(x_r-\sqrt{x_r^2-\mu_r^2}) \le 1+\rho_r \le (\frac{2x_r}{\mu_r^2}-1)(x_r+\sqrt{x_r^2-\mu_r^2}),$$

$$\rho_r > 0,\ \rho_1+\rho_2+\rho_3 = 1,\ r=1,2,3, \qquad (12)$$

where the inequality is strict for $x_r > \mu_r$. If (12) is fulfilled, the scattering amplitude is the boundary value of a single analytic function. For example, (12) holds whenever $m_1=m_2=m_3=m$ and $\sqrt{s} > 4.8\,m$ [2].

We next improve the above stated result. It follows immediately from (12) that the scattering amplitude is the limiting value of a single analytic function at the physical point $p \in Q$ if $(x_1, x_2) \in D_2(s)$, where $D_2(s)$ consists of all points (x_1, x_2) such that $\Phi(x,s) > 0$, $x_3 = 1 - x_1 - x_2$, with

$$\Phi(x,s) = 4 - \sum_{r=1}^{3} (\frac{2x_r}{\mu_r^2} - 1)(x_r - \sqrt{x_r^2 - \mu_r^2}). \qquad (13)$$

We will show that the region $D(s) \cap D_2(s)$ is convex ($D_2(s)$ extends when s increases) is not empty for

$$s > (\frac{1}{2} + \frac{1}{\sqrt{3}})(m_1 + m_2 + m_3)^2, \qquad (14)$$

and contains the whole Dalitz plot (i.e. $D(s) \subset D_2(s)$) for

$$\sqrt{s} \ge 3.335 \max_{r=1,2,3} m_r, \qquad \eta = \max_{r,r'=1,2,3} \frac{m_r}{m_{r'}} \le 9. \qquad (15)$$

If $x_r = \mu_r \sigma$, $\sigma = (\mu_1 + \mu_2 + \mu_3)^{-1}$, $r = 1,2,3$, then

$$\operatorname{sgn} \Phi(x,s) = \operatorname{sgn}(12\sigma^4 - 12\sigma^2 - 1). \tag{16}$$

In view of (16), the region $D(s) \cap D_2(s)$ is nonvoid if s satisfies (14). Notice now that $\Phi(x,s)$ is a strictly increasing concave function of s for $x_r < \frac{1}{2}(1+\mu_r^2)$. Since the restrictions of both $\max \Gamma(x,s)$ and $\Phi(x,s)$ to $D(s)$ are strictly increasing functions of s, the region $D(s) \cap D_2(s)$ exists when s increases and $s > s_2$; here s_2 is the unique solution of the equation $\chi(s_2) = 0$, where $\chi(s) = \max \Phi(x,s)$ on $D(s)$. The last property is satisfied because $\chi(s)$ is a strictly increasing continuous function of s (its continuity follows from its concavity /10/), $\chi(s) = -1$ at the threshold $x_r = \mu_r$, $r = 1,2,3$, and $\chi(s) > 0$ when (14) holds. Moreover, the function $\Phi(x,s)$ is concave at x for $x_r \geq \mu_r$ because its Hesse matrix is a diagonal one with strictly negative diagonal coefficients. Then $D_2(s)$ is a convex domain. By the convexity of $D(s)$, the region $D_2(s) \cap D(s)$, is also convex.

We now prove that (15) implies $D(s) \subset D_2(s)$. Let θ denote the mapping which carries every point $x = (x_1, x_2, x_3)$ to the point $y = (y_1, y_2, y_3)$ defined by $y_r = \sqrt{x_r^2 - \mu_r^2}$, $r = 1,2,3$. The image $\theta(D(s))$ of the Dalitz plot under θ is given by

$$\sum_{r=1}^{3} \sqrt{y_r^2 + \mu_r^2} = 1, \quad y_{r_1} + y_{r_2} \geq y_{r_3} \tag{17}$$

for all cyclic permutations (r_1, r_2, r_3) of $(1,2,3)$. Let A_r be the tangent point of $D(s)$ with the straight line given by $x_r = \mu_r$ (see Fig. 1c)). Denote by A_r' the image of A_r under θ and define $\Psi(y,s) = \Phi(x,s)$ for $x = \theta(y)$. Since (17) is a portion of a convex surface, given any $y \in \theta(D(s))$, there exists a number λ, $0 < \lambda \leq 1$, such that the point $(\lambda y_1, \lambda y_2, \lambda y_3)$ belongs to the triangle $A_1' A_2' A_3'$. But the function $\Psi(y,s)$ is increasing at y_r for $y_r > 0$. Hence the minimum of the function $\Psi(y,s)$ on the triangle $A_1' A_2' A_3'$ is smaller than its minimum on $\theta(D(s))$. $\Psi(y,s)$ is concave at y. The minimum of a concave function on a triangle occurs at one of its vertices. Therefore the minimum of $\Phi(x,s)$ on the Dalitz plot $D(s)$ occurs at one of the points A_r, $r=1,2,3$. Moreover, $D(s)$ is contained in $D_2(s)$ if $\Phi(x,s)>0$ at any A_r. An estimate of this condition for A_3 gives

$$\mu_1^2\mu_2^2(1-\mu_3)^4 < [(1-\mu_3)^4 - 2(\mu_1^2+\mu_2^2)(1-\mu_3)^2 + (\mu_1^2-\mu_2^2)^2] \times \tag{18}$$

$$\times [\mu_3(\mu_1^2+\mu_2^2)(1-\mu_3)^2 + \mu_1^2\mu_2^2(1-\mu_3)^2 - \mu_3(\mu_1^2-\mu_2^2)^2].$$

Replacing in (18) μ_1 and μ_2 by $\mu_{12} = \max\{\mu_1, \mu_2\}$, we have

$$2\mu_3(1-\mu_3)^2 + \mu_{12}^2(\mu_3^2 - 10\mu_3 - 4\mu_{12}^2) > 0. \tag{19}$$

Replacing in (19) μ_{12} and μ_3 by $\mu = \max\{\mu_1, \mu_2, \mu_3\}$, we obtain

$$2 - 4\mu - 8\mu^2 - 3\mu^3 > 0. \tag{20}$$

It is easy to see that (18) is implied by (19). Moreover, **if $\eta \leq 9$ any inequality obtained by permuting indices 1,2,3 in (18) and (19) follows from (20). Then (20) implies (15) and the above considered assertion is proved.***

We now return to Fig. 1. The shaded area shows the region $D_2(s)$. The zeros of $\Phi(x,s)$ lie on the thick full curve. In Fig. 1a), $D_2(s)$ appears at $\sqrt{s}=2.053$ GeV. Notice that the corresponding value of s is smaller than r.h.s. of (14). Figure 1b) shows that $D_2(s)$ at $\sqrt{s}=2.205$ GeV is dominant in the Dalitz triangle. The l.h.s. of (19) vanishes at $\sqrt{s}=3.000(+2\times10^{-4})$ GeV when $D(s)$ is contained

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***Using (19) and (20) with μ_{12} replaced by μ and μ_3 replaced by $\min(\mu_1, \mu_2, \mu_3)$, we have $D(s) \subset D_2(s)$ for any $\eta \geq 1$ provided either (15) hold or $\sqrt{s} \geq a \max_{r=1,2,3} m_r$ with $a = (4\eta)^{-1}[\eta + 1 + (16\eta^3 + 5\eta^2 + 2\eta + 1)^{1/2}] > 3.335$.**

in $D_2(s)$ excepting the point A_3 where the boundaries of $D_2(s)$ and $D(s)$ are tangent (see Fig. 1c)). Notice that the momentum of particle a_3 is vanishing at A_3. Finally, we remark that the minimum of $\Phi(x,s)$ on the hexagon circumscribed to the Dalitz plot $D(s)$ occurs to the vertices C and C′ shown in Fig. 1c). This hexagon is contained in $D_2(s)$ whenever $\sqrt{s} > 3.006$ GeV.

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