СООБЩЕНИЯ
ОБЪЕДИНЕННОГО
ИНСТИТУТА
ЯДЕРНЫХ
ИССЛЕДОВАНИЙ

ДУБНА

E2 - 7461

B.L.Aneva, S.Ch.Mavrodiev, L.K.Hadjiivanov

LORENTZ-INVARIANT EXPANSION OF THE SCATTERING AMPLITUDE FOR PARTICLES OF ANY SPIN

1973

ЛАБОРАТОРИЯ
ТЕОРЕТИЧЕСКОЙ ФИЗИКИ

E2 - 7461

B.L.Aneva*, S.Ch.Mavrodiev, L.K.Hadjiivanov*

LORENTZ-INVARIANT EXPANSION OF THE SCATTERING AMPLITUDE FOR PARTICLES OF ANY SPIN

* On leave of absence from Institute for Nuclear Research and Nuclear Energetic, Bulgarian Academy of Sciences, Sofia.

CONTENTS

1. Introduction

The problem of the expansion of the scattering amplitude for particles of arbitrary spin in covariant structures dates from the very begining of the development of quantum field theory. In the usual exposition [1,2,3], such covariant expansions are derived by a specially devised procedure for each combination of spins encountered in practical calculations.

On the basis of full Lorentz invariance (including invariance under C,P and T) Hepp [4] proposed a general method for constructing covariant expansions. Hepp's solution however was not quite explicit and the problem continues to attract attention (see for instance [5,6], and further references cited there).

In the present paper, following an unpublished work of Oksak and Todorov, we attach the general problem of covariant expansions, by exploiting the manifestly covariant technique of ref. [7,8]. It consists, roughly, in substituting the spin-tensors with two-component indices, used in ref. [4], by homogeneous polynomials of pairs of complex variables associated with each particle with spin.

The Lorentz invariant expansion is written as a sum of

$$(2s_1+1)(2s_2+1)(2s_3+1)(2s_4+1)$$

terms (for binars processes). The full Lorentz invariance leads to further reduction of the number of independent invariant structures.

In Sec.2 the problem of the Lorentz-covariant expansion of the scattering amplitude is reduced to the problem of a Lorentz-invariant expansion of a homogeneous function of two-component complex spinors. The degree of homogeneity is related to the spins of the incoming and outgoing particles. The apparatus of the finite dimensional representations of the group SL(2,C) in the space of the two-component complex spinors is used. From the point of view of representation theory the general formula for the Lorentz-invariant expansion of the scattering amplitude is a decomposition of the direct product of the SL(2,C) finite-dimensional representations $[2s_a, 0]$ into irreducible representations (this fact was noticed by Hepp [4]).

Further on we find out identities for the invariant structures, with the help of which the general expansion can be written as a function of structures associated with a single channel (s - , t or u).

In Sec.3 A and B the conditions of P- and T-invariance of the amplitude are used for deriving the respective transformation laws for the invariant structures.

In Sec.3C the transformation laws for the invariant structures under CPT are derived.

In Sec.4 an illustration of the developed method for the cases of pion- nucleon and nucleon-nucleon scattering is given. As expected our results agree with the well-known formulae for these processes.

In Appendix 1 we give an explicit realization of the Dirac and Bargmann-Wigner formalism used in the main body of the paper.

Appendix 2 contains a table of the transformation laws for the Lorentz invariant structures under P and T operation.

2. Lorentz-invariant expansion of the scattering amplitude

2A. Proof of the main formula

Consider a process presented on Fig.1

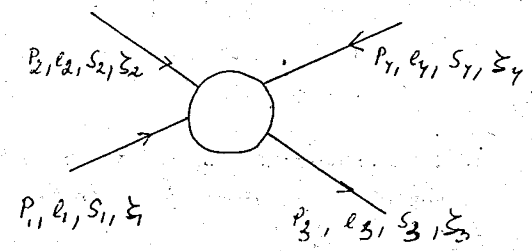

Fig.1

where p, e, s, ζ denotes four-momentum, charge, spin and spin projection of the a[th] particle. The conservation of momentum gives

$$p_1 + p_2 + p_3 + p_4 = 0. \tag{1}$$

All particles are on the mass shell

$$p_a^2 \equiv p_a^{0\,2} - \underline{p}_a^2 = M_a^2, \tag{2}$$

$$(a = 1,2,3,4).$$

We introduce the Mandelstam variables

$$S_1 \equiv S = P_1 + P_2 , \quad S^2 = s ,$$
$$S_2 \equiv T = P_1 + P_3 , \quad T^2 = t , \qquad (3)$$
$$S_3 \equiv U = P_1 + P_4 , \quad U^2 = u ,$$

with the well-known property:

$$\sum_{i=1}^{3} S_i^2 \equiv s + t + u = \sum_{a=1}^{4} M_a^2 .$$

The scattering amplitude of flat process is written on the form

$$T_{s_1 s_2 s_3 s_4}(P_1, e_1, e_1\zeta_1 ; \dots ; P_3, e_3, e_3\zeta_3 ; \dots). \qquad (4)$$

Let $\mathcal{U}^{(e)\,\alpha_1 \dots \alpha_{2s}}_{s,\zeta}(P)$ be a Bargmann-Wigner spinor for a particle of four-momentum P, charge e, spin s and spin projection $e\zeta$, $\{\alpha_1, \dots, \alpha_{2s}\} = 1,2,3,4$, which satisfies the equations x)

$$(\not{P}^{(2s)} - eM)\, \mathcal{U}^{(e)}_{s,\zeta}(P) = 0 ,$$

where

$$\not{P}^{(2s)} \equiv \overbrace{\not{P} \otimes \dots \otimes \not{P}}^{2s} ,$$

and

$$\underline{S}^2\, \mathcal{U}^{(e)}_{s,\zeta}(P) = s(s+1)\, \mathcal{U}^{(e)}_{s,\zeta}(P) ,$$
$$S_3\, \mathcal{U}^{(e)}_{s,\zeta}(P) = e\zeta\, \mathcal{U}^{(e)}_{s,\zeta}(P) . \qquad (5)$$

---

x) We omit the spinor indices in this and similar formulae.

(We have used a somewhat unusual notation for the spin projection, which allows us to write down all four solutions of the free Dirac equation in one formula).

In analogy with the case of spin 1/2 particle we introduce a Dirac conjugated spinor

$$\widetilde{\mathcal{U}^{(e)}_{s,\zeta}(P)} = \overline{\mathcal{U}^{(e)}_{s,\zeta}(P)}\, \gamma^{0\,(2s)} \qquad (6)$$

(the bar stands for complex conjugation). The Dirac conjugation transforms upper indices into lower indices.

In the basis on spinor space, used in Appendix 1, in which γ_5 is diagonal we introduce also the four-component complex spinors

$$\xi = \begin{pmatrix} Z \\ 0 \end{pmatrix} , \quad \tilde{\xi} = \xi C ,$$

where

$$C = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix} , \quad \varepsilon = i\,\sigma_2 , \quad Z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} ,$$

z_1, z_2 are complex numbers.

We associate with the physical amplitude on a given channel (12 $\rightarrow$ 34) a Bargmann-Wigner spintensor:

$$T_{s_1 s_2 s_3 s_4}(P_1, e_1 ; \dots ; P_3, e_3 ; \dots) =$$
$$= \mathcal{U}^{(e_1)}_{s_1,\zeta_1}(P_1)\, \mathcal{U}^{(e_2)}_{s_2,\zeta_2}(P_2)\, T_{s_1 s_2 s_3 s_4}(P_1, e_1, e_1\zeta_1 ; \dots) \times \qquad (7)$$
$$\times\, \mathcal{U}^{(e_3)}_{s_3,\zeta_3}(-P_3)\, \mathcal{U}^{(e_4)}_{s_4,\zeta_4}(-P_4)$$

In eq. (7) a sumation is to be carried over each repeated index ζ_a from $-s_a$ to s_a (a = 1,2,3,4). The minus signs of arguments of the Bargmann-Wigner spinor are dictated by the choice of the channel.

Further after multiplying by (7) from left and right with the necessary number of $\tilde{\xi}'s$ and $\xi's$, respectively, we obtain the invariant amplitude ;

$$\begin{aligned} T_{s_1s_2s_3s_4}(p_1,e_1,z_1;\dots;p_3,e_3,z_3;\dots) = \\ = \overset{(2s_1)}{\tilde{\xi}_1}\, \mathcal{U}^{(e_1)}_{s_1,\zeta_1}(p_1)\, \overset{(2s_2)}{\tilde{\xi}_2}\, \mathcal{U}^{(e_2)}_{s_2,\zeta_2}(p_2)\, T_{s_1s_2s_3s_4}(p_1,e_1,e\zeta;\dots)\times \\ \times \widetilde{\mathcal{U}^{(e_3)}_{s_3,\zeta_3}(-p_3)}\, \overset{(2s_3)}{\xi_3}\, \mathcal{U}^{(e_4)}_{s_4,\zeta_4}(-p_4)\, \overset{(2s_4)}{\xi_4} \end{aligned} \qquad (8)$$

The amplitude (8) satisfies the invariance condition :

$$T_{s_1s_2s_3s_4}(p_1,e_1,z;\dots) = T_{s_1s_2s_3s_4}(\Lambda(A)p_1, e_1, Az_1;\dots),$$

where

$$A \in SL(2,C)$$

and

$$A\,\underset{\sim}{p}\,A^* = \underset{\sim}{\Lambda(A)p} .$$

It is a homogeneous polynomial of degree $2s_a$ with respect to z_a (a = 1,2,3,4).

From the theory of the finitedimensional representations of the Lorentz group in the space of two-component complex spinors it follows that the amplitude (8) is a polynomial of invariants of the type ;

$$u_0(z_a,z_b), \quad u_j(z_a,z_b) = \tfrac{1}{2}\varepsilon_{jkl}\, z_a\, \varepsilon\, \underset{\sim}{S'_k}\, \tilde{S}'_l\, z_b ,$$

where z_a are two-component complex spinors and

$$\underset{\sim}{S_i} \equiv S_i^{\mu}\sigma_\mu = S_i^0 - \mathbf{S}_i\cdot\boldsymbol{\sigma} ,$$

$$\tilde{S}_i \equiv S_i'^{\mu} g_{\mu\mu}\sigma_\mu = S_i'^0 + \mathbf{S}'_i\cdot\boldsymbol{\sigma} .$$

So we write :

$$\begin{aligned} T_{s_1s_2s_3s_4}(p_1,e_1,z_1;\dots;p_3,e_3,z_3;\dots) = \\ = \sum_{j_1=|s_1-s_3|}^{s_1+s_3}\ \sum_{j_2=|s_2-s_4|}^{s_2+s_4}\ \sum_{j=|j_1-j_2|}^{j_1+j_2} P^{j_1}_{s_1s_3}(z_1,z_3;\partial_u)\times \\ \times P^{j_2}_{s_2s_4}(z_2,z_4;\partial_v)\, P^{j}_{j_1j_2}(u,v;\partial_z)\times \\ \times F^{(e_1e_2e_3e_4)}_{j_1j_2j}(z;S',T,U). \end{aligned} \qquad (9)$$

The monomials P and the functions F satisfy the homogeneity conditions

$$P^{j}_{j_1j_2}(\lambda_1 u,\lambda_2 v;\lambda_3\partial_z) = \lambda_1^{2j_1}\lambda_2^{2j_2}\lambda_3^{2j}\cdot P^{j}_{j_1j_2}(u,v;\partial_z),$$

$$F^{(e_1e_2e_3e_4)}_{j_1j_2j}(\lambda z;S',T,U) = \lambda^{2j} F^{(e_1e_2e_3e_4)}_{j_1j_2j}(z;S',T,U).$$

The monomials P have the form (cf. ref. [7]);

$$P^{j}_{j_1 j_2}(u,v;\partial_z) = N^{j}_{j_1 j_2}(u\varepsilon v)^{j_1+j_2-j}(u\partial_z)^{j_1-j_2+j}(v\partial_z)^{j_2-j_1+j} . \qquad (10)$$

The normalization constant can be determined from following requirement: If $f_j(z)$ and $F_{j_1 j_2}(u,v)$ are homogeneous polynomials of z and u,v of degree $2j$ and $2j_1, 2j_2$ respectively and

$$F_{j_1 j_2}(u,v) = \sum_{j=|j_1-j_2|}^{j_1+j_2} P^{j}_{j_1 j_2}(u,v;\partial_z) f_j(z) , \qquad (11a)$$

then $f_j(z)$ can be expressed conversely in terms of $F_{j_1 j_2}(u,v)$ by

$$f_j(z) = P^{j}_{j_1 j_2}(\partial_u,\partial_v;z) F_{j_1 j_2}(u,v) . \qquad (11b)$$

Conditions (11) are equivalent to the operator equations:

$$\sum_j P^{j}_{j_1 j_2}(u,v;\partial_z) P^{j}_{j_1 j_2}(\partial_u,\partial_v;z) = 1 \qquad (12)$$

and

$$P^{j}_{j_1 j_2}(\partial_u,\partial_v;z) P^{j'}_{j_1 j_2}(u,v;\partial_z) = \delta^{jj'} .$$

If we apply the operator equations (12) to the function

$$\frac{(au)^{2j_1}(bv)^{2j_2}}{(2j_1)!\,(2j_2)!} ,$$

take into account that

$$P^{j}_{j_1 j_2}(\partial_u,\partial_v;z)\frac{(au)^{2j_1}(bv)^{2j_2}}{(2j_1)!\,(2j_2)!} = P^{j}_{j_1 j_2}(a,b;z)$$

and compare both sides using the differentiation formulae

$$(a\partial_z)(bz) = (ab),$$

$$\frac{(a\partial_z)^n (bz)^m}{n!\;m!} = \frac{(ab)^n (bz)^{m-n}}{n!\;(m-n)!} ,$$

$$\frac{(a\partial_z)^{\alpha}(bz)^{\beta}(cz)^{\gamma}}{\alpha!\,\beta!\,\gamma!} = \sum_{k=0}^{\alpha}\frac{(ba)^{k}(ca)^{\alpha-k}(bz)^{\beta-k}(cz)^{\gamma-\alpha+k}}{k!\,(\alpha-k)!\,(\beta-k)!\,(\gamma-\alpha+k)!} ,$$

$$\frac{(a\partial_z)^{\alpha}(bz)^{\beta}(cz)^{\gamma}(dz)^{\delta}}{\alpha!\,\beta!\,\gamma!\,\delta!} = \sum_{s+t+u=\alpha}\frac{(ba)^{s}(ca)^{t}(da)^{u}}{s!\,t!\,u!}\times$$

$$\times\frac{(bz)^{\beta-s}(cz)^{\gamma-t}(dz)^{\delta-u}}{(\beta-s)!\,(\gamma-t)!\,(\delta-u)!} ,$$

the binomial equality

$$\sum_{k=a}^{b}\binom{k}{a}\binom{b}{k}\binom{c}{k} = \binom{b}{a}\binom{c+b-a}{b} ,$$

and the important identity

$$(u\varepsilon v)(a\varepsilon b) = (ua)(vb) - (ub)(va), \qquad (13)$$

we obtain

$$N^{j}_{j_1 j_2} = \sqrt{\frac{(2j+1)}{(j_1+j_2+j+1)!\,(j_1+j_2-j)!\,(j_1-j_2+j)!\,(j_2-j_1+j)!}}$$

The U- and V- differentiation in eq. (9) gives

$$T_{s_1 s_2 s_3 s_4}(p_1, e_1, z_1; \dots ; p_3, e_3, z_3; \dots) = \sum_{j_1, j_2, j} P^{j_1 j_2 j}_{s_1 s_2 s_3 s_4}(z_1, z_2, z_3, z_4; \partial_z) \times F^{(e_1 e_2 e_3 e_4)}_{j_1 j_2 j}(z; S, T, U), \tag{14}$$

where

$$P^{j_1 j_2 j}_{s_1 s_2 s_3 s_4}(z_1, z_2, z_3, z_4; \partial_z) = N^{j_1 j_2 j}_{s_1 s_2 s_3 s_4} (z_1 \varepsilon z_3)^{s_1+s_3-j_1} (z_2 \varepsilon z_4)^{s_2+s_4-j_2} \times \sum_{\substack{\alpha+\beta+\gamma+\delta= \\ = s_1+s_3-j}} \frac{(z_1 \varepsilon z_2)^{\alpha}}{\alpha!} \frac{(z_1 \varepsilon z_4)^{\beta}}{\beta!} \frac{(z_3 \varepsilon z_2)^{\gamma}}{\gamma!} \frac{(z_3 \varepsilon z_4)^{\delta}}{\delta!} \times \frac{(z_1 \partial_z)^{s_1-s_3+j_1-\alpha-\beta}}{(s_1-s_3+j_1-\alpha-\beta)!} \frac{(z_2 \partial_z)^{s_2-s_4+j_2-\alpha-\gamma}}{(s_2-s_4+j_2-\alpha-\gamma)!} \times \frac{(z_3 \partial_z)^{s_3-s_1+j_1-\gamma-\delta}}{(s_3-s_1+j_1-\gamma-\delta)!} \frac{(z_4 \partial_z)^{s_4-s_2-j_2-\delta-\beta}}{(s_4-s_2-j_2-\delta-\beta)!} \tag{15}$$

The factor $N^{j_1 j_2 j}_{s_1 s_2 s_3 s_4}$ has the form

$$N^{j_1 j_2 j}_{s_1 s_2 s_3 s_4} = \sqrt{\frac{(2j_1+1)(s_1-s_3+j_1)!\,(s_3-s_1+j_1)!\,(2j_2+1)(s_2-s_4+j_2)!\,(s_4-s_2+j_2)!}{(s_1+s_3+j_1+1)!\,(s_1+s_3-j_1)!\,(s_2+s_4+j_2+1)!}} \times \sqrt{\frac{(2j+1)(j_1+j_2-j)!\,(j_1-j_2+j)!\,(j_2-j_1+j)!}{(s_2+s_4-j_2)!\,(j_1+j_2+j+1)!}}, \tag{16}$$

The function $F^{(e_1, e_2, e_3, e_4)}_{j_1 j_2 j}(z; S, \dots)$ is determined from the homogeneity condition and the possibilities to form Lorentz invariants from z and S, T, U:

$$F^{(e_1 e_2 e_3 e_4)}_{j_1 j_2 j}(z; S, T, U) = \sum_{k=0}^{j} f^{(e_1 e_2 e_3 e_4)}_{j_1 j_2 j k}(s, t, u)\, u_1^{k}(z)\, u_2^{j-k}(z) + u_3(z) \sum_{k=0}^{j-1} g^{(e_1 e_2 e_3 e_4)}_{j_1 j_2 j k}(s, t, u)\, u_1^{k}(z)\, u_2^{j-1-k}(z). \tag{17}$$

Higher powers of $u_3(z)$ do not appear in the right-hand side of eq. (17) since they can be expressed in terms of powers of $u_1(z)$ and $u_2(z)$ from the identity

$$\left[S_1 u_1(z) + S_2 u_2(z) + S_3 u_3(z) \right]^2 = 0. \tag{18}$$

In the right-hand side of eq. (14) the z - differentiation can be done with the help of formulae of the type

$$\frac{(a\partial_z)^n}{n!} \frac{(zAz)^{\alpha}}{\alpha!} = \sum_{k=0}^{[\frac{n}{2}]} \frac{(zAz)^{\alpha-n+k}}{(\alpha-n+k)!} \frac{(aAz+zAa)^{n-2k}}{(n-2k)!} \frac{(aAa)^{k}}{k!},$$

$$\frac{(a\partial_z)^n (zAz)^\alpha (zBz)^\beta}{n!\;\alpha!\;\beta!} =$$

$$= \sum_{k=0}^{n}\;\sum_{\ell=0}^{[\frac{n-k}{2}]}\;\sum_{m=0}^{[\frac{k}{2}]} \frac{(zAz)^{\alpha-n+k+\ell}}{(\alpha-n+k+\ell)!} \times$$

$$\times \frac{(aAz+zAa)^{n-k+2\ell}}{(n-k+2\ell)!}\,\frac{(aAa)^{\ell}}{\ell!} \times$$

$$\times \frac{(aBz+zBa)^{k+2m}}{(k+2m)!}\,\frac{(aBa)^{m}}{m!} .$$

As a result the amplitude (14) is a polynomial of the Lorentz invariants:

$$u_0(z_a, z_b) = z_a \varepsilon z_b ,$$

$$u_j(z_a, z_b) = \frac{1}{2}\varepsilon_{jk\ell}\, z_a\, \varepsilon(\sigma_k \tilde{\sigma}_\ell - \sigma_\ell \tilde{\sigma}_k) z_b ,$$

where

$$\{a, b\} = 1, 2, 3, 4 .$$

On the other hand the way we do the transition from the spinor amplitude to the scalar one shows that the amplitude should be a function of structures associated with a given channel (s-, t- or u-channel) correspond to pairs of variables (12) (34), (13) (24) or (14) (23), respectively. As it can be seen from eq. (9) in the present paper we have chosen to work in the t- channel.

2B. Identities for the invariant structures

The remaining structures can be eliminated with the help of the identities:

$$u_0(z_1,z_3)u_0(z_2,z_4) - u_0(z_1,z_2)u_0(z_3,z_4) - u_0(z_1,z_4)u_0(z_2,z_3) = 0 ,$$

$$u_0(z_1,z_3)u_j(z_2,z_4) - u_0(z_1,z_2)u_j(z_3,z_4) - u_j(z_1,z_4)u_0(z_2,z_3) = 0 ,$$

$$\varepsilon_{kj\ell}\,\delta_k^{2}\, u_0(z_1,z_3)u_k(z_2,z_4) - u_j(z_1,z_2)u_\ell(z_3,z_4) + \qquad (19a)$$

$$+ u_j(z_2,z_3)u_\ell(z_1,z_4) = 0 ,$$

$$\delta_\ell^{2}\delta_k^{2}\, u_0(z_1,z_3)u_0(z_2,z_4) - u_j(z_1,z_2)u_j(z_3,z_4) +$$

$$+ u_j(z_1,z_4)u_j(z_2,z_3) = 0 ,$$

$$(k, j, \ell = 1, 2, 3)$$

which follow from the modified identity (13)

$$(z_1\varepsilon\sigma_{\mu_1}\sigma^{\mu_3}z_3)(z_2\varepsilon\sigma_{\mu_2}\sigma^{\mu_4}z_4) -$$

$$-(z_1\varepsilon\sigma_{\mu_1}\sigma^{\mu_2}z_2)(z_3\varepsilon\sigma_{\mu_3}\sigma^{\mu_4}z_4) - (z_1\varepsilon\sigma_{\mu_1}\sigma^{\mu_4}z_4)(z_2\varepsilon\sigma_{\mu_2}\sigma^{\mu_3}z_3) = 0 ,$$

$$\{\mu_1, \mu_2, \mu_3, \mu_4\} = 0, 1, 2, 3 .$$

Another necessary identity is found when the operator

$$(z_1\partial_z)(z_2\partial_z)(z_3\partial_z)(z_4\partial_z)$$

acts on eq. (18):

$$\sum_{j=1}^{3} \delta_j^{\,2}\left[u_j(z_1,z_3)\,u_j(z_2,z_4) + u_j(z_1,z_4)\,u_j(z_2,z_3) + u_j(z_1,z_2)\,u_j(z_3,z_4)\right] +$$

$$+\sum_{j,k,l=1}^{3} (\varepsilon_{jkl})^2 (\delta_k, \delta_l)\left[u_k(z_1,z_2)\,u_l(z_3,z_4) + \right.$$

$$\left. + u_k(z_1,z_3)\,u_l(z_2,z_4) + u_k(z_1,z_4)\,u_l(z_2,z_3)\right] = 0 . \qquad (19b)$$

Obviously, taking different differential operators and higher powers of equation (18), we can obtain identities relating more general structures.

3. Discrete operations

For obtaining the CPT-invariant expansion let us analyze the right-hand side of eq. (8)

$$T_{s_1 s_2 s_3 s_4}(p_1, q, z; \ldots) =$$
$$= \tilde{\zeta}_1^{(\alpha s)}\, u^{(e_1)}_{s_1 \zeta_1}(p_1) \ldots \overline{T}_{\zeta_1 s_2 s_3 s_4}(p_1, q, q\zeta_1; \ldots) \ldots \qquad (8)$$

Here we have two types of scalar products: one in the spin space coming from the summation over ζ's and another in Dirac space. So, the product

$$\tilde{\tilde{\zeta}}^{(\alpha s)}\, u^{(e)}_{s,\zeta}(p) \qquad \left(\widetilde{u^{(e)}_{s,\zeta}(p)}\; \tilde{\zeta}^{(\alpha s)}\right)$$

is a scalar in Dirac space and a spinvector in spin space.

The transformation properties of each single ζ under space and time reflection will be determined from the requirements

$$I\tilde{\zeta}^{(\alpha s)}\, u^{(e)}_{s,\zeta}(p) = \tilde{\zeta}^{(\omega)}\, I u^{(e)}_{s\zeta}(p)$$

or

$$I\tilde{\zeta}^{(\omega)}\, u^{(e)}_{s,\zeta}(p) = \widetilde{I u^{(e)}_{s\zeta}(p)}\;\tilde{\zeta}^{(\omega)}$$

when I involves time reversal.

3A. Space reflection

The physical amplitude (4) transforms as

$$I_s\, T_{s_1 s_2 s_3 s_4}(p_1, q, q\zeta_1; \ldots) = \eta_s\, T(I_s p_1, q, q\zeta_1; \ldots),$$

where η_s is the phase factor of space reflection (which is a product of four such factors, one for each particle), and

$$I_s p \equiv (p_0, -\underline{p}) .$$

Using the space-invariant condition

$$\widetilde{I_s \zeta}^{(\alpha s)}\, u^{(e)}_{s,\zeta}(p) = \tilde{\tilde{\zeta}}^{(\alpha s)}\, I_s\, u^{(e)}_{s,\zeta}(p)$$

with the help of the identity for the Bargmann-Wigner spinor (see Appendix I)

$$\mathcal{U}^{(e)}_{s,\zeta}(p) = \left(\gamma_0 \frac{\overset{*}{p}}{M}\right)^{(2s)} \mathcal{U}^{(e)}_{s,\zeta}(I_s p)$$

we derive

$$I_s z_a = \frac{\tilde{p}_a}{M_a} z_a \quad (a=1,2,3,4) \tag{20}$$

The invariants $\mathcal{U}_\mu(z_a, z_b)$ transform as

$$I_s \mathcal{U}_0(z_a,z_b) = \frac{1}{M_a M_b} z_a \varepsilon \underset{\sim}{p_a} \tilde{p}_b z_b ,$$

$$I_s \mathcal{U}_j(z_a,z_b) = \frac{1}{2} \varepsilon_{jkl}\frac{1}{M_a M_b} z_a \varepsilon \underset{\sim}{p_a}\left(\tilde{s}_k \underset{\sim}{s}_l - \tilde{s}_l \underset{\sim}{s}_k\right)\tilde{p}_b z_b \tag{21}$$

(See Appendix 2).

The amplitude will be invariant under space reflection if and only if

$$T_{s_1 s_2 s_3 s_4}(p_1,q_1,z_1;\dots;p_3,q_3,z_3;\dots) = (-1)^{2s_3+2s_4}\eta_s T_{s_1 s_2 s_3 s_4}(I_s p_1,q_1,I_s z_1;\dots;I_s p_3,q_3;I_s z_3;\dots) \tag{22}$$

where the phase factor $(-1)^{2s_3+2s_4}$ results from the minus signs of the arguments in eq. (8).

3B. Time reversal

Under time-reversal $I = I_t$ the amplitude (4) transforms as

$$I_t T_{s_1 s_2 s_3 s_4}(p_1,q_1,q_1\zeta_1;\dots;p_3,q_3,q_3\zeta_3;\dots) = \eta_t \prod_{a=1}^{4}(-1)^{s_a-\zeta_a} T_{s_3 s_4 s_1 s_2}(I_s p_3,q_3,-q_3\zeta_3;\dots;I_s p_1,q_1,-q_1\zeta_1;\dots),$$

where η_t is the phase factor of time reversal. Using then the T- invariance condition

$$\widetilde{I_t \zeta}^{(2s)}\, \mathcal{U}^{(e)}_{s,\zeta}(p) = \widetilde{I_t\, \mathcal{U}^{(e)}_{s,\zeta}(p)}\, \zeta^{(2s)}$$

with the help of the identity for the Bargmann-Wigner spinor (see Appendix 1) :

$$\mathcal{U}^{(e)}_{s,\zeta}(p) = (-1)^{s-\zeta}\, \widetilde{\mathcal{U}^{(e)}_{s,-\zeta}(I_s p)} \left(\frac{\overset{*}{p}}{M}\, {}^{t}V_T\right)^{(2s)}$$

we derive the time-invariant condition for the amplitude

$$T_{s_1 s_2 s_3 s_4}(p_1,q_1,z_1;\dots;p_3,q_3,z_3;\dots) = (-1)^{2s_3+2s_4}\eta_t T_{s_3 s_4 s_1 s_2}(I_s p_3,q_3,I_s z_3;\dots;I_s p_1,q_1,I_s p_1;\dots). \tag{23}$$

Hence, we have

$$I_t\, z_1, z_2 = I_s\, z_{3,4}$$

$$I_t\, z_2, z_4 = I_s\, z_1, z_3 \qquad (24)$$

and

$$I_t\, \underset{\sim}{S_1} = -\widetilde{S_1} \qquad (25)$$

$$I_t\, \underset{\approx}{S_2} = \widetilde{S_2} \qquad I_t\, \underset{\sim}{S_3} = -\widetilde{S_3}$$

The invariants $\mathcal{U}_\mu(z_a, z_b)$ transform under time-reversal as follows

$$I_t\, \mathcal{U}_0(z_a, z_b) = \frac{1}{M_a M_b}\, z_{a'}\, \varepsilon\, \underset{\sim}{p_{a'}}\, \widetilde{p_{b'}}\, z_{b'} ,$$

$$I_t\, \mathcal{U}_j(z_a, z_b) = (-1)^j \varepsilon_{jkl} \frac{1}{2M_{a'}M_{b'}}\, z_{a'}\, \varepsilon\, \underset{\sim}{p_{a'}} \left(\underset{\approx}{\widetilde{S_k}}\, \underset{\sim}{S_l} - \widetilde{S_l}\, \underset{\sim}{S_k}\right) p_{b'}\, z_{b'} .$$

3C. CPT-transformation

Under CPT-transformation $I = I_{CPT}$ the amplitude (4) transforms as:

$$I_{CPT}\, T_{s_1 s_2 s_3 s_4}(p_1, e_1, e_1\xi_1; \dots; p_3, e_3, e_3\xi_3; \dots) =$$

$$= \eta_{CPT} \prod (-1)^{s_a - \xi_a}\, T_{s_3 s_4 s_1 s_2}(p_3, -e_3, e_3\xi_3; \dots; p_1, -e_1, e_1\xi_1; \dots)$$

Using the identity

$$C^{(\omega)}\, \mathcal{U}^{(e)}_{s,\xi}(p) = -(-1)^{s-\xi}\, \widetilde{\mathcal{U}^{(-e)}_{s,\xi}(p)}$$

in the same way like in the previous subsections we find, that the CPT invariance reduces to invariant of the amplitude (8)

$$T_{s_1 s_2 s_3 s_4}(p_1, e_1, z_1; \dots; p_3, e_3, z_3, \dots)$$

under the transformations

$$z_1 \leftrightarrow z_3, \quad z_2 \leftrightarrow z_4; \qquad (26)$$

$$e_1 \leftrightarrow -e_3, \quad e_2 \leftrightarrow e_4; \qquad (27)$$

and

$$S_1 \to -S_1, \quad S_2 \to S_2, \quad S_3 \to -S_3. \qquad (28)$$

Obviously, the invariance under (27) imposes restrictions only on the invariant scalar amplitudes $f(s, t, u)$.

4. Application to the case of πN and NN scattering

Pion-nucleon scattering

Using the formulae (9), (14), (15), (16) and (17), we find with $S_1 = S_3 = \frac{1}{2}$, $S_2 = S_4 = 0$

$$T_{\pi N}(p_1, q_1, z_1; \dots) = \tag{29}$$

$$= f_0(s,t,u)\, U_0(z_1,z_3) + f_1(s,t,u)\, U_1(z_1,z_3) + f_2(s,t,u)\, U_2(z_1,z_3) + f_3(s,t,u)\, U_3(z_1,z_3),$$

where

$$U_0(z_1,z_3) = \tilde{z}_1 \varepsilon z_3,$$

$$U_j(z_1,z_3) = \varepsilon_{jkl}\, \tilde{z}_1 \varepsilon \underset{\sim}{S_k} \tilde{S}_l z_3$$

We apply the space reflection operation to eq. (30) and require that the amplitude satisfies the P- invariance condition

$$T_{\pi N}(p_1, q_1, z_1; \dots) = -T_{\pi N}(I_s p_1, q_1, I_s z_1; \dots).$$

It follows that only two of the invariant scalar functions $f(s,t,u)$ are linear independence. So the even parity amplitude can be written in the form

$$T_{\pi N}(p_1, q_1, z_1; \dots) =$$

$$= \frac{A}{4}\left[2(s+u)\, U_0(z_1,z_3) + U_1(z_1,z_3) - U_3(z_1,z_3)\right] + \tag{30}$$

$$+ \frac{MB}{2}\left[2(s-u)\, U_0(z_1,z_3) - U_1(z_1,z_3) - U_3(z_1,z_3)\right],$$

where

$$\frac{A}{2} = f_1 - f_3, \quad MB = -f_1 - f_3,$$

and M is the nucleon mass.

Eq. (30) agrees exactly with the familiar result (see ref.[2,3])

$$A + B \hat{Q},$$

where

$$Q = \frac{p_2 + p_4}{2},$$

as it is easy to verify with the help of the formula

$$T_{\pi N}(p_1, q_1, z_1; \dots) =$$

$$= \tilde{\bar{z}}_1 (\hat{p}_1 + M)(A + B\hat{Q})(-\hat{p}_3 + M) \bar{z}_3.$$

Further in the same way we find the odd parity πN- amplitude

$$T_{\pi N}(\dots) =$$

$$= A_1\left[-2t\, U_0(z_1,z_3) + U_1(z_1,z_3) - U_3(z_1,z_3)\right] +$$

$$+ A_2\, U_2(z_1,z_3).$$

For the even T-invatiant πN- amplitude we obtain

$$T_{\pi N}(\dots) =$$

$$= \frac{A}{4}\left[2(s+u)\,\mathcal{U}_0(z_1,z_3) + \mathcal{U}_1(z_1,z_3) - \mathcal{U}_3(z_1,z_3)\right] +$$

$$+ \frac{-MB}{2}\left[2(u-s)\,\mathcal{U}_0(z_1,z_3) + \mathcal{U}_1(z_1,z_3) + \mathcal{U}_3(z_1,z_3)\right] + \tag{31}$$

$$+ MC\,\mathcal{U}_2(z_1,z_3),$$

where

$$A = 2(f_1 - f_3), \quad MB = -f_1 - f_3, \quad MC = f_2 .$$

Hence, the P- and T-invariant πN- amplitude is given by (30).

Nucleon-nucleon scattering

In the case of $s_1 = s_2 = s_3 = s_4 = 1/2$ we find from (9), (14), (15), (16) and (17) the Lorentz invariant expansion of the amplitude x)

$$T_{NN}(p_1, z_1; \dots ; p_3, z_3; \dots) =$$

$$= \frac{1}{2} f_1\, \mathcal{U}_0(13)\mathcal{U}_0(24) + \frac{1}{2\sqrt{3}} f_2\left[\mathcal{U}_0(12)\mathcal{U}_0(34) - \mathcal{U}_0(14)\mathcal{U}_0(23)\right] +$$

$$+ \frac{1}{2} f_3\, \mathcal{U}_0(13)\mathcal{U}_1(24) + \frac{1}{2} f_4\, \mathcal{U}_0(13)\mathcal{U}_2(24) + \frac{1}{2} f_5\, \mathcal{U}_0(13)\mathcal{U}_3(24) +$$

$$+ \frac{1}{2} f_6\, \mathcal{U}_1(13)\mathcal{U}_0(24) + \frac{1}{2} f_7\, \mathcal{U}_2(13)\mathcal{U}_0(24) + \frac{1}{2} f_8\, \mathcal{U}_3(13)\mathcal{U}_0(24) +$$

x) We use the abbreviate notation $\mathcal{U}_\mu(z_a, z_b) \to \mathcal{U}_\mu(a,b)$.

$$+ \frac{1}{4\sqrt{2}} f_9\left[\mathcal{U}_0(12)\mathcal{U}_1(34) + \mathcal{U}_1(12)\mathcal{U}_0(34) - \mathcal{U}_0(23)\mathcal{U}_1(14) + \mathcal{U}_1(23)\mathcal{U}_0(14)\right] +$$

$$+ \frac{1}{4\sqrt{2}} f_{10}\left[\mathcal{U}_0(12)\mathcal{U}_2(34) + \mathcal{U}_2(12)\mathcal{U}_0(34) - \mathcal{U}_0(23)\mathcal{U}_2(14) + \mathcal{U}_2(23)\mathcal{U}_0(14)\right] +$$

$$+ \frac{1}{4\sqrt{2}} f_{11}\left[\mathcal{U}_0(12)\mathcal{U}_3(34) + \mathcal{U}_3(12)\mathcal{U}_0(34) - \mathcal{U}_0(23)\mathcal{U}_3(14) + \mathcal{U}_3(23)\mathcal{U}_0(14)\right] +$$

$$+ \frac{1}{2\sqrt{6}} f_{12}\left[\mathcal{U}_1(12)\mathcal{U}_1(34) + \mathcal{U}_1(13)\mathcal{U}_1(24) + \mathcal{U}_1(23)\mathcal{U}_1(14)\right] +$$

$$+ \frac{1}{2\sqrt{6}} f_{13}\left[\mathcal{U}_2(12)\mathcal{U}_2(34) + \mathcal{U}_2(13)\mathcal{U}_2(24) + \mathcal{U}_2(23)\mathcal{U}_2(14)\right] + \tag{32}$$

$$+ \frac{1}{4\sqrt{6}} f_{14}\left[\mathcal{U}_1(12)\mathcal{U}_2(34) + \mathcal{U}_1(13)\mathcal{U}_2(24) + \mathcal{U}_1(14)\mathcal{U}_2(23) + \mathcal{U}_2(12)\mathcal{U}_1(34) + \right.$$
$$\left. + \mathcal{U}_2(13)\mathcal{U}_1(24) + \mathcal{U}_2(14)\mathcal{U}_1(23)\right] +$$

$$+ \frac{1}{4\sqrt{6}} f_{15}\left[\mathcal{U}_2(12)\mathcal{U}_3(34) + \mathcal{U}_2(13)\mathcal{U}_3(24) + \mathcal{U}_2(14)\mathcal{U}_3(23) + \mathcal{U}_3(12)\mathcal{U}_2(34) + \right.$$
$$\left. + \mathcal{U}_3(13)\mathcal{U}_2(24) + \mathcal{U}_3(14)\mathcal{U}_2(23)\right] +$$

$$+ \frac{1}{4\sqrt{6}} f_{16}\left[\mathcal{U}_3(12)\mathcal{U}_1(34) + \mathcal{U}_3(13)\mathcal{U}_1(24) + \mathcal{U}_3(14)\mathcal{U}_1(23) + \mathcal{U}_1(12)\mathcal{U}_3(34) + \right.$$
$$\left. + \mathcal{U}_1(13)\mathcal{U}_3(24) + \mathcal{U}_1(14)\mathcal{U}_3(23)\right].$$

As it can be seen in the invariant expression (33) all three kinds of structures (13),(24),(12),(34) and (14),(23) appear. But, because we are dealing with the t-channel, our amplitude should be a function of the structures (13),(24) only. With the help of the identities (19) all structures in (32) can be expressed in terms of the sixteen linear independent structures $\mathcal{U}_\mu(13)\ \mathcal{U}_\nu(24)$ $(\{\mu,\nu\} = 0,1,2,3)$:

$$
\begin{aligned}
T_{NN} &= \tfrac{1}{2} f_1\, u_0(13)u_0(24) + \\
&+ \frac{1}{2\sqrt{3}}\,\frac{f_2}{s_1 s_2 s_3}\Big(s_1\, u_1(13)u_1(24) + s_2\, u_2(13)u_2(24) + s_3\, u_3(13)u_3(24)\Big) + \\
&+ \tfrac{1}{2} f_3\, u_0(13)u_1(24) + \tfrac{1}{2} f_4\, u_0(13)u_2(24) + \tfrac{1}{2} f_5\, u_0(13)u_3(24) + \\
&+ \tfrac{1}{2} f_6\, u_1(13)u_0(24) + \tfrac{1}{2} f_7\, u_2(13)u_0(24) + \tfrac{1}{2} f_8\, u_3(13)u_0(24) + \\
&+ \frac{1}{2\sqrt{2}}\,\frac{f_9}{s_1}\Big[u_2(13)u_3(24) - u_3(13)u_2(24)\Big] + \\
&+ \frac{1}{2\sqrt{2}}\,\frac{f_{10}}{s_2}\Big[u_3(13)u_1(24) - u_1(13)u_3(24)\Big] + \\
&+ \frac{1}{2\sqrt{2}}\,\frac{f_{11}}{s_3}\Big[u_1(13)u_2(24) - u_2(13)u_1(24)\Big] + \\
&+ \frac{1}{2\sqrt{6}}\,\frac{f_{12}}{s_1^2}\Big[2s_1^2\, u_1(13)u_1(24) + s_2^2\, u_2(13)u_2(24) + s_3^2\, u_3(13)u_3(24)\Big] + \\
&+ \frac{1}{2\sqrt{6}}\,\frac{f_{13}}{s_2^2}\Big[s_1^2\, u_1(13)u_1(24) + 2s_2^2\, u_2(13)u_2(24) + s_3^2\, u_3(13)u_3(24)\Big] + \\
&+ \frac{3}{4\sqrt{6}} f_{14}\Big[u_1(13)u_2(24) + u_2(13)u_1(24)\Big] + \\
&+ \frac{3}{4\sqrt{6}} f_{15}\Big[u_2(13)u_3(24) + u_3(13)u_2(24)\Big] + \\
&+ \frac{3}{4\sqrt{6}} f_{16}\Big[u_3(13)u_1(24) + u_1(13)u_3(24)\Big].
\end{aligned}
\tag{33}
$$

The P-invariance of the amplitude means

$$
T_{NN}(p_a, z_a) = T_{NN}(I_s\, p_a,\; I_a\, z_a)\,.
$$

This requirement imposed with the help of the transformation laws in Appendix 2 leads to relations between the scalar function $f_K(s,t,u)$ $(K = 1, 2, \dots, 16)$ so that only eight of them $(G_m(s,t,u),\ m = 1,2,\dots,8)$ are linear by independent:

$$
\begin{aligned}
T_{NN} &= \\
&= G_1 \Big\{ (s+u)^2 u_0(13)u_0(24) + \\
&\qquad + \tfrac{1}{2}(s+u)\Big[u_0(13)u_1(24) + u_1(13)u_0(24)\Big] + \\
&\qquad + \tfrac{1}{2}(s+u)\Big[u_0(13)u_3(24) - u_3(13)u_0(24)\Big] + \\
&\qquad + \tfrac{1}{4}\Big[u_1(13)u_3(24) - u_3(13)u_1(24) + u_1(13)u_1(24) - u_3(13)u_3(24)\Big]\Big\} + \\
&+ M G_2 \Big\{ (s^2 - u^2)\, u_0(13)u_0(24) + \\
&\qquad + \tfrac{1}{2} u\Big[u_0(13)u_1(24) + u_1(13)u_0(24)\Big] + \\
&\qquad + \tfrac{1}{2} s\Big[u_0(13)u_3(24) - u_3(13)u_0(24)\Big] + \\
&\qquad - \tfrac{1}{4}\Big[u_1(13)u_1(24) + u_3(13)u_3(24)\Big]\Big\} + \\
&+ M^2 G_3\; u_2(13)u_2(24) +
\end{aligned}
$$

$$
\begin{aligned}
&+M^2 G_4 \Big\{ (s-u)^2 u_0(13)\, u_0(24) - \\
&\quad - \tfrac{1}{2}(s-u)\big[u_0(13)u_4(24) + u_4(13)u_0(24)\big] + \\
&\quad + \tfrac{1}{2}(s-u)\big[u_0(13)u_3(24) - u_3(13)u_0(24)\big] + \\
&\quad + \tfrac{1}{4}\big[-u_4(13)u_3(24) + u_3(13)u_4(24) + u_4(13)u_4(24) - u_3(13)u_3(24)\big]\Big\} + \\
&+ G_5 \Big\{ \tfrac{1}{4} t^2 u_0(13)\, u_0(24) + \\
&\quad + \tfrac{1}{2} t \big[-u_0(13)u_4(24) - u_4(13)u_0(24) - \\
&\qquad - u_0(13)u_3(24) + u_3(13)u_0(24)\big] + \\
&\quad + \tfrac{1}{4}\big[u_4(13)u_3(24) - u_3(13)u_4(24) + \\
&\qquad + u_4(13)u_4(24) - u_3(13)u_3(24)\big]\Big\} + \\
&+ M G_6 \Big\{ \tfrac{1}{2} s \big[u_0(13)u_4(24) - u_4(13)u_0(24)\big] - \\
&\quad - \tfrac{1}{2} u \big[u_0(13)u_3(24) + u_3(13)u_0(24)\big] - \\
&\quad - \tfrac{1}{4}\big[u_4(13)u_3(24) + u_3(13)u_4(24)\big]\Big\} + \\
&+ G_7 \Big\{\big[-2t\, u_0(13) + u_4(13) - u_3(13)\big]\, u_2(24)\Big\} + \\
&+ G_8 \Big\{ u_2(13)\big[-2t\, u_0(24) + u_4(24) + u_3(24)\big]\Big\} ,
\end{aligned}
\qquad (34)
$$

where for convenience the substitutions are used
$s_1 \to s,\ s_2 \to u,\ s_3 \to t$.

The T- invariance requirement implies in addition the vanishing of the functions G_7 and G_8 .

Further the structure with coefficient G_6 does not satisfy the symmetry property for identical particles , i.e. it is not symmetric under the substitutions $1 \leftrightarrow 2$ and $3 \leftrightarrow 4$. In fact, it changes sign under these substitutions:

$$
\begin{aligned}
u_4(13) &\leftrightarrow u_4(24) , & S_1 &\to S_1' , \\
u_2(13) &\leftrightarrow -u_2(24) , & S_2' &\to -S_2' , \\
u_3(13) &\leftrightarrow -u_3(24) , & S_3 &\to -S_3 .
\end{aligned}
$$

So, it follows that

$$G_6 = 0 .$$

Thus, the full Lorentz reflection invariant amplitude for NN scattering has the form (34) with

$$G_6 = G_7 = G_8 = 0 .$$

This expression agrees with the well-known result for NN scattering[1] :

$$
\begin{aligned}
T_{NN}(P_c) = {} & G_1\, 1 \otimes 1 + G_2 \big[\not{K} \otimes 1 + 1 \otimes \not{P}\big] + \\
& + G_3\, \not{K} \otimes \not{P} + G_4\, (i\gamma_5 \not{K}) \otimes (i\gamma_5 \not{P}) + \\
& + G_5\, i\gamma_5 \otimes i\gamma_5 ,
\end{aligned}
$$

where

$$K = p_2 - p_4 , \quad P = p_1 - p_3 ,$$

as it can be seen with the help of the formula

$$T_{NN}(p_a, z_a) =$$
$$= \tilde{\xi}_1 (\hat{p}_1 + M) \tilde{\xi}_2 (\hat{p}_2 + M) T_{NN}(p_a) (-\hat{p}_3 + M) \xi_3 (-\hat{p}_4 + M) \xi_4 .$$

5. Conclusions

Summarizing, we may assert, that a presoription for the oovariant expansion of the soattering amplitude for particles of arbitrary spin is found. Furthermore we can oonstruot the modulus of the amplitude, whioh is related to the oross seotion of a soattering prooess

$$\left| T_{s_1 s_2 s_3 s_4}(p_a, l_a) \right|^2 = \frac{1}{(2s_1)!(2s_2)!(2s_3)!(2s_4)!} \times$$
$$\times \overline{T}_{s_1 s_2 s_3 s_4}(p_a, l_a, \partial_{z_a}) T_{s_1 s_2 s_3 s_4}(p_a, l_a, z_a) ,$$

where the oomplex conjugation concerns only the invariant soalar amplitudes.

In order to find the explioit covariant expansion of the amplitude and hence the cross seotion for particles of any spin it might be useful to apply the so-called therminal systems of computer teohnique (see ref. [8]) .

Acknowledgements:

The authors are pleased to thank Professor I.T.Todorov for his oonstruotive advices,useful disoussions and for oritical reading of the manusoript.

Appendix I

The Dirac formalism and identities for Bargmann-Wigner spinors

Let

$$\{ \gamma_\mu , \gamma_\nu \} = 2 g_{\mu\nu} ,$$

where

$$(g_{\mu\nu}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

For the Dirac γ - matrices we choose the representation ;

$$\gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \gamma^j = \begin{pmatrix} 0 & -\sigma_j \\ \sigma_j & 0 \end{pmatrix} , \quad \gamma^5 \equiv \gamma^0 \gamma^1 \gamma^2 \gamma^3 = -i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

We use also the matrices

$$C = - C^{-1} = - {}^{t}C = - \overset{*}{C} = i \gamma^0 \gamma^2 = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix} ,$$

$$V_T = - V_T^{-1} = - {}^{t}V_T = - \overset{*}{V}_T = i \gamma^0 \gamma^5 C = \begin{pmatrix} 0 & \varepsilon \\ \varepsilon & 0 \end{pmatrix} ,$$

with the properties ;

$$\gamma^0 \gamma^\mu \gamma^0 = g^{\mu\mu} \gamma^\mu = \overset{*}{\gamma}{}^\mu ,$$

$$C \gamma^\mu C^{-1} = - {}^{t}\gamma^\mu ,$$

$$V_T \gamma^\mu V_T^{-1} = \overline{\gamma^\mu} .$$

As usual we denote

$$\not{p} \equiv p_\mu \gamma^\mu = \begin{pmatrix} 0 & \underset{\sim}{p} \\ \tilde{p} & 0 \end{pmatrix} ,$$

where

$$\underset{\sim}{p} \equiv p^\mu \sigma_\mu = p_0 - \underline{p} . \underline{\sigma} , \quad \tilde{p} \equiv g_{\mu\mu} p^\mu \sigma_\mu = p_0 + \underline{p} . \underline{\sigma} .$$

The two-dimensional representation of the Lorentz boost is

$$B (\underset{\sim}{p}) = \frac{M + \underset{\sim}{p}}{\sqrt{2M(\omega + M)}} ,$$

where

$$\omega = + \sqrt{M^2 + \underline{p}^2} = | p_0 | .$$

It is seen that the boost matrix is positive definite. One readily verifies that

$$[B (\underset{\sim}{p})]^2 = \frac{\underset{\sim}{p}}{M}$$

or equivalently

$$\underset{\sim}{p} \; B (\tilde{p}) = M \; B (\underset{\sim}{p}) .$$

The useful relations

$$\varepsilon \; \underset{\sim}{p} \; \varepsilon^{-1} = {}^{t}\tilde{p} = \overline{\tilde{p}}$$

follow from

$$ {}^{t}\sigma_{\mu} = \varepsilon\, \sigma^{\mu}\, \varepsilon^{-1} , $$

The solutions of the Dirac equation

$$ (\not{p} - e M)\, \mathcal{U}^{(e)}_{\xi}(p) = 0 $$

where the charge $e = \pm 1$ and the spin projection $\xi = \pm 1/2$, have the form

$$ \mathcal{U}^{(e)}_{\xi}(p) = \begin{pmatrix} B(p)\, \ell_{e\xi} \\ e\, B(\tilde{p})\, \ell_{e\xi} \end{pmatrix} . $$

The two-component spinors $\ell_{e\xi}$ are determined from the equations

$$ \sigma_3\, \ell_{\xi} = \xi\, \ell_{\xi} , $$

$$ \varepsilon\, \ell_{\xi} = -(-1)^{\frac{1}{2}-\xi}\, \ell_{-\xi} , $$

with the normalization

$$ \overset{*}{\ell}_{\xi}\, \ell_{\xi'} = M\, \delta_{\xi\xi'} . $$

It follows then that the spinors $\mathcal{U}^{(e)}_{\xi}(p)$ satisfy the conditions

$$ \overset{*}{\mathcal{U}}{}^{(e)}_{\xi}(p)\, \mathcal{U}^{(e)}_{\xi'}(p) = 2 p_0\, \delta_{\xi\xi'} , $$

$$ \widetilde{\mathcal{U}^{(e)}_{\xi}(p)}\, \mathcal{U}^{(e)}_{\xi'}(p) = 2 M e\, \delta_{\xi\xi'} $$

and the completeness relation

$$ \sum_{\xi} \mathcal{U}^{(e)}_{\xi}(p) \otimes \widetilde{\mathcal{U}^{(e)}_{\xi}(p)} = \not{p} + e M $$

It is not difficult to verify the following identities:

$$ \mathcal{U}^{(e)}_{\xi}(p) = \gamma^0\, \frac{\overset{*}{\not{p}}}{M}\, \mathcal{U}^{(e)}_{\xi}(I_s p) , $$

$$ \widetilde{\mathcal{U}^{(e)}_{\xi}(p)} = \widetilde{\mathcal{U}^{(e)}_{\xi}(I_s p)}\, \frac{\overset{*}{\not{p}}}{M}\, \gamma_0 ; $$

$$ \mathcal{U}^{(e)}_{\xi}(p) = (-1)^{\frac{1}{2}-\xi}\, \widetilde{\mathcal{U}^{(e)}_{-\xi}(I_s p)}\, \frac{\overset{*}{\not{p}}}{M}\, {}^{t}V_T , $$

$$ \widetilde{\mathcal{U}^{(e)}_{\xi}(p)} = (-1)^{\frac{1}{2}-\xi}\, V_T\, \frac{\overset{*}{\not{p}}}{M}\, \mathcal{U}^{(e)}_{-\xi}(I_s p) ; $$

$$ \mathcal{U}^{(e)}_{\xi}(p) = (-1)^{\frac{1}{2}-\xi}\, C\, \widetilde{\mathcal{U}^{(-e)}_{\xi}(p)} , $$

$$ \widetilde{\mathcal{U}^{(e)}_{\xi}(p)} = (-1)^{\frac{1}{2}-\xi}\, {}^{t}C\, \mathcal{U}^{(-e)}_{\xi}(p) . $$

with the help of the relations :

$$\gamma^{0}\, u_{\xi}^{(e)}(p) = e\, u_{\xi}^{(e)}(I_s p)\,,$$

$$i\gamma_5\, u_{\xi}^{(e)}(p) = u_{-\xi}^{(-e)}(p)\,,$$

$$C\, u_{\xi}^{(e)}(p) = -e\,(-1)^{\frac{1}{2}-\xi}\, \overline{u_{\xi}^{(-e)}(I_s p)}\,,$$

$$V_T\, u_{\xi}^{(e)}(p) = -(-1)^{\frac{1}{2}-\xi}\, \overline{u_{-\xi}^{(e)}(p)}\,.$$

The relations given above can be verified also without making use of the explicit realization of the solutions of the Dirac equation.

One can easy check that they are fulfiled for the Bargmann-Wigner spinors also, if the Dirac algebra is constructed like in eq. (5) and eq. (6) :

$$\gamma_0^{(2s)}\, u_{s,\xi}^{(e)}(p) = e\, u_{s,\xi}^{(e)}(I_s p)\,,$$

$$i\gamma_5^{(2s)}\, u_{s,\xi}^{(e)}(p) = u_{s,-\xi}^{(-e)}(p)\,,$$

$$C^{(2s)}\, u_{s,\xi}^{(e)}(p) = -e\,(-1)^{s-\xi}\, \overline{u_{s,\xi}^{(-e)}(I_s p)}\,,$$

$$V_T^{(2s)}\, u_{s,\xi}^{(e)}(p) = -(-1)^{s-\xi}\, \overline{u_{s,-\xi}^{(e)}(p)}\,,$$

Appendix 2

Table of the P and T-transformation laws for the invariant structures

If

$$M_1 = M_3 = M\,, \qquad M_2 = M_4 = m\,,$$

we have the following transformation properties for space reflection :

$$I_s u_0(13) = \frac{1}{4M^2}\left[(-s+t-u)\,u_0(13) - u_1(13) + u_3(13)\right],$$

$$I_s u_1(13) = \frac{1}{4M^2}\left[-4t\,(u+\delta(U))\,u_0(13) - (s+t-u)\,u_1(13) - 2(u+\delta(U))\,u_3(13)\right],$$

$$I_s u_2(13) = u_2(13)\,,$$

$$I_s u_3(13) = \frac{1}{4M^2}\left[4t\,(s+\delta(U))\,u_0(13) - 2(s+\delta(U))\,u_1(13) + (s-t-u)\,u_3(13)\right];$$

$$I_s u_0(24) = \frac{1}{4m^2}\left[(s-t-u)\,u_0(24) - u_1(24) - u_3(24)\right],$$

$$I_s u_1(24) = \frac{1}{4M^2}\left[-4t\,(u+\delta(U))\,u_0(24) - (s+t-u)\,u_1(24) + 2(u+\delta(U))\,u_3(24)\right],$$

$$I_s u_2(24) = u_2(24)\,,$$

$$I_s u_3(24) = \frac{1}{4m^2}\left[-4t\,(s+\delta(U))\,u_0(24) + 2(s+\delta(U))\,u_1(24) + (s-t-u)\,u_3(24)\right];$$

and for time reversal:

$$I_t u_0(13) = -I_s u_0(13)\,, \qquad I_t u_0(24) = -I_s u_0(24)\,,$$

$$I_t u_1(13) = -I_s u_1(13)\,, \qquad I_t u_1(24) = -I_s u_1(24)\,,$$

$$I_t u_2(13) = I_s u_2(13)\,, \qquad I_t u_2(24) = I_s u_2(24)\,,$$

$I_t u_3(13) =$ [illegible]$\,,$ $\qquad I_t u_3(24) = -I_s u_3(24)\,.$

References:

1. M.L.Goldberger, Y.Nambu, R.Oehme, Ann.of Phys., 2, 226-282 (1957).
2. G.Kallen, Elementary Partiole Physics, Addison-Wesley Publ. C., Ino., London.
3. D.B.Новожилов, Введение в теорию элементарных частиц, Наука, Москва (1972).
4. K.Hepp, Helv.Phys.Aota, 36, 355 (1963).
5. B.O.Enflo, B.E.Laurent, Invariant Spinors and Their Use for Expansion of Scattering Amplitudes, Preprint, Stockholm University (1972).
6. A.O.Barut, S.A.Baran, B.C.Unal, Theory of Nucleon-Nucleon Scattering, Trieste, Preprint IC/73/47.
7. A.I.Oksak, I.T.Todorov, Commun.Math.Phys., 14, 271 (1969), Sec 3; I.T.Todorov et al.Private Communication.
8. I.T.Todorov, R.P.Zaikov, J.Math.Phys. 10, 2014 (1964).
9. Proceedings of the IFIP, Ljubljana, Yugoslavia (1971).

Received by Publishing Department
on September 20, 1973

Анева Б.Л., Мавродиев С.Ш., Хаджииванов Л.К.

E2 - 7461

Лоренти-инвариантное разложение амплитуды рассеяния для частиц с произвольным спином

На основе представлений полной группы Лоренца в терминах комплексных спиноров сделано разложение амплитуды рассеяния для частиц с произвольным спином по инвариантным структурам.

Сообщение Объединенного института ядерных исследований
Дубна, 1973

Aneva B.L., Mavrodiev S.Ch.,
Hadjiivanov L.K.

E2 - 7461

Lorentz-Invariant Expansion of the Scattering Amplitude for Particles of Any Spin

Expansion of the scattering amplitude for particles of any spin in invariant structures is performed in terms of complex spinors on the basis of full Lorentz group representations.

Communications of the Joint Institute for Nuclear Research.
Dubna, 1973