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CDD POLES, INELASTICITIES AND OPTIMIZATION OF N/D EQUATIONS

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I. INTRODUCTION

An important problem often encounted in particle physics is to construct an analytic amplitude subjected to unitarity on its physical cut γ , starting from its (error-affected) boundary values on the left-hand cut Γ .

This is also the goal of the conventional N/D equations which start from the left-hand cut jump of the amplitude; l.h.c. boundary values (and so the jump) can be found in relativistic scattering by an analytic extrapolation from the physical region of the crossed reactions. Nevertheless in finding such an amplitude, one has to be very careful in choosing the actual equations to be used, as it is well known that equivalent mathematical methods may have different degrees of sensitivity towards the errors of the input data; this is the more so, as the data may be not known at all (apart from some boundedness condition) on the far region of the l.h.c. We would like to stress that this instability occurs even if we use the complete extrapolated amplitude as input data, not only its jump across Γ .

In a previous paper [1], using the method of construction based on the classical N/D equations, slightly modified to accomodate the whole amplitude on Γ , we have managed to find, among all the tautological equations that one which yields results least affected by the lack of exact knowledge concerning the input data. As it will be shown in section 2, this optimization was performed by noticing [1] that the kernels of the various tautological equations differ among them by a function $F_t(z(\omega))$

(eq. 2.17b of [1]) holomorphic in the s''-complex plane cut only along Γ and otherwise arbitrary. Then the most insensitive equation to the uncertainties of the initial data is found by solving an extremal L^1-norm problem, namely that of finding that function $F_z(z'')$ for which the norm $\|F_z + G_z\|_{L^1}$ is least, where $G_z(z'')$ is a given, fixed function (see further eq.(2.8)) appearing in the integral kernels. (This norm multiplies the initial errors, to yield the error of the output). As $G_z(z'')$ defined by (2.8) has a right-hand cut in the s plane which $F_z(z'')$ does not exhibit, one <u>cannot</u> simply take F_z equal to $-G_z$; therefore the norm $\|F_z + G_z\|_{L^1}$ can never be zero but rather attains a minimal value, namely for that F_z which turns one of the Cauchy kernels of the classical integral equation for $D(z)$ into a suitably weighted Poisson kernel.

We have not discussed so far any questions concerning CDD ambiguities and inelasticity. These will be the main purpose of the present note. After a short review of the relevant results of paper [1] in section 2, we discuss how the CDD ambiguities may be controlled by requiring that the constructed amplitude get inside the error corridor of the l.h. cut data. This problem did not arise in the classical *N/D* equations, as the imaginary part of the amplitude alone is too poor an information to perform such a feedback. In section 4 we treat the inelastic case using a weight function of the Froissart type, while in section 5 we discuss the extension of the method to the matrix many-channel case.

2. OPTIMAL N/D EQUATIONS

As in the paper [1], we use the canonical mappings $z(s)$ and $\zeta(s)$; $z(s)$ transforms the cut plane onto the unit disk, cut between the zero and 1, so that the l.h.c. Γ comes onto the unit circle, and γ — the physical cut — onto the cut between 0 and 1, whereas $\zeta(s)$, to be used in section 4, maps both cuts onto the unit circle, so that Γ comes on the left semicircle, whereas γ comes onto the right one (see figs. 1 and 2).

Let $\mathcal{A}_\ell(s)$ be a partial wave amplitude, supposed to be holomorphic in the cut unit circle of fig. 1, and satisfying the elastic unitarity condition:

$$\mathrm{Im}\ \mathcal{A}_\ell(s) = \varrho(s)\,|\mathcal{A}_\ell(s)|^2$$

on γ. Here $\varrho(s)$ is the corresponding phase space factor ($\varrho(s)=\sqrt{(s-4)/s}$ in the equal mass case). Then, if $\mathcal{A}_\ell(s)$ vanishes at infinity and if it observes the usual threshold behaviour $\mathcal{A}_\ell \sim q^{2\ell}$ at $q \to 0$, it is useful to introduce the "reduced" partial wave:

$$A_\ell(z) \equiv \mathcal{A}_\ell(z)/[z^\ell(1-z)^2] \qquad (2.1a)$$

$$\mathrm{Im}\ A_\ell(z) = \varrho_\ell(z)\,|A_\ell(z)|^2 \ , \quad \varrho_\ell(z) \equiv z^\ell(1-z)^2\varrho(z) \qquad (2.1b)$$

and the corresponding data function $\mathcal{A}_\ell(e^{i\theta})$ as its approximant on Γ, within an error corridor $\varepsilon(\theta)$. (The latter are obtained by dividing the estimated extrapolation noise by the factor appearing in (2.1)). Hence

$$|A_\ell(e^{i\theta}) - \mathcal{A}_\ell(e^{i\theta})| \leq \varepsilon(\theta) \qquad (2.2)$$

On $\Gamma_2 \subset \Gamma$ where no information at all is known about the amplitude, we simply take $\mathcal{A}_\ell(e^{i\theta})=0$, and $\varepsilon(\theta) = M(\theta)$ -- a (known) function bound for the amplitude.

We can reduce this variable error channel to a constant one of width ε (ε is, for instance, a mean value of the errors) by multiplying both $\mathcal{A}_\ell(e^{i\theta})$ and $A_\ell(e^{i\theta})$ by the limiting values of an outer " C -function" [2]

$$C(z) = \exp\left\{\frac{1}{2\pi}\oint d\theta \, \frac{e^{i\theta}+z}{e^{i\theta}-z} \ln\left(\varepsilon/\varepsilon(\theta)\right)\right\} \tag{2.3}$$

such that

$$\left|C(e^{i\theta})\right| = \varepsilon/\varepsilon(\theta) \qquad \text{on} \quad \Gamma. \tag{2.3a}$$

The resulting amplitudes

$$\tilde{A}_\ell(z) \equiv A_\ell(z)\, C(z) \tag{2.3b}$$

are approximants to the modified data function $\tilde{\mathcal{A}}_\ell(e^{i\theta}) \equiv \mathcal{A}_\ell(e^{i\theta})\, C(e^{i\theta})$ within a constant error ε:

$$\left|\tilde{A}_\ell(e^{i\theta}) - \tilde{\mathcal{A}}_\ell(e^{i\theta})\right| \leq \varepsilon \qquad \text{on} \quad \Gamma, \tag{2.4}$$

and also observe on γ a modified unitarity condition

$$\operatorname{Im} \tilde{A}_\ell(x) = \frac{\rho_\ell(x)}{C(x)} \left|\tilde{A}_\ell(x)\right|^2, \quad x \in \gamma \equiv [0,1]. \tag{2.5}$$

In this and the following section we shall assume only elastic unitarity (on γ), whereas in section 4 we shall reduce the general inelastic case to this one, via a Froissart weight function.

This done, to achieve the actual construction at interior points of an analytic and unitary reduced amplitude from the data

$\mathcal{A}_\ell(z)$ given on the unit circle, one looks for a reduced amplitude of the form $N(z)/D(z)$, with $N(z)$ holomorphic in the unit circle and $D(z)$ holomorphic in the whole z-plane, except for the cut between 0 and 1 (see fig. 3).

Let us now run quickly over the results of paper [1] . If the CDD poles are absent, the dispersion relations for $N(z)$ and $D(z)$ read ($A_\ell \equiv N/D$, $\tilde{A}_\ell \equiv \tilde{N}/D$ and $\tilde{N} = NC$)

$$\tilde{N}(z') = \frac{1}{2\pi i} \oint_\Gamma dz'' \frac{\tilde{A}_\ell(z'') D(z'')}{z'' - z'}$$
$$D(z) = 1 + \frac{1}{\pi} \int_0^1 dz' \frac{\operatorname{Im} D(z')}{z' - z} \tag{2.6a}$$

or using (2.5),

$$D(z) = 1 - \frac{1}{\pi} \int_0^1 dz' \frac{\rho_\ell(z') N(z')}{C(z')(z' - z)} \tag{2.6b}$$

and one gets straightforwardly one of the possible (tautological) N/D equations $D = 1 + \mathbb{K} D$:

$$D(z) = 1 + \frac{1}{2\pi i} \oint_\Gamma dz'' \tilde{A}_\ell(z'') G_z(z'') D(z'') , \tag{2.7}$$

where

$$G_z(z'') \equiv \frac{1}{\pi} \int_0^1 dz' \frac{\rho_\ell(z')}{C(z')} \frac{1}{(z'-z)(z'-z'')} \tag{2.8}$$

is a well defined function, analytic in z'' apart from the cut γ ; the main point of paper [1] is that one can find a tautological integral equation, by adding to $G_z(z'')$ any function $F_z(z'')$ holomorphic in the unit disk in z'' and of arbitrary (continuous) dependence on z . This is so since the factor $\tilde{A}_\ell(z'') D(z'') \equiv \tilde{N}(z'')$ appearing under the integral sign in (2.7) is holomorphic inside the unit circle and hence $\oint_\Gamma dz'' \tilde{N}(z'') F_z(z'') \equiv 0$, so that the

the integral equation written with the kernel

$$K_F(z,z'') = \tilde{A}_\ell(z'')\{G_z(z'') + F_z(z'')\}$$

instead of $K_0(z,z'')$ of (2.7), is equivalent to the former. Of course, this happens <u>if and only if</u> $\tilde{A}_\ell(e^{i\theta})$ are really the boundary values of a function which enjoys the analytic and unitary properties of true (reduced) amplitude. Now, in practical computations we are forced to replace the boundary values $\tilde{A}_\ell(e^{i\theta})$ of the reduced amplitude by the data function $\mathcal{A}_\ell(e^{i\theta})$ (the kernels K_F become $\mathcal{K}_F$); then, the functions $F_z(z'')$ <u>will alter</u> the solutions of the approximate equations and it is not at all a priori clear which of these solutions is closest to the solution of the exact equation.

The task of paper [1] was to construct that F_z for which the norm of the difference between the exact and the approximate kernels is least $^{+)}$:

$$\sup_A \|\mathbb{K}_F - \mathcal{K}_F\| \equiv \sup_A \sup_{z\in\Gamma} \frac{1}{2\pi}\oint_\Gamma |dz''|.|\tilde{A}_\ell(z'')-\tilde{\mathcal{A}}_\ell(z'')|.|G_z(z'')+F_z(z'')| \quad (2.9)$$

$$\leq \frac{\varepsilon}{2\pi} \sup_{z\in\Gamma} \oint_\Gamma |dz''|.|G_z(z'')+F_z(z'')| \rightarrow \text{least} \;.$$

The optimal $F_z(z'')$ is such that the optimized integral equation for $D(z)$ reads

$$\mathcal{D}(z) = 1 + \frac{1}{2\pi^2 i}\oint_\Gamma \frac{dz''}{z''}\mathcal{D}(z'')\mathcal{A}_\ell(z'')C(z'')\int_0^1 dz'\frac{\rho_\ell(z)}{C(z)(z'-z)}\mathcal{P}(z',z''), \quad (2.10)$$

$^{+)}$ When we are for from the eigenvalues, this does mean that the solution of the approximate equation is close to the exact one.

where

$$\mathcal{P}(z', z'') \equiv \operatorname{Re} \frac{z'+z''}{z'-z''}$$

is the Poisson kernel replacing the Cauchy kernel $1/(z'-z'')$ of the conventional equation (2.7) - (2.8).

Using this solution $\mathcal{D}(z)$ of the optimized equation (in (2.10) both z'' and z are on the unit circle) one can go back and construct estimates of the amplitude at every interior point. This can be done in more than one way:

<u>i) The (non-analytic) estimate $\hat{A}_\ell(z)$</u>

Starting from the "data function" $\mathcal{N}(e^{i\theta}) \equiv \mathcal{A}_\ell(e^{i\theta})\, \mathcal{D}(e^{i\theta})$ we can construct the best dispersion relation estimate for $N(z)$, denoted by $\hat{\mathcal{N}}'(z)$ (see [3]):

$$\hat{\mathcal{N}}'(z) = \frac{1}{2\pi i\, C'(z)} \oint_\Gamma \frac{dz''}{z''} \mathcal{P}(z'', z)\, \mathcal{A}_\ell(z'')\, C'(z'')\, \mathcal{D}(z''), \tag{2.11}$$

where the outer function $C'(z)$ is such that

$$\left| \left(A_\ell(e^{i\theta}) - \mathcal{A}_\ell(e^{i\theta}) \right) C'(e^{i\theta})\, \mathcal{D}(e^{i\theta}) \right| < \varepsilon \;,$$

i.e.

$$C'(z) = \exp\left\{ \frac{1}{2\pi} \oint d\theta\, \frac{e^{i\theta}+z}{e^{i\theta}-z} \ln\left[\varepsilon / (\varepsilon(\theta)\, \mathcal{D}(e^{i\theta})) \right] \right\} . \tag{2.12}$$

For practical purposes one can use almost as successfully $\hat{\mathcal{N}}(z)$ defined by (2.11) with $C(z)$ given by (2.3) instead of $C'(z)$; $\hat{\mathcal{N}}(z)$ has the advantage of being directly related to the jump of the optimal $\mathcal{D}(z)$ across the cut γ .

Although $\hat{\mathcal{N}}'(z)$ is the best dispersion relation estimate, it is <u>not</u> holomorphic in the unit disk ($\hat{\mathcal{N}}'(z)$ is in fact, [3] the envelope of the family of analytic functions, every one of them being the best approximant of $N(z)$ at a fixed point z).

The functions

$$\hat{A}_\ell(z) \equiv \hat{\mathcal{N}}(z)/\mathcal{D}(z) \qquad (2.13)$$

as well as

$$\hat{A}'_\ell(z) \equiv \hat{\mathcal{N}}'(z)/\mathcal{D}(z) \qquad (2.13b)$$

($\mathcal{D}(z)$ for interior points can be found inserting in the r.h. side of (2.10) the solution $\mathcal{D}(e^{i\theta})$ and computing the integral for every z of the cut unit disk) <u>do coincide</u> on the boundary Γ with the data function $A_\ell(e^{i\theta})$, due to the fact that $\hat{\mathcal{N}}'(z)C'(z)$ as well as $\hat{\mathcal{N}}(z)C(z)$ are harmonic (see 2.12) inside the unit disk. However, owing to the nonanalyticity of $\hat{\mathcal{N}}'(z)$ and $\hat{\mathcal{N}}(z)$, $\hat{A}'_\ell(z)$ and $\hat{A}_\ell(z)$ <u>do not</u> have the analytic properties of the true amplitude. On the other hand, it may happen (see below) that the error-channel condition (2.4) is such that there are no amplitudes at all satisfying both (2.4) and (2.6): a drawback of this (i) extrapolation method is then that $\hat{A}'_\ell(z)$ (or $\hat{A}_\ell(z)$) can always be written down, irrespective of the existence or nonexistence of such amplitudes.

ii) In many problems it is preferable to deal with a holomorphic extrapolation for the amplitude. We would have then to replace $\hat{\mathcal{N}}(z)$ of eq. (2.12) by some holomorphic function $\check{\mathcal{N}}(z)$ in such a way as to preserve unitarity. Since for the solution $\mathcal{D}(z)$ of the optimized integral equation (2.10) unitarity is expressed in terms of the nonanalytic $\hat{\mathcal{N}}(z)$ (see comment following eq. (2.12))

$$\operatorname{Im} \mathcal{D}(z) = -\rho_\ell(z)\hat{\mathcal{N}}(z) \quad , z \in \gamma \tag{2.14}$$

we might try to find an analytic extension $\check{\mathcal{N}}(z)$ of $\hat{\mathcal{N}}(z \in \Gamma)$ to the whole unit disk. This extension might have well not existed but it happily does. Indeed, for real z (and for $z'' = e^{i\theta}$)

$$\hat{\mathcal{N}}(z) = \frac{1}{2\pi i C(z)} \oint_\Gamma \frac{dz''}{z''} \frac{1-|z|^2}{|z-z''|^2} \mathcal{A}_\ell(z'') C(z'') \mathcal{D}(z'') = \frac{1}{2\pi i C(z)} \oint_\Gamma dz'' \frac{1-z^2}{1-z z''} \frac{\mathcal{A}_\ell(z'') C(z'') \mathcal{D}(z'')}{z''-z} \tag{2.15}$$

since on the unit circle $z''^* = 1/z''$. Therefore, on γ, $\hat{\mathcal{N}}(z)$ coincides with the holomorphic function

$$\check{\mathcal{N}}(z) \equiv \frac{1}{2\pi i C(z)} \oint_\Gamma dz'' \frac{1-z^2}{1-z z''} \frac{\mathcal{A}_\ell(z'') C(z'') \mathcal{D}(z'')}{z''-z} \equiv \tag{2.16}$$

$$\equiv \frac{1}{C(z)} \left\{ (\mathcal{A}_\ell C \mathcal{D})_+(z) + (\mathcal{A}_\ell C \mathcal{D})_-(1/z) \right\},$$

where $(\ldots)_+(z)$ and $(\ldots)_-(z)$ are the Fourier positive and negative frequency parts of the function $(\ldots)(z)$.

The corresponding amplitude +)

$$\check{A}_\ell(z) \equiv \check{\mathcal{N}}(z) / \mathcal{D}(z) \tag{2.17}$$

has correct analytic and unitary properties but the price to be paid for this is that its boundary values, unlike those of $\hat{A}_\ell(z)$ are no longer identical with the data function $\mathcal{A}_\ell(e^{i\theta})$, and so one might run into trouble with the error condition, i.e. with

+) One might wonder how one is to construct $\mathcal{D}(z)$ in interior points of the cut unit disk (the optimized integral equation defines $\mathcal{D}(z)$ on the unit circle). One can either resort to the integral equation (2.10) as a means of analytic extension , or, which is exactly the same, computing from the unit circle values either $\hat{\mathcal{N}}(z)$ or $\check{\mathcal{N}}(z)$ on γ (where they are identical) and using then for $\mathcal{D}(z)$ the dispersion relation (2.6b).

$$
\left| \check{A}_\ell(z)C(z) - \mathcal{A}_\ell(z)C(z) \right|_\Gamma \equiv \left| \frac{(\mathcal{A}_\ell C\mathcal{D})_+(z) + (\mathcal{A}_\ell C\mathcal{D})_-(1/z)}{\mathcal{D}(z)} - \frac{(\mathcal{A}_\ell C\mathcal{D})(z)}{\mathcal{D}(z)} \right|_\Gamma
$$

$$
\equiv \left| \frac{(\mathcal{A}_\ell C\mathcal{D})_-(z) - (\mathcal{A}_\ell C\mathcal{D})_-(1/z)}{\mathcal{D}(z)} \right|_\Gamma \leq \varepsilon . \tag{2.18}
$$

As this may well happen +), we are thus naturally led to consider the CDD poles.

3. CDD POLES.

3.1. Theory for Zero Errors

In this subsection we will forget for a while about errors and study only the "ideal" equations, i.e. with $\mathcal{A}_\ell$ -exact limiting values of (unitary and meromorphic) amplitudes.

A. The "CDD-Class"

As we showed in [1] , "any" ++) amplitude $A_\ell(z)$ meromorphic in the cut unit disk can be written as $N_c(z)/D_c(z)$ with the "canonical" N_c and D_c defined as

$$
D_c(z) = \prod_{j=1}^{n_c} (z - a_j) \cdot e^{-\frac{1}{\pi}\int_0^1 \frac{\delta(z')}{z'-z} dz'} \tag{3.1}
$$

$$
N_c(z) = A_\ell(z) D_c(z) ,
$$

+) The negative frequency parts appearing on the left-hand side of ineq. (2.18) are due both to the inaccuracy of the data in the region where they are known and, mainly, because we took $\mathcal{A}_\ell(z)=0$ on the remote parts of Γ . (It is well known from the theory of Fourier series that a function which is zero on a part of the circle cannot have only positive coefficients). Of course, if $\mathcal{A}_\ell(z)$ are the exact limiting values of the true amplitude, $\mathcal{A}_\ell(z)\mathcal{D}(z)$ would have been limiting values of an analytic function in the unit disk (and so $\mathcal{A}_\ell(z)C(z)\mathcal{D}(z)$) and the left-hand side of ineq. (2.18) would vanish identically.

++) See eq. (2.5) (and comments) of paper [1] .

where the a_i's are the poles (both input and output) of the amplitude and $\delta(z)$ is the physical phase of $A_\ell(z)$, chosen such that $\delta(0)=0$. As it stands, $D_c(z)$ fails in general to satisfy a dispersion relation of the type (2.6), firstly because the $z \cdot a_j$ factors in front spoil the normalization to 1 at infinity which was assumed there, and, secondly, because the phase $\delta(z)$ might give rise to logarithmic singularities in the integral at $z=1$. Indeed, since in general $\delta(1)\neq 0$, near $z=1$, $\frac{1}{\pi}\int_0^1 \frac{\delta(z')}{z'-z}dz' \sim \delta(1)/\pi \, \ln(z-1)$ (if $\delta(z')$ is Hölder continuous near and at $z=1$, [4] [+]), and so $D_c(z) \sim (z-1)^{-\delta(1)/\pi}$ which precludes the convergence of the dispersion integrals if $\delta(1)/\pi \geqslant 1$.

The remedy for this is well-known [5] - one writes dispersion integrals not for $D_c(z)$ but rather for

$$D(z) = D_c(z)\,\Phi(z) \tag{3.2a}$$

with $\Phi(z)$ a real rational function chosen as to cancel the singularities of $D_c(z)$ at 1 and infinity and with its poles d_j <u>outside</u> the unit circle, so as to keep the dispersion relation for

$$N(z) = N_c(z)\,\Phi(z) \tag{3.2b}$$

unchanged; for instance

$$\Phi_{min}(z) = (z-1)^{(\delta(1)/\pi)} \Big/ \prod_{j=1}^{n_{min}} (z-d_j) \quad , \quad n_{min} = n_\ell + (\delta(1)/\pi), \tag{3.3}$$

[+] Many delicate questions, related to the precise conditions under which the N/D method is meaningful, and also the eigenvalue problem, are all treated in the exhaustive paper on the N/D mathematics due to Lyth [4] .

where $(\delta(1)/\pi)$ is the smallest integer greater than or equal to $\delta(1)/\pi$. There are clearly many functions $\Phi(z)$ annihilating the singularities of $D_c(z)$ at 1 and infinity, but the one above is minimal in the sense that the degree n_{min} of its denominator is the smallest one required to ensure a dispersion relation for $D(z)$ defined in (3.2a). The points d_j – the CDD poles of $D(z)$ – have arbitrary positions : we can choose them to lie for instance in 2, 3, 4,..., but then their residues g_j are well determined by the initial $D_c(z)$. $D_{min}(z) \equiv D_c(z)\,\Phi_{min}(z)$ thus satisfies the equation (see eq. (2.7))

$$D_{min}(z) = 1 + \sum_{j=1}^{n_{min}} \frac{g_j}{z-d_j} + \frac{1}{2\pi i} \oint_{\Gamma} dz''\, D_{min}(z'')\, A_\ell(z'') \int_0^1 dz' \frac{\rho_\ell(z')}{(z'-z)(z'-z'')} . \quad (3.4)$$

In the following we shall refer to n_{min} as the "CDD-class" of $A_\ell(z)$, it being uniquely determined by $A_\ell(z)$.

B. Spurious CDD's

All other possible D's (all other choices of $\Phi(z)$) could contain a number of (spurious) CDD poles greater than n_{min}, but certainly not less. To keep the normalization at infinity of these D's right, more zeros will also have to be included, so that in general

$$n_{CDD} = n'_{zeros} + n_{min} \equiv n_{zeros} + (\delta(1)/\pi) \quad (3.5)$$

$$(n_{zeros} = n'_{zeros} + n_{bound})$$

which can also be seen as an expression of the Levinson's theorem for D round the cut (0.1). However, in their corresponding equation the residua g_j need no longer be precisely defined numbers, because their magnitude depends upon the position of the n'_{max} spurious zeroes, and, is , so to say, at our disposal +).

C. Removal of CDD Ambiguities

We would like to stress again that all this argument concerning the determination of the residues of the CDD poles is rather academic, since it assumes the knowledge of the amplitude in the whole cut unit disk. However, it is worth noticing that, in contradistinction to the conventional N/D equations where only the imaginary part of the amplitude along Γ is given, if we take as input the whole amplitude, in the limit of zero errors this completely determines the residue of the (minimal number of) CDD-poles.

Indeed, starting from the boundary values $A_\ell(z)$ —which we suppose here as hundred per cent exact — let $D_j(z)$ be the solutions of the Fredholm integral equations

$$D_j(z) = \frac{1}{z-d_j} + \frac{1}{2\pi^2 i}\oint_\Gamma dz'' D_j(z'') A_\ell(z'') \int_0^1 dz' \frac{\rho_\ell(z')}{(z'-z)(z'-z'')} \qquad (3.6)$$

$$d_1 = 2 \; , \; d_2 = 3 \; , \; \ldots$$

and $D_0(z)$ the solution of the eq. (2.10). Then we have

$$D_{min}(z) = \sum_{j=1}^{n_{max}} g_j D_j(z) + D_0(z) \, , \qquad (3.7)$$

+) For instance, if $\Phi(z) = \Phi_{min}(z).(z-a)/(z-d_{n+1})$, the new residues g'_j depend upon the arbitrary a . Indeed, $g'_j = g_j (d_j - a)/(d_j - d_{n+1})$ for $j \leq n$ and $g'_{n+1} = (D_c \Phi_{min})(d_{n+1}) . (d_{n+1} - a)$

where both the constants g_i and their number n_{min} are yet unknown but can easily be determined (if n_{min} is finite) from the error corridor condition (2.18) with ε set equal to zero. One can take, for instance, increasing numbers n_{min} in (3.7) and impose that the first n_{min} negative Fourier coefficients of ACD appearing [+)] in (2.8) vanish. This provides us with a set of algebraic linear equations which yields the required numerical values for g_i . The correctness of a particular choice for n_{min} is verified by the fact that all the following $(> n_{min})$ negative Fourier coefficients vanish identically.

We come back to these questions in the case of non-zero errors in sect. 3.2A.

D. Problem of Eigenvalues

At the end of this subsection we turn to the case when the kernel of the integral equation for $D(z)$ happens to have 1 as an eigenvalue. This causes trouble in the optimization procedure, which only works (see footnote near eq. (2.9)) when one is far from eigenvalues.

Now, in the "exact case", eigenvalues can crop up <u>when and only when</u> the CDD-class of $A_\ell(z)$ is negative. Indeed, if the class of $A_\ell(z)$ is negative, $D_\ell(z)$ has a sufficiently strong zero at 1 , so that, dividing it by

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[+)] In the exact case $(\varepsilon = 0)$, $C(z)$ can be taken equal to 1 .

$$(z-1)^{(\delta(1)/\pi)-n} \qquad \text{with} \quad 0 \leq n \leq |n_{\_}|-1$$

we find functions $D(z)$ vanishing at infinity and for which dispersion relations are still valid, so that they satisfy the homogeneous equation

$$D(z) = \frac{1}{2\pi^2 i} \oint_{\Gamma} dz'' \, D(z'') \, A_\ell(z'') \int_0^1 dz' \frac{\rho_\ell(z')}{(z'-z'')(z'-z)} \,. \tag{3.8}$$

The converse is also true: homogeneous equations imply negative CDD-class. Also, as it is easily seen, all the degenerate D's produce <u>the same</u> amplitude upon its reconstruction as N/D.

Happily, eigenvalues should really cause no problems, because if one defines instead of $A_\ell(z)$ a "new" amplitude

$$A_\ell^{(1)}(z) = \frac{A(z)}{z-z_0} \tag{3.9a}$$

(with z_0 real, inside the cut unit disk) obeying a modified unitarity condition with

$$\rho_\ell^{(1)}(z) = \rho_\ell(z)(z-z_0) \tag{3.9b}$$

one can conclude from the very definition of the canonical $D_c^{(1)}$ of $A_\ell^{(1)}$ (see eq. (2.19)) that the CDD class of $A_\ell^{(1)}(z)$ is lifted by one with respect to that of $A_\ell(z)$. (Sufficiently many such divisions will result in an integral equation with no eigenvalue equal to one). In the zero error case, the artificial pole of $A_\ell^{(1)}(z)$ will certainly be found among the zeroes of its $D^{(1)}(z)$ (uniqueness is guaranteed because its equation is no longer degenerate) and will be canceled then by the factor relating $A_\ell^{(1)}(z)$ back to $A_\ell(z)$.

3.2. Finite Errors

A. Allowed Islands for CDD Residua

At the end of section 2, it was pointed out that the analytic and unitary amplitude $\check{A}_\ell(z)=\check{\mathcal{N}}(z)/\mathcal{D}(z)$, constructed there (and defined so as to coincide with the "optimal", but nonanalytic $\hat{A}_\ell(z)=\hat{\mathcal{N}}(z)/\mathcal{D}(z)$ on the real axis) might fail to get into the error, channel, when returning back to Γ . This is a direct consequence of the fact that the limiting values of functions defined by Cauchy kernels do not necessarily coincide with the imput on the boundary. (This is to be contrasted with the properties of the Poisson kernel).

The reason for this failure may be twofold:

i) the presence of negative Fourier coefficients born by our lack of knowledge of the far region of the l.h. cut and by the inaccuracy of the data.

ii) a wrong choice of the $\mathcal{D}$ -equation, i.e. an equation with an insufficient number of CDD poles and /or with wrong residues.

Since, in practice, one does not know a priori the CDD class of the true amplitude, one may reasonably hope to get $\check{\mathcal{N}}/\mathcal{D}$ amplitudes consistent with the error corridor by gradually increasing the number of CDD poles and then patiently scanning in residue space for an appropriate choice. However, due to the linearity of the equations this amounts to the not very formidable task of finding a set of real constants g_i (the residues of the CDD poles) so that:

$$|\Delta(g_1,g_2..g_n;z)|_\Gamma \equiv \left| C(z)\,\frac{\sum_0^n g_j \check{\mathcal{N}}_j(z)}{\sum_0^n g_j \mathcal{D}_j(z)} - C(z)\mathcal{A}_\ell(z)\right|_\Gamma \equiv$$

$$(g_0 \equiv 1) \qquad \equiv \left| \frac{\sum_0^n g_j\left[(\mathcal{A}_\ell C\mathcal{D}_j)_-(z) - (\mathcal{A}_\ell C\mathcal{D}_j)_-(1/z)\right]}{\sum_0^n g_j \mathcal{D}_j(z)} \right|_\Gamma \qquad (3.10)$$

gets smaller than the (constant) error corridor ε . The quantities $\mathcal{D}_j(z)$ are defined by an equation similar to (2.24), using the optimal +) Poisson kernel :

$$\mathcal{D}_j(z) = \frac{1}{z-d_j} + \frac{1}{2\pi^2 i}\oint_\Gamma dz''\mathcal{A}_\ell(z'')C(z'')\mathcal{D}(z'')\int_0^1 dz'\frac{\rho_\ell(z')}{(z'-1)}\mathcal{P}(z',z'') \qquad (3.11)$$

and the $\check{\mathcal{N}}_j$'s are defined by equation (2.16) from their corresponding $\mathcal{D}_j$.

In the second form of $\Delta(g;z)$ in (3.10) it is the negative Fourier coefficients $\alpha_{j;-k}$ of $\mathcal{A}_\ell C\mathcal{D}_j$ that are involved, so that, in practice, one could take as an approximate way of fulfilling the error channel condition the minimization of their sum of squares in the numerator of (3.10):

$$\sum_{k=1}^{n}\left(\sum_{j=1}^{n_{CDD}} g_j\alpha_{j;-k}\right)^2 \longrightarrow \text{minimal}\ ,\ (g_0 \equiv 1). \qquad (3.12)$$

One should recall (see sect. 3.1) that when errors are present (exact case), this sum is identically zero.

When the scanning is performed, one usually finds only a minimal CDD class consistent with the error channel, i.e. as the number of degrees of Freedom (of the g_j's) increases, the more probable it is to satisfy the error condition ++).

+) The optimality of the Poisson kernel when CDD poles are introduced should be taken to mean that the Poisson kernel renders the difference between the exact and approximate $\mathcal{D}$ minimal (see [1], chapter 1) with respect to all the amplitudes of the same CDD class and with the same CDD residues which get inside the error channel.

++) Once the minimal number of CDD poles is found, the admissible range of residues for any number of CDD poles larger than it is necessarily infinite, since, as was seen in sect. 3.1., the "spurious" CDD's give rise to a whole linear variety of possible residues for the same amplitude A_ℓ .

B. The ε_0 Limitation

However, at this point, one should be cautioned against a rather nontrivial fact: if one is looking for amplitudes holomorphic in the cut unit disk, one must keep in mind, of course, that the N/D method cannot prevent ghosts or antiresonances from appearing on the physical sheet. Now, given $\mathcal{A}_\ell(z)$, one knows [6] that, in general, there are no holomorphic and unitary amplitudes arbitrarily close to it, (in the sense of the L^∞ norm on Γ). Indeed, defining the outer function $C_\rho(\zeta)$ by $|C_\rho(\zeta)|_\gamma = 2\rho(\zeta)$, $|C_\rho(\zeta)|_\Gamma = 1$ (see fig. 2 for the ζ variable), unitarity on γ is equivalent to:

$$|C_\rho(\zeta)\tilde{A}_\ell(\zeta) + i\,|_\gamma = 1 \qquad (3.13a)$$

while the error channel condition (2.4) reads:

$$|\,C_\rho(\zeta)\,(\tilde{A}_\ell(\zeta) - \tilde{\mathcal{A}}_\ell(\zeta))|_\Gamma \leq \varepsilon \qquad (3.13b)$$

However, <u>there are no</u> analytic functions satisfying (3.13a,b) with ε less than a certain number ε_{∞} which is found by solving the equation

$$\varepsilon_{\infty} = \varepsilon_0[h; 1/\varepsilon_\infty] \qquad (3.13c)$$

Here $\varepsilon_0[h;1/\varepsilon]$ is a functional of $h(\zeta)$ (equal to $-i$ on γ and to $C_\rho\tilde{\mathcal{A}}_\ell$ on Γ), being the norm of the matrix:

$$\varepsilon_0[h;1/\varepsilon] = \left\| \begin{pmatrix} c_{-1} & c_{-2} & c_{-3} & \cdots \\ c_{-2} & c_{-3} & \cdots & \\ c_{-3} & \cdots & & \\ \cdots & & & \end{pmatrix} \right\| , \qquad (3.13d)$$

where c_{-n} are the negative frequency Fourier coefficients of $h(\zeta)C_0(\zeta;\varepsilon)$, with the outer function C_0 defined by $|C_0(\zeta)|_\gamma = \varepsilon$ and $|C_0(\zeta)|_\Gamma = 1$.

So, in trying to render the sum $\Delta(g;z)$ of (3.12) very small, one should be aware that as soon as the left hand side of inequality (3.13b) gets smaller than ε_∞, it is sure that poles are already present on the physical sheet (inside the unit circle)!

C. Eigenvalues in the Finite Error Case

Some words are now in order about what might happen when the true amplitude is of negative CDD class (i.e. the "exact" — but unknown — kernel has an eigenvalue 1) or, which is the same, when there exist amplitudes of negative CDD class passing inside the error channel. We mean that although the probability for the data to exactly hit upon an eigenvalue is nil, they may nevertheless be such as to give a too large norm to the Fredholm resolvent. Now, refering to sect. 3.1D one knows that if one divides an amplitude of, say, CDD class - 1 by a factor $z-z_0$, with z_0 inside the unit circle, one is likely to move away from near the eigenvalue.

However, when the data are "erroneous" , one runs into trouble because there is no assurance that the $\mathcal{D}^{(1)}(z)$ obtained by solving the resulting modified equation:

$$\mathcal{D}^{(1)}(z) = 1 + \frac{1}{2\pi^2 i} \oint_{\Gamma} \frac{dz''}{z''} \mathcal{D}^{(1)}(z'') \frac{A_\ell(z'')}{z''-z_0} C(z'') \int_0^1 dz' \frac{\rho_\ell(z')(z'-z_0)}{C(z')(z'-z)} \mathcal{P}(z';z'') \qquad (3.14)$$

(modified according to prescriptions (3.9)) will indeed exhibit a zero at the right place, i.e. in z_0 . (Clearly, when $\varepsilon \to 0$, $\mathcal{D}^{(1)}(z)$ <u>must</u> have a zero there, by the uniqueness of the canonical decomposition (3.1) and the uniqueness of the solution of (3.14)). So, one cannot be sure that when one turns back to the initial amplitude a pole in the neighbourhood of z_0 does not survive, as it is no longer killed by the factor $z-z_0$, and this is, of course, undesirable (z_0 was chosen arbitrary, without any significance). So far, we do not know of any fundamental way of getting out of this difficulty, but one may still try to modify slightly the data $\mathcal{A}_\ell(z)$ staying nontheless in the error channel, so that the zero of the resulting $\mathcal{D}^{(1)}(z)$ fall at the right place. To this end one can, for instance, add to the "data" $\mathcal{A}_\ell(z)C(z)/(z-z_0)$ a small function, of the form $\lambda\, e(z)/\mathcal{D}^{(1)}(z)$ ($e(z)$ can be even taken equal to 1) which has the advantage that, upon its introduction, it is only the free term of the equation that is modified (by a known quantity):

$$\lambda E(z) = \frac{\lambda}{2\pi i} \oint_\Gamma \frac{dz''}{z''}\, e(z'') \int_0^1 dz' \frac{\rho_\ell(z')(z'-z_0)}{z'-z}\, \mathcal{P}(z',z'') \qquad (3.15)$$

So the final $\mathcal{D}^{(1)}(z)$ is changed to

$$\mathcal{D}^{(1)\prime}(z) = \mathcal{D}^{(1)}(z) + \lambda\, \mathcal{D}^{(1)}_E(z)$$

$$\mathcal{D}^{(1)}_E(z) = E(z) + \frac{1}{2\pi i} \oint_\Gamma \frac{dz''}{z''} \mathcal{D}^{(1)}_E(z'') \frac{\mathcal{A}_\ell(z'')}{z''-z_0} C(z'') \int_0^1 dz' \frac{\rho_\ell(z')(z'-z_0)}{C(z')(z'-z)} \mathcal{P}(z',z''). \qquad (3.16)$$

If $\mathcal{D}^{(1)}(z)$ really had a zero in a neighbourhood of z_0 , it is to be expected that small λ's will suffice to move it back to z_0 . Then,

$$\check{\mathcal{N}}^{(1)\prime}(z) = \check{\mathcal{N}}^{(1)}(z) + \lambda\, \check{\mathcal{N}}^{(1)}_E(z) \qquad (3.17a)$$

with

$$\check{\mathcal{N}}^{(1)}_E(z) = \frac{1}{2\pi i} \frac{1}{C(z)} \oint_\Gamma dz'' \frac{1-z^2}{1-zz''} \frac{e(z'')}{z''-z} \qquad (3.17b)$$

and so, the amplitude thus obtained $\check{A}^{(1)\prime}(z)$ has a pole precisely at $z=z_0$ and the initial amplitude $\check{A}^{(1)\prime}(z)\,(z-z_0)$ will indeed be of CDD class -1 and show <u>no</u> insignificant pole.

D. Input Bound States

It is often desired to use the data on the left-hand cut in conjunction with some precise information (see section 2 of ref. [7]) about the position of a bound state, lying, say, at z_0 , in the region (-1,0). As usual, this is done by solving the N/D equations for a new amplitude $A'_\ell(z)=(z-z_0)A_\ell(z)$, which obeys a modified unitarity with $\rho'_\ell(z)=\rho_\ell(z)/(z-z_0)$. $A'_\ell(z)$ will, in general, have no zero at z_0 , and so $A_\ell(z)$ will necessarily have a pole precisely there. However, this causes the CDD class of $A'_\ell(z)$ to decrease by one with respect to that of $A_\ell(z)$. This procedure runs into trouble if it happens that the class of the initial $A_\ell(z)$ is zero. But then, one can directly manufacture a zero for $\mathcal{D}(z)$ at the right place, by varying the data as in 3,2C.

E. Short Conclusions

In conclusion, it might be worth pointing out that, in contrast to the usual N/D setting when one feeds in complete data on the left-hand side, the CDD ambiguity is totally removed in the limit of zero errors (naturally) and its persistence in "real", error-affected equations is only due to the uncertainties existing in the data. Even then, however, the feedback exerted by the complete data (sect. 3.2.A) may severely restrict the allowed range of CDD classes and residues. Certainly, no such constraints can be performed by the imaginary part only, in the usual N/D method.

4. INELASTICITIES AND THRESHOLD BEHAVIOUR

So, far, we have treated only the elastic case, but inelasticity can easily be brought into our scheme. Certainly, the ideal way of treating inelasticity is to consider simultaneously all the coupled channels. The problem of optimizing in the same way as we did so far, the matrix N/D equations (which include, of course, only those channels corresponding to binary reactions) will be dealt with in section 5. However, for practical purposes inelasticity is most often taken into account globally, via two alternative parametrizations:

a) The Chew-Mandelstam parametrization $R_\ell(z)$,[8] :

$$\mathrm{Im}\, A_\ell(z) = \rho_\ell(z) R_\ell(z) |A_\ell(z)|^2. \tag{4.1}$$

In our optimization scheme , clearly all the results stay unchanged, since this amounts only to a change $\rho_\ell(z) \to \rho_\ell(z) R_\ell(z)$.

b) The Froissart parametrization [9] (also [5]): $\eta_\ell(z)$;

In the inelastic region (see fig. 2)

$$| S_\ell(z)|_{\gamma_2} = \eta_\ell(z) \leq 1 \ . \tag{4.2}$$

In the spirit of Froissart's original derivation [9] , this η_ℓ is included by working with a modified S -matrix:

$$S_\ell'(z) = S_\ell(z) C_F(z) \tag{4.3}$$

with $C_F(z)$ an outer function of modulus $1/\eta_\ell(z)$ on the inelastic region and 1 on the elastic one, so that $S_\ell'(z)$ be unitary on the whole right-hand cut. There are however two additional constraints to be palced on the function $C_F(z)$:

i) The threshold behaviour of $S_\ell'(z)$ should be the same as that of $S_\ell(z)$: for the ℓ -th partial wave

$$S_\ell(z) \sim 1 + O(z^{\ell+1/2}). \tag{4.4}$$

Clearly, this is only possible if the function $C_F(z)$ has the same threshold behaviour as $S_\ell(z)$.

ii) At least on the region,where data are available ($\Gamma_1 = \Gamma \setminus \Gamma_2$), the modulus of $C_F(z)$ should be 1, so as not to disturb the S -matrix analog of the error-channel condition (2.4).

$$\left| S_\ell(z) - \mathcal{S}_\ell(z) \right|_{\Gamma_1} \leq \varepsilon'(z). \tag{4.5}$$

If we impose $|C_F(z)| = 1$ everywhere outside γ_2, such a function may not exist, However, one overcomes this by using the freedom left in prescribing the definition of the bound $M(z)$ on the remote ("unknown") left hand cut Γ_2.

With this freedom there are many functions satisfying these restrictions. We shall construct one of them (fulfilling all the requirements) and geometrically more intuitive, in the ζ -plane (fig. 2):

$$C_F(\zeta) = \exp\left\{ -\frac{w^\ell(\zeta)}{2\pi} \oint_{\Gamma+\gamma} d\theta \, \frac{e^{i\theta}+\zeta}{e^{i\theta}-\zeta} \, \frac{\ln \eta_\ell(\theta)}{w^\ell(e^{i\theta})} \right\}, \tag{4.6}$$

where $w(\zeta)$ is the function transforming the unit circle of the ζ -plane onto the domain of fig. 4 :

$$w(\zeta) = -\frac{\sqrt{s_M+4} - \sqrt{s_M + 2(1+\zeta)^2/(1+\zeta^2)}}{\sqrt{s_M+4} + \sqrt{s_M + 2(1+\zeta)^2/(1+\zeta^2)}} \tag{4.7}$$

(s_M is the beginning of Γ_2 in the s-plane)

and $\eta_\ell(\theta)$ is a function which is taken equal to 1 outside the inelasticity region (so that the integrand vanishes except on γ_2).

The threshold behaviour $C_F(z) \sim 1 + O(z^{\ell+1/2})$ is ensured by the factor $w^\ell(\zeta)$, which vanishes in $\zeta = 1$ ($w(\zeta) = O(z^2)$ near $\zeta = 1$). On the other hand, the Schwartz-Villat kernel in (4.6) constructs a function holomorphic in the unit ζ disk, having its real part on the boundary equal to $\ln \eta_\ell(\theta)/w^\ell(e^{i\theta})$; this last quantity is indeed real everywhere on the boundary, since on γ_1, where $\ln \eta_\ell(\theta) \neq 0$, $w^\ell(e^{i\theta})$ is real. So, the integral in the exponent is purely imaginary on γ_1 , Γ_1 and Γ_2 . Therefore, $|C_F(\zeta)| = 1$ on γ_1 and Γ_1 , since here the factor $w^\ell(\zeta)$ is real and so the whole exponent is purely imaginary. On Γ_2 , $w^\ell(\zeta)$ is complex and so,

$$|C_F(\zeta)|_{\Gamma_2} = \exp\left\{-i\,\mathrm{Im}(w^\ell(e^{i\theta}))/2\pi \int_{\gamma_2} d\theta' \frac{e^{i\theta'}+e^{i\theta}}{e^{i\theta'}-e^{i\theta}} \frac{\ln \eta_\ell(\theta')}{w^\ell(e^{i\theta'})}\right\} \equiv M_1(\theta) \neq 1$$

As already remarked, the fact that $|C_F(\zeta)|_{\Gamma_2} \neq 1$ should not bother one too much because the data are unknown here, and this will only amount to changing $M'(\theta)$ by a known factor $M_1(\theta)$.

On the inelasticity region γ_2 , $C_F(\zeta)$ fulfills its purpose of making the modulus of $S'_\ell(z)$ in (4.2) equal to one, since $w^\ell(e^{i\theta})$ is real here and the real part of the Schwartz-Villart kernel reproduces $-\ln \eta_\ell(\theta)/w^\ell(e^{i\theta})$ on the boundary, and so,

$$|C_F(\zeta)|_{\gamma_2} = 1/\eta_\ell(\theta) . \tag{4.8}$$

Now, it is easy to go on constructing new amplitudes $A'_\ell(z)$ and data functions $\mathcal{A}_\ell(z)$ from $S'_\ell(z)$ and $\mathcal{S}'_\ell(z)$:

$$S'_\ell(z) = 1 + 2i\,\rho_\ell(z) A'_\ell(z) \tag{4.9a}$$

$$\mathcal{S}'_\ell(z) = 1 + 2i\,\rho_\ell(z)\mathcal{A}'_\ell(z) , \tag{4.9b}$$

These satisfy on Γ_1

$$|A'_\ell(e^{i\theta}) - \mathcal{A}'_\ell(e^{i\theta})|_{\Gamma_1} \leq \varepsilon(\theta) \equiv \varepsilon'(\theta)/|2\rho_\ell(e^{i\theta})| \tag{4.10a}$$

and on Γ_2 (as usual [1] , we put here $\mathcal{A}'_\ell(e^{i\theta}) = 0$)

$$\left| A_{\ell}^{1}(e^{i\theta}) \right|_{\Gamma_L} \leq M(\theta) \equiv \frac{M'(\theta) M_1(\theta) + 1}{|2 \rho_r(e^{i\theta})|}$$

and so we have returned to formulae similar to those of the beginning (see (2.2)), from which one proceeds in a manner identical to that of section 2.

5. COUPLED CHANNELS

Happily, the optimization procedure described in [1] can be generalized to include the many channel two-body processes described by the matrix N/D equations [10]. We would like to warn the reader that one needs a rather good acquaintance with the type of proofs of paper [1] to go through this section, but the results of practical relevence, basically similar to those of [1], can be found gathered at its end (formula (5.17)ff).

As usual, we start with a symmetrized amplitude (see, for instance [11], [12]

$$\mathbb{A}(z) = \mathbb{N}(z)\, \mathbb{D}^{-1}(z) \,, \tag{5.1}$$

where $\mathbb{N}(z)$ and $\mathbb{D}(z)$ are $n \times n$ matrices, with the same analytic properties as the scalar $N(z)$ and $D(z)$ of the preceding sections, and the matrix $\mathbb{D}(z)$ can be chosen such that :

$$\mathbb{D}_{ik}(z) \longrightarrow \delta_{ik} \qquad \text{as} \quad z \to \infty \tag{5.2}$$

For a general proof of the existence of such decompositions we refer to the papers of R.L.Warnock [13], the results of which can be immediately adapted to our particular analytic structure +).

+) We restrict ourselves to the case of no CDD poles.

The unitarity condition for $\mathbb{A}(z)$ reads :

$$\operatorname{Im} \mathbb{A}(z) = \mathbb{A}(z-i\epsilon)\,\rho(z+i\epsilon)\,\mathbb{A}(z+i\epsilon)\,, \tag{5.3a}$$

where $\rho(z+i\epsilon)$ is the diagonal matrix :

$$\rho_{ik}(z) = \delta_{ik}\,\Theta(s(z)-s_i(z))\sqrt{\frac{s(z)-s_i}{s(z)}} \tag{5.3b}$$

with $s(z)$ the mapping function of fig. 1, s_i -the thresholds for the various two-body channels and $\Theta(s-s_i)$ -the Heaviside side function.

5.1. The Reduction to Constant Errors

As in paper [1], we assume that a "data matrix" $\mathcal{A}_{ij}(z)$ is given on the left-hand cut, i.e. on the unit circle in the z -plane, so that

$$\left|A_{ij}(z) - \mathcal{A}_{ij}(z)\right|_{\Gamma} \leq \varepsilon_{ij}(z)\,. \tag{5.4}$$

The functions $\varepsilon_{ij}(z)$ also contain the boundedness conditions assumed on the "unknown", remote part of Γ (see section 2).

At first sight, one is tempted to work, as in [1] , with a modified amplitude, obtained by multiplying every amplitude $A_{ij}(z)$ by a suitable outer function $C_{ij}(z)$, but, this brings about complications in the treatment of the unitarity condition.

We shall instead use the freedom offered by the tautology group, in a rather nontrivial way, and finally reduce the problem to one equivalent to the scallar case.

As in [1] , the different tautological kernels of the integral equations can be found by writing all the possible dispersion relations for $\mathbb{N}(z)$ +):

+) Unless explicitly stated, an index appearing twice means summation. If it appears three times, this means nothing.

$$N_{ij}(z') = \frac{1}{2\pi i} \oint_\Gamma \frac{A_{\alpha k}(z'') D_{k\beta}(z'')}{z''-z'} g_{\alpha\beta,ij,z,z'}(z'')\, dz'' , \qquad (5.5)$$

where the functions $g_{\alpha\beta,ij;z,z'}(z'')$ have arbitrary dependence on z, z' but are holomorphic in z'' in the unit disk. Also, they must be such that:

$$g_{\alpha\beta,ij;z,z'}(z') = \delta_{\alpha i}\,\delta_{\beta j} . \qquad (5.6)$$

The "tautology" (equivalence) of all these dispersion relations is ensured by the fact that $A_{\alpha k}(z'') D_{k\beta}(z'')$ is a function holomorphic inside the unit disk. In practice, of course, when $A_{\alpha k}(z'')$ is replaced by $\mathcal{A}_{\alpha k}(z'')$, the tautology is broken — the dispersion relations yield results depending upon the chosen function $g_{\alpha\beta,ij,z,z'}(z'')$ and hence the problem of optimizing among the various "tautologies" makes sense.

Upon introducing (5.5a) in the dispersion relation for $\mathbb{D}(z)$:

$$D_{ij}(z) = 1 - \frac{1}{\pi} \int_0^1 \frac{\rho_{ii}(z') N_{ij}(z')}{z'-z} dz' \qquad (5.7)$$

we get the equation for $D_{ij}(z)$:

$$D_{ij}(z) = 1 + \frac{1}{2\pi^2 i} \oint_\Gamma dz'' A_{\alpha k}(z'') D_{k\beta}(z'') \int_0^1 \frac{\rho_{ii}(z')}{(z'-z)(z'-z'')} g_{\alpha\beta,ij;z,z'}(z'') dz' . \qquad (5.8)$$

Following the philosophy of [1] , we shall try to find that $g_{\alpha\beta,ij;z,z'}(z'')$ for which the norm of the difference between the kernels of equation (5.8) and of the equation obtained by replacing in (5.8) $A_{\alpha k}(z)$ by $\mathcal{A}_{\alpha k}(z)$, is minimal.

In the space of $n\times n$ matrix functions continuous on the unit circle Γ , the norm of the difference between the kernels of the two equations reads : (see also appendix 1 of [1]):

$$\| \mathbb{K}_A - \mathcal{K}_{\mathcal{A}} \| = \max_{i,j} \sup_z \sum_{\beta,k} \oint_\Gamma |dz''| . \left| (A_{\alpha k}(z'') - \mathcal{A}_{\alpha k}(z'')) \int_0^1 dz' \frac{\rho_{ii}(z')}{(z'-z)(z'-z'')} g_{\alpha\beta,ij;z,z'}(z'') \right| \qquad (5.9)$$

Since no information is assumed about the phase of the difference $A_{\alpha k}(z'') - \mathcal{A}_{\alpha k}(z'')$, we try to minimize not $\| \mathbb{K}_A - \mathcal{K}_A \|$ but $\sup_A \| \mathbb{K}_A - \mathcal{K}_A \|$, where the supremum is taken with respect to all amplitudes A passing through the error channels. So

$$\sup_A \| \mathbb{K}_A - \mathcal{K}_A \| \leqslant \max_{i,j} \sup_z \sum_{\beta k} \oint_\Gamma |dz''| . \, \varepsilon_{\alpha k}(z'') \left| \int_0^1 dz' \frac{\rho_{ii}(z')}{(z'-z)(z'-z'')} g_{\alpha\beta,ij;z,z'}(z'') \right| \equiv \tag{5.10a}$$

$$\equiv \max_{i,j} \sup_z \sum_\beta \oint_\Gamma |dz''| . \, \sigma_\alpha(z'') . \left| \int_0^1 dz' \frac{\rho_{ii}(z')}{(z'-z)(z'-z'')} g_{\alpha\beta,ij;z,z'}(z'') \right| ,$$

where the last equality follows from the observation that the sum over k bears only on the functions $\varepsilon_{\alpha k}(z'')$ so that one can introduce a new function

$$\sigma_\alpha(z'') \equiv \sum_k \varepsilon_{\alpha k}(z'') . \tag{5.10b}$$

Now, the fact that the errors appear in (5.10a) in a form depending only on the index α allows one to split a factor $C_i^{-1}(z')C_\alpha(z'')$ off the tautology function $g_{\alpha\beta,ij;z,z'}(z'')$, with the C_i's chosen such that

$$|C_\alpha(z'')| \sigma_\alpha(z'') = \varepsilon \qquad (\text{ no summation ! }) \quad , z'' \in \Gamma \tag{5.11}$$

for every α . The formulae giving these functions are analogous to (2.3) (see, further (5.17b)).

Thus, with

$$\tilde{g}_{\alpha\beta,ij;z,z'}(z'') \equiv C_\alpha^{-1}(z'') g_{\alpha\beta,ij;z,z'}(z'') C_i(z') \qquad (\text{ no summation!}) \tag{5.12}$$

our equations become successively :

$$\sup_A \| \mathbb{K}_A - \mathcal{K}_A \| \leqslant \max_{i,j} \sup_z \sum_{\alpha,\beta} \oint_\Gamma |dz''| . \left| \int_0^1 dz' \frac{\tilde{\rho}_{ii}(z')}{(z'-z)(z'-z'')} \tilde{g}_{\alpha\beta,ij;z,z'}(z'') \right| \tag{5.13a}$$

$$\equiv \max_{i,j} \sup_z \sum_{\alpha,\beta} \oint_\Gamma |dz''| . \left| \int_0^1 dz' \frac{\tilde{\rho}_{ii}(z')}{(z'-z)(z'-z'')} \left(\delta_{\alpha i}\delta_{\beta j} + (z'-z'') f_{\alpha\beta,ij;z,z'}(z'') \right) \right| \equiv$$

$$\equiv \max_{i,j} \sup_{z} \sum_{\alpha,\beta} \oint_{\Gamma} |dz''| \left| G_{i;z}(z'')_{i\alpha} \delta_{\beta j} + F_{\alpha\beta,ij;z}(z'') \right| ,$$

where

$$\tilde{\rho}_{ii}(z') \equiv \rho_{ii}(z')\, C_i^{-1}(z') \tag{5.13b}$$

$$G_{i;z}(z'') \equiv \frac{1}{\pi} \int_0^1 dz' \frac{\tilde{\rho}_{ii}(z')}{(z'-z)(z'-z'')} \tag{5.13c}$$

$G_{i,z}(z'')$ — is a matrix function holomorphic in z'' in the <u>cut</u> unit disk — and

$$F_{\alpha\beta,ij;z}(z'') \equiv \frac{1}{\pi} \int_0^1 dz' \frac{\tilde{\rho}_{ii}(z')}{z'-z} f_{\alpha\beta,ij;z,z'}(z'') \tag{5.13d}$$

which is a function <u>holomorphic</u> in z'' in the <u>whole</u> unit disk, but otherwise <u>arbitrary.</u>

5.2. The Optimization Procedure

The last equality of (5.13a) has cast our problem in a form very similar to the scalar one discussed in [1], and reminded of in the Introduction of the present paper (cf. formula (3.3) of [1]): namely, that of finding functions $F_{\alpha\beta,ij;z}(z'')$ holomorphic in z'' and of arbitrary dependence on z, so that $\sup_{A} \|\mathbb{K}_A - \mathcal{H}_A\|$ on the left hand side of (5.13a) becomes least.

Following now closely paper [1] , one can verify for oneself that the minimax theorem of section 3A, [1], goes through unchanged; i.e.

$$\inf_{F}.\max_{i,j}.\sup_{z} (\ldots\ldots) = \max_{ij} \sup_{z} \inf_{F}. (\ldots\ldots) \tag{5.14}$$

This theorem then leads us to recognize the problem just stated as that of finding the best holomorphic approximant in the sense of the L^1 -norm of the given nonholomorphic (cut between 0 and 1) function $G_{i;z}(z'')\delta_{\alpha i}\delta_{\beta j}$.

Now it is easy to see that for fixed i, j the given nonholomorphic part is nonzero only for both $\alpha = i$ and $\beta = j$, i.e. the best approximant $F_{\alpha\beta, ij; z}(z'')$ must be zero, unless $\alpha = i$ and $\beta = j$. (Clearly, those F's with $\alpha \neq i$, $\beta \neq j$ can only enhance the sum of moduli by positive quantities!). So F depends only upon i and j and the sums over α and β disappear.

Moreover, since the nonholomorphic part to be approximated does not depend on j , the optimal function F is the same for every j and so depends effectively only upon i . So, one is finally left with evaluating:

$$\max_{i} \sup_{z} \inf_{F} \oint_{\Gamma} |dz''| . \left| G_{i;z}(z'') + F_{i;z}(z'') \right| . \tag{5.15}$$

Again following [1] , the minimization over F is turned into its dual problem which allows the construction of an upper bound for $\inf_{F} \| G_{i;z} + F_{i;z} \|_{L^1}$:

$$m_{0,i} = \frac{1}{\pi} \int_0^1 dz' \, \frac{\tilde{\rho}_{ii}(z')}{1-z'} . \tag{5.16}$$

This bound can be shown to be actually attained for $z = 1$, at which point one can construct explicitly the optimal function; the latter turns out [1] to be such as to turn one of the Cauchy denominators $1/(z'-z'')$ into a Poisson kernel.

So, the final optimized equations read:

$$\mathcal{D}_{ij}(z) = 1 + \frac{1}{2\pi^2 i} \oint_{\Gamma} \frac{dz''}{z''} \mathcal{A}_{ik}(z'') C_i(z'') \mathcal{D}_{kj}(z'') \int_0^1 dz' \frac{\rho_{ii}(z')}{C_i(z')(z'-z)} \mathcal{P}(z', z'') . \tag{5.17a}$$

The reader is reminded of the definitions employed:

$C_i(z'')$ is an outer function in the unit circle of modulus

$$\left| C_i(z'') \right|_\Gamma = \varepsilon / \sigma_i(z'')$$

on the circle, with

$$\sigma_i(z'') = \sum_k \varepsilon_{ik}(z''),$$

where $\varepsilon_{ik}(z'')$ are the errors on the symmetrized "data matrix" $\mathcal{A}_{ik}(z'')$:

$$C_i(z'') = \exp\left\{ \frac{1}{2\pi} \oint_\Gamma d\theta \, \frac{e^{i\theta} + z''}{e^{i\theta} - z''} \, \ln\left(\varepsilon / \sigma_i(e^{i\theta})\right) \right\} \tag{5.17b}$$

ε is arbitrary, equal, say, to the average error.

Also, the Poisson kernel is :

$$\mathcal{P}(z', z'') = \mathrm{Re}\, \frac{z' + z''}{z' - z''} \equiv \frac{1 - r'^2}{1 - 2r' \cos(\theta'' - \theta') + r'^2} \, . \tag{5.17c}$$

6. OUTLOOK

As explained in the Introduction, it might make a great deal of difference for the results of the N/D equations whether the input on the left-hand cut is exact limiting values of holomorphic and unitary amplitudes or it just consists of approximate data. In this connection, it is essential to notice that the exact equations, i.e. those with holomorphic and unitary input admit of a whole set of tautologisal transformations that leave their solution strictly unchanged: namely, those obtained by adding to the usual N/D kernel functions with the holomorphy properties of $N(z)$. (We should remind the reader that we use not only the imaginary part on the left-hand cut as input, but the complete amplitude). Since this tautology invariance of the exact equations is broken when approximate data are used — which is the case in practice — one might look for that one of the possible equations whose results are least perturbed by the departure of the input from "exact" data.

It was the purpose of this paper and of [1] to show how one can obtain such optimal kernels and to investigate all the problems they raise in connection with CDD poles and the inclusion of inelasticities.

Section 2 of this paper recalls the main result of [1] — namely, that for no CDD poles, the optimal kernel is obtained by simply replacing the Cauchy kernel of the dispersion relation for $N(z)$ by a suitably weighted Poisson kernel. (see eq. (2.10)). An optimal nonanalytic estimate of the amplitude at interior points can then be constructed (eq. (2.11))-(2.13)) but one can also produce an analytic extension $\check{A}_\ell(z)$ off the unitarity cut (eq. (2.16)). The error channel condition is rewritten then (eq.(2.18)) in terms of the negative Fourier coefficients of $\mathcal{A}_\ell(z)C(z)\mathcal{D}(z)$.

Section 3 explores along well-worn lines the question of CDD poles for our particular choice of holomorphy domains for $N(z)$ and $D(z)$.

In Section 3.1. we treat the ideal case of zero errors, i.e. data "exact" in the above sense and essentially show that, in this limit, as expected from the principle of the uniqueness of analytic continuation, the CDD ambiguities are completely removed. In subsection 3.1A we introduce the concept of CDD class of a given amplitude, which is the minimal number of CDD poles required for the dispersion relations to hold, and then show (subsection 3.1B) that the use of a larger number of CDD poles than the minimal one leads to corresponding degrees of freedom in the choice of their residua. In subsection 3.1C' we dwell a bit on the question of the actual determination of the residua of the CDD poles from the Fourier coefficients of the boundary data and finnally (subsection 3.1D), show how the degeneracy appearing necessarily when the CDD class is negative is removed.

In Section 3.2A we first show to what extent the presence of finite errors allows CDD ambiguities to persist in the solution, by finding allowed domains for the residua of the possible CDD poles, (eqs. (3.10-3.12)). Subsection 3.2B is concerned with the important question of the appearance of ghosts and proves, in relation with results of extremal problems of analytic function theory, that if the constructed unitary amplitude gets closer to the data than a certain number ε_{co} , uniquely determined by the data themselves (eq. (3.13)), ghosts (or bound states) are bound to appear on the physical sheet. Nonzero errors lead to some trouble in the procedure of 3.1D for the removal of degeneracy, and this difficulty is approximately solved in 3.2C. Subsection 3.2D shortly treats the problem of input bound d states.

In Section 4 we move on to the treatment of inelasticities. The Chew-Mandelstam parametrization is trivially included in the optimization scheme (eq. (4.1)), but the Froissart parameter γ requires a more careful construction of a suitable outer function to include it without affecting the assumed threshold behaviour (eq. (4.6)).

Finally, section 5 considers the case of binary coupled channels and shows that the optimization results go through essentially unchanged, although one has to treat carefully (subsection 5.1) the question of variable error channels.

The final formulae to be used in practice (including the new outer function deviced in section 5.1) are gathered together in equations (5.17).

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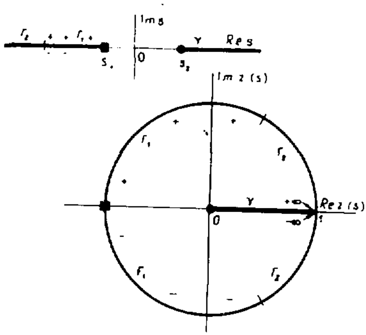

Fig. 1. The conformal mapping $z=(\sqrt{s_1-s}-i\sqrt{s_2-s_1})/(\sqrt{s_1-s}+i\sqrt{s_2-s_1})$ leading from the cut energy plane to the cut unit circle.

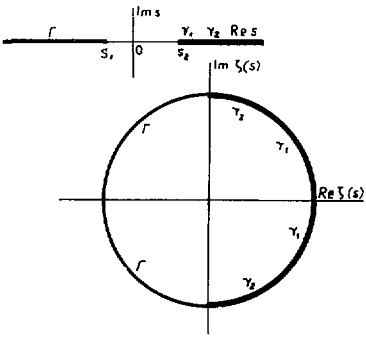

Fig. 2. The canonical mapping $\zeta(s)=(\sqrt{1+u}-\sqrt{1-u})/(\sqrt{1+u}+\sqrt{1-u})$, $u=(2s-s_1-s_2)/(s_2-s_1)$ transforming the l.h. cut Γ onto the left unit semicircle and the r.h. cut γ onto the right semicircle

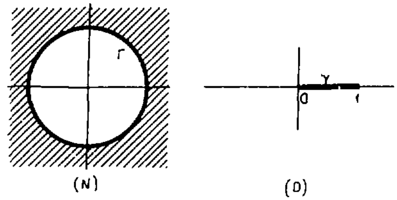

Fig. 3. Holomorphy domains for $N(z)$ and $D(z)$, respectively.

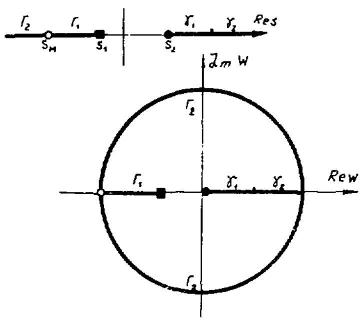

Fig. 4. The domain onto which the unit ζ-disk of fig. 2 is mapped by the function $w(\zeta) = -\dfrac{\sqrt{s_M+4} - \sqrt{s_M+2(1+\zeta)^2/(1+\zeta^2)}}{\sqrt{s_M+4} + \sqrt{s_M+2(1+\zeta)^2/(1+\zeta^2)}}$.