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SIMPLE PROOF OF THE BOGOLUBOV-PARASIUK THEOREM

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SIMPLE PROOF OF THE BOGOLUBOV-PARASIUK THEOREM

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1. Introduction

Renormalization procedure in the quantum field theory has achieved its final expression in the R-operation by Bogolubov and Parasiuk [1,2]. Bogolubov-Parasiuk theorem on renormalization is a key-stone of quantum field theory in Lagrangian form. Indeed, this theorem guarantees finiteness of an arbitrary Feynman graph, and thus gives way to every real calculation. It allows correct analysis of unitarity and causality of the scattering matrix, gives rise to the equations and relations in terms of renormalized Green functions [5], etc.

However, all the proofs of this most important theorem we have at our diposal at present [1-4] are very complicated, because they are essentially based on the recurrence relations describing the involved combinatorial structure of the R-operation. Use of the requrrence relation necessary leads to the methods of mathematical induction and combinatorics becomes an intrinsic part of the proof.

These recurrence relations may be explicitly solved though. This solution has been obtained as early as in 1964-65 independently in [7] and [8]. In [8] the following formulae were obtained for the graphs having no overlapping divergencies:

$$R = (1-M_1)\dots(1-M_{k-1})(1-M_k). \tag{1}$$

M_j here is an operator, which maps a coefficient function of the j-th divergent subgraph Γ_j into the sum of definite number of junior terms of MacLaurin series of this function. In the same article it has been shown how to alter this formula in the case of arbitrary graphs.

Namely, performing the multiplication of all the factors we obtain a sum, in which we have to cross off all the terms containing any pair of operators M_i, M_j corresponding to the overlapping divergencies Γ_i, Γ_j. This simple rule is completely equivalent to the formal procedure recently formulated by Zimmermann [5].

Equation (1) provides a convenient basis for obtaining explicit representation for renormalized Feynman graphs. Such representations have also been obtained in various forms in [7] and [8]. For our purposes the formulas of the type [8] - giving a generalization of the well-known α-representation to divergent graphs - are most convenient. (Note, by the way, that these formulas also were many times rediscovered later; see e.g. [9]).

It is interesting to notice that from this integral representation the conclusion is easily drawn that the R-operation has in fact the same form given by Eq. (1) for graphs containing overlapping divergencies, because the superfluous terms in the sum automatically vanish.

In this article, beginning with this parametric representation and slightly generalizing it, we give a direct proof of the Bogolubov-Parasiuk theorem, which is - due to the reasons just discussed - much simpler than proofs known before [1-4].

For the sake of brevity and clarity we restrict ourselves here to the scalar case. A generalization to nonscalar theories is straightforward and does not include any serious difficulties. This case may be obtained as a simple consequence of some more general statement, which would be a subject of the next article.

2. Basic Definitions and Equations

In describing the R-operation different authors use close or even the same notions giving them different content. Therefore we are compelled to begin with some definitions.

$\Gamma(L, N, F)$ is a Feynman graph with L internal lines, N vertices and F disjoint connected components.

$\gamma(\ell,n,f) \subset \Gamma(L,N,F)$ is a subgraph of Γ with ℓ, n, f internal lines, vertices and components respectively. A number

$$\omega_\gamma = \ell - 2n + 2f$$

will be called a divergence index of the subgraph γ. Note, that $\gamma(\ell,n,f)$ contains exactly

$$\mathfrak{N} = \ell - n + f$$

independent loops and $\omega_\gamma = 2\mathfrak{N}_\gamma - \ell$.

Let us consider a set of lines from Γ. Adding to this set all the vertices incident to these lines we obtain a subgraph $\gamma(\ell,n,f) \subset \Gamma$. In the case $\omega_\gamma \geq 0$ we call this set of lines a divergent structure and $\gamma(\ell,n,f)$ -divergent subgraph. There is a one-to-one correspondence between divergent structures of a graph and its divergent subgraphs. It is necessary to underline, that our definition of a divergent subgraph differs from the usual. We do not require a divergent subgraph to be a "generalized vertex", i.e. to be one-particle irreducible and to include all the lines internal with respect to its vertices.

At last we call a q-tree of a subgraph $\gamma(\ell,n,f)$ its subgraph containing all the n vertices of the graph, which have no loops and exactly q connected components. Evidently q-tree contains exactly $(n-q)$ lines and $q \geq f$.

By the Feynman rules every graph $\Gamma(L,N,F)$ corresponds to the product

$$\tilde{F}^{\epsilon}(\ldots x \ldots) = \prod_{\ell} \Delta^{c}_{\epsilon,\ell}(x_{\ell_i} - x_{\ell_f})$$

of causal propagators $\Delta^{c}_{\epsilon,\ell}(x_{\ell_i} - x_{\ell_f})$, where $\Delta^{c}_{\epsilon,\ell}$ corresponds to the ℓ-th line connecting vertices ℓ_i and ℓ_f. This product of generalized functions in general is not defined at the points of coincidence of the arguments. Moreover if the graph Γ contains at least one divergent structure the formal integral of Feynman amplitude provisionally defined by the symbolic equation

$$F^{\epsilon}(\ldots p \ldots)\delta(\Sigma p_i) = \int e^{i\Sigma p_i x_i}\, \tilde{F}^{\epsilon}(\ldots x \ldots) dx \qquad (2)$$

is divergent and has no meaning.

Redefinition of F^{ϵ} in this case is given by the R-operation

$$F^{\epsilon} \to RF^{\epsilon}. \tag{3}$$

Let us briefly describe now the parametric α-representation for renormalized amplitude RF^{ϵ} mentioned in Section 1, which serves us as a basis for the proof of Bogolubov-Parasiuk theorem.

Ascribe to each line ℓ a parameter α_ℓ in such a way that Fourier-transform $\tilde{\Delta}^c_{\epsilon,\ell}$ of a function $\Delta^c_{\epsilon,\ell}$ has a form

$$\tilde{\Delta}^c_{\epsilon,\ell}(p_\ell) = i^{-1} \int_0^\infty d\alpha_\ell \exp\{ i\alpha_\ell (p_\ell^2 - m_\ell^2 + i\epsilon)\}. \tag{4}$$

Let $\{\Gamma_1(L_1, N_1, F_1), \ldots, \Gamma_k(L_k, N_k, F_k)\}$ be a set of all divergent subgraphs of Γ (In particular one of Γ_j may coincide with Γ; we use capitals for divergent subgraphs and small letters for general subgraphs; divergence index of Γ_j would shortly be written as ω_j). Let us ascribe to each Γ_j a parameter ζ_j. Then construct new parameters β_ℓ in such a way that

$$\begin{aligned} &\beta_\ell = \alpha_\ell, \text{ if a line } \ell \text{ does not enter into any of divergent subgraphs } \Gamma_1, \ldots, \Gamma_k; \\ &\beta_\ell = \zeta_{i_\ell} \cdots \zeta_{j_\ell} \alpha_\ell \text{ if a line } \ell \text{ enters into subgraphs } \Gamma_{i_\ell}, \ldots, \Gamma_{j_\ell}. \end{aligned} \tag{5}$$

Remind that parameters α are introduced by Eq. (4) in order to perform integrations over internal momenta in Eq. (2) reducing it to the Gauss integrals. Parameters ζ provide, via Schlemilch formula

$$f(p) - \sum_{n_1+\ldots+n_m \le k} \prod_{i=1}^{m} \frac{(p_i)^{n_i}}{n_i!} \left(\frac{\partial^{n_i}}{\partial p_i^{n_i}} f(p) \right) \Big|_{p=0} =$$

$$= \frac{1}{k!} \int_0^1 d\zeta (1-\zeta)^k \frac{\partial^{k+1}}{\partial \zeta^{k+1}} f(\zeta p)$$

necessary subtractions in right places prescribed by the R-operation.

This consideration leads to the following representation of RF^{ϵ}

$$RF^{\epsilon} \cong \int_0^{\infty} d\alpha_1 \ldots d\alpha_L \int_0^1 d\zeta_1 \ldots d\zeta_k \frac{(1-\zeta_1)^{\omega_1}}{\omega_1 !} \cdots \frac{(1-\zeta_k)^{\omega_k}}{\omega_k !} \times \tag{6}$$

$$\times \left(\frac{\partial}{\partial \zeta_1}\right)^{\omega_1+1} \cdots \left(\frac{\partial}{\partial \zeta_k}\right)^{\omega_k+1} \frac{\zeta_1^{2\mathcal{N}_1} \cdots \zeta_k^{2\mathcal{N}_k}}{D^2(\underline{\beta})} e^{i \frac{A(\underline{p},\underline{\beta})}{D(\underline{\beta})}} \times$$

$$\times e^{-i \sum_{\ell=1}^{L} (m_\ell^2 - i\epsilon)\alpha_\ell} .$$

Equation (6) includes standard homogeneous polynomials $D(\underline{\beta})$ and $A(\underline{p}, \underline{\beta})$ which are constructed by the following rules: Consider certain 1-tree of the graph Γ. Construct a product of all β corresponding to the chords of this tree, i.e. to all the lines of Γ which do not belong to this tree. $D(\underline{\beta})$ is sum of such products over all possible 1-trees of the graph Γ:

$$D(\underline{\beta}) = \sum_{1-\text{trees}} \left(\prod_{\text{chords}} \beta_i \right) \tag{7}$$

$D(\underline{\beta})$ is homogeneous of degree $\mathcal{N}$ in β.

Consider now a 2-tree and construct a product of β corresponding to its chords. Multiply each of these products by the square of a sum of external momenta $(\Sigma p_i)^2$ entering in the vertices of any components of a 2-tree. The sum of these expressions over all possible 2-trees of Γ gives $A(\underline{p}, \underline{\beta})$:

$$A(\underline{p}, \underline{\beta}) = \sum_{2-\text{trees}} \left(\prod_{\text{chords}} \beta_i \right) (\Sigma p_i)^2 . \tag{8}$$

Thus $A(\underline{p}, \underline{\beta})$ is a quadratic form in P and is homogeneous of degree $\mathcal{N}+1$ in β.

Equation (6) - representation of a divergent Feynman graph - was obtained in /8/ under some restrictions on the structure of a set of divergent subgraphs. But it holds without any restrictions. In particular so-called overlapping divergences should be treated in quite general grounds and do not lead to any difficulties. Moreover we state that eq. (6) remains valid and equivalent to the prescriptions of the R-operation when divergent subgraph is understood in a generalized sense described above. The proof in fact is almost trivial and follows from the observation that under the action of operators $\int d\kappa \frac{(1-\kappa)^{\omega}}{\omega!} \frac{\partial^{\omega+1}}{\partial \zeta^{\omega+1}}$, corresponding to generalized vertices, additional subtractions allowed in our formula are reduced to identity operation.

Bogolubov-Parasiuk theorem states the existence and finiteness of the integral RF. It consists of two parts: statement of the finiteness of RF^{ϵ} for finite ϵ and the proof that the limit $\lim_{\epsilon \to 0} RF^{\epsilon}$ exists. The second part is not too complicated and may be borrowed from /1,2,3/.

Essentially non trivial is the first part of the theorem. The remainder of this article is devoted to this proof.

THEOREM. The integral RF^{ϵ} exists.

3. Proof of the Theorem

We split the proof of the Theorem into some simple lemmas. The convergence of the integral (6) at the upper limit $a_i \to \infty$ is controlled by the factor $\{-\Sigma a_i \epsilon\}$, and therefore the region of the small a is essentially dangerous. The following lemma accounts for this fact.

LEMMA 1. Let $\tilde{\Phi}(a_1,...,a_L)$ be such that the integral

$$\int_0^{\tilde{a}_1} \dots \int_0^{\tilde{a}_L} da_1 \dots da_L \, \tilde{\Phi}(a_1,\dots,a_L)$$

exists for any positive $\tilde{a}_1,...,\tilde{a}_L$ and is polynomially bounded in $\tilde{a}_1,...,\tilde{a}_L$. Then for any $\epsilon_1 > 0,..., \epsilon_L \geq 0$ the integral

$$\int_0^\infty d\alpha_1 \cdots \int_0^\infty d\alpha_L \tilde{\Phi}(\alpha_1, \ldots, \alpha_L) \exp\{-\Sigma \alpha_i \epsilon_i\}$$

exists.

Proof of this statement is elementary. It reduces to integrating by parts repeatedly.

In accordance with Lemma 1 it is sufficient to convince oneself of the existence and polynomial boundedness of the integral

$$I = \int_0^{\tilde{\alpha}_1} \cdots \int_0^{\tilde{\alpha}_L} d\alpha_1 \ldots d\alpha_L \int_0^1 d\zeta_1 \ldots d\zeta_k \tilde{\Phi}(\underline{\alpha}, \underline{\zeta}, \underline{p}), \tag{9}$$

where

$$\tilde{\Phi}(\underline{\alpha}, \underline{\zeta}, \underline{p}) = \prod_{i=1}^{k} \frac{(1-\zeta_i)^{\omega_i}}{\omega_i!} e^{-i\Sigma m_j \alpha_j} \times \tag{10}$$

$$\times \prod_{s=1}^{k} \left(\frac{\partial}{\partial \zeta_s}\right)^{\omega_s+1} \frac{\zeta_s^{2\eta_s}}{D^2} e^{i\frac{A}{D}}.$$

The first two factors $\Pi(1-\zeta_i)^{\omega_i}(\omega_i!)^{-1}$ and $\{-i\Sigma m_j \alpha_j\}$ in (10) are less than one in the absolute value and may only improve the convergence of the integral. Therefore

$$|I| \le J_{00} = \int_0^{\tilde{\alpha}_1} \cdots \int_0^{\tilde{\alpha}_L} d\underline{\alpha} \int_0^1 d\underline{\zeta} |\tilde{\Phi}(\underline{\alpha}, \underline{\zeta}, \underline{p})| \tag{11}$$

and it is sufficient to check absolute convergence of the integral with the integrand

$$\Phi(\underline{\alpha}, \underline{\zeta}) = \prod_{i=1}^{k} \frac{\partial^{\omega_i+1}}{\partial \zeta_i^{\omega_i+1}} \prod_{j=1}^{k} \zeta_j^{2\eta_j} \frac{1}{D^2(\underline{\beta})} e^{i\frac{A(p,\underline{\beta})}{D(\underline{\beta})}}. \tag{12}$$

When all β_j or some combinations of β_j are going to zero $D(\underline{\beta})$ has in general nonintegrable singularities. This fact reflects of course the well-known ultraviolet divergencies of quantum field theory. The analysis of these singularities and the proof that they are compensated in $\Phi(\underline{\alpha}, \underline{\zeta})$ due to the R-operation, i.e. due to the action of the derivatives $(\partial/\partial\zeta_i)^{\omega_i+1}$ on the respective expressions is just our task.

Notice, by the way, that this analysis cannot be reduced to the simple power counting with some set of parameters simultaneously going to zero, as it is sometimes stated (see e.g. [9]). Thus an integral

$$\int_0 d\alpha\, d\kappa \frac{\kappa}{(\alpha+\kappa^2)^2}$$

which satisfies the conditions of the power counting "theorem", evidently diverges. Therefore considerations based on this trick do not have, in general, a convincing power.

First we consider an integral with cutoffs at the lower limits in $\underline{\alpha}$ and $\underline{\zeta}$:

$$J_{\underline{r},\underline{\delta}} = \int_{r_1}^{\tilde{\alpha}_1} d\alpha_1 \ldots \int_{r_L}^{\tilde{\alpha}_L} d\alpha_L \int_{\delta_1}^{1} d\zeta_1 \ldots \int_{\delta_k}^{1} d\zeta_k |\Phi(\underline{\alpha},\underline{\zeta})|, \quad (r_i > 0,\ \delta_i > 0), \tag{13}$$

which corresponds to the product of regularized propagators. $J_{\underline{r},\underline{\delta}}$ is absolutely convergent and we have full right to change an order of integration in it, to use any variable substitutions, etc.

Then we show that

$$\lim_{\substack{\underline{r}\to 0\\ \underline{\delta}\to 0}} J_{\underline{r},\underline{\delta}} = J_{0,0} \tag{14}$$

exists and is polynomially bounded.

The very form of the function Φ (Eq. (12)) makes it evident that the natural variables in it are not $\underline{\alpha}$ and $\underline{\zeta}$, but $\underline{\beta}$ -defined by Eq. (5)- and $\underline{\zeta}$.

LEMMA 2

After substitution

$$\begin{aligned} &\zeta_i \to x_i = \zeta_i, && (i \in \{1,2,\ldots,k\}) \\ &\alpha_\ell \to \beta_\ell = \zeta_{i_\ell}\ldots\zeta_{j_\ell}\alpha_\ell \equiv \pi_\ell \alpha_\ell, && (\ell \in \{1,\ldots,L\}) \end{aligned} \tag{15}$$

(a product $\zeta_{i_\ell} \ldots \zeta_{j_\ell}$ is taken over all divergent graphs containing a line ℓ) an integral $J_{\underline{r},\underline{\delta}}$ may be written in the form

$$J_{\underline{r},\underline{\delta}} = \int_{\delta_1}^{1} \frac{dx_1}{x_1} \ldots \int_{\delta_k}^{1} \frac{dx_k}{x_k} \, |\Psi_{\underline{r}}(\underline{x})| . \tag{16}$$

Here

$$\Psi_{\underline{r}}(x) = \int_{\pi_1 r_1}^{\pi_1 \tilde{a}_1} d\beta_1 \ldots \int_{\pi_L r_L}^{\pi_L \tilde{a}_L} d\beta_L \, \mathcal{L}_1 \ldots \mathcal{L}_k \frac{1}{D^2(\underline{\beta})} e^{i\frac{A(\beta)}{D(\beta)}} , \tag{17}$$

and operators $\mathcal{L}_j$ are of the form

$$\mathcal{L}_j = \prod_{s=0}^{\omega_j} (2\mathfrak{N}_j - s + \sum_{i \in \Gamma_j} \beta_i \frac{\partial}{\partial \beta_i}) . \tag{18}$$

P r o o f: The substitution - Eq. (15) - introduces the Jacobian

$$\frac{\partial(\underline{a}, \underline{\zeta})}{\partial(\underline{\beta}, \underline{x})} = x_1^{-L_1} \ldots x_k^{-L_k} . \tag{19}$$

The operators $(\partial/\partial\zeta_j)^{\omega_j+1}$ in terms of the new variables are

$$\left(\frac{\partial}{\partial \zeta_j}\right)^{\omega_j+1} \to \left\{ \frac{1}{x_j} \left(x_j \frac{\partial}{\partial x_j} + \sum_{i \in \Gamma_j} \beta_i \frac{\partial}{\partial \beta_i}\right) \right\}^{\omega_j+1} \tag{20}$$

Taking $x_j^{2\mathfrak{N}_j}$ to the left of respective operator one obtains $x_j^{2\mathfrak{N}_j - (\omega_j+1)} = x_j^{L_j - 1}$ which alongwith $x_j^{-L_j}$ from the Jacobian gives a factor x_j^{-1} in Eq. (16). Under this operation a power $(\omega_j + 1)$ (Eq. (20)) splits out into a product of $\omega_j + 1$ factors, which we meet in the definition, Eq. (18), of the operator $\mathcal{L}_j$.

After having proved this Lemma we have, roughly speaking, to prove that the function $\Psi_0 = \lim_{\varepsilon \to 0} \Psi_\varepsilon$ exists, that it is polynomially bounded in $\tilde{e}$ and that the integral Eq ... over $\underline{x}$ converges, at the ... wer limit.

Singularities of the function $D^{-2}(\beta)\exp\{iA/D\}$ on the hyper planes $\beta_j = 0, \ldots, \beta_\ell = 0$ have rather complicated structure. But we know, that [3] in the sectors of the form $\beta_{j_1} \leq \ldots \leq \beta_{j_L}$ these singularities are effectively factorized. Consider for example a sector

$$\beta_1 \leq \beta_2 \leq \ldots \leq \beta_L \tag{21}$$

and introduce a substitution

$$\begin{aligned} t_1 &= \ldots \\ &\vdots \\ t_{L-1} &= \beta_{L-1}/\beta_L \\ t_L &= \ldots \end{aligned} \tag{22}$$

The functions D and A expressed in terms of the new variables $\underline{t}$ would be distinguished by the dash. Above mentioned factorization of singularities is demonstrated by the following:

LEMMA 3 (23)

The equality

$$\frac{1}{D'^2(\underline{t})}\, e^{i\frac{A'(\underline{t})}{D'(\underline{t})}} = \frac{W(\underline{t})}{t_1^{2\pi_1} \ldots t_L^{2\pi_L}}$$

takes place. Here $W(\underline{t})$ is s function of the variables $t_1, \ldots, t_L$ holomorphic in all points, where $t_1 \geq 0, \ldots, t_L \geq 0$ and π_j are numbers of independent loops in the special subgraphs. Namely, each π_j corresponds to subgraph constructed precisely by lines 1,2,...,j without omitting any line. The proof of this Lemma, first formulated by Speer [4], may be easily obtained from initial definitions of D and A .

Equation (23) exhibits an origin of ultraviolet divergencies in the sector. Indeed, the Jacobian of the substitution (22) is

$$\frac{\partial(\ldots\beta\ldots)}{\partial(\ldots t\ldots)} = t_2 (t_3)^2 \ldots (t_L)^{L-1} . \tag{24}$$

Thus, if in Eq. (17) the "subtracting" operators $\mathcal{L}_j$ were absent, and a set of lines $1,2,\ldots,j$ would be such that $\omega_j = 2\,\Pi_j - j \geq 0$, the integral over t_j, $\int dt_j (t_j)^{-\omega_j - 1}$ would be badly divergent. Therefore only special subgraphs mentioned in the formulation of Lemma 3 are "dangerous" in the sector, and we should see that they are regularized by "subtractors" $\mathcal{L}_j$.

In the strict terms this discussion is expressed by the following:

LEMMA 4

A function $\Psi_{\underline{r}}(\underline{x})$ entering Eq. (17) exists for all $\underline{r} \geq 0$ and admits an estimate

$$|\Psi_{\underline{r}}(\underline{x})| \leq P(\tilde{a}_L)(x_1^{L_1} \ldots x_k^{L_k})^{1/L} (\tilde{a}_1 \ldots \tilde{a}_L)^{1/L} ,$$

where $P(\tilde{a}_L)$ is some polynomial in $\tilde{a}_L$.

Proof: Consider the intersection of the integration region in (17) with some sector $\beta_{i_1} \leq \beta_{i_2} \leq \ldots \leq \beta_{i_L}$. A contribution to the integral from this region would be designated by $\Psi_{\underline{r}}^s(x_1,\ldots,x_k)$. Evidently it is enough to prove Lemma only for $\Psi_{\underline{r}}^s$. Without loosing generality we may consider a sector /21/. Using substitution (22), Jacobian (24) and Lemma 3 we may write

$$\Psi_{\underline{r}}^s(\underline{x}) = \int_{a_L}^{b_L} dt_L \ldots \int_{a_1}^{b_1} dt_1 \, \chi(\underline{t}) , \tag{25}$$

where

$$a_i = r_i \pi_i / t_{i+1} \ldots t_L \; ; \quad b_i = \max\{a_i , \min(1, \tilde{a}_i \pi / t_{i+1} \ldots t_L)\}$$
$$i < L$$

$a_L = r_L \pi_L$; $b_L = \max\{a_L, \tilde{\alpha}_L \pi_L\}$

and

$$\chi(\underline{t}) = t_2 (t_3)^2 \dots (t_L)^{L-1} \mathcal{L}'_1 \dots \mathcal{L}'_k \times \frac{W(\underline{t})}{t_1^{2\pi_1} \dots t_L^{2\pi_L}} . \qquad (26)$$

Operators $\mathcal{L}'_j$ are of the form:

$$\mathcal{L}'_j = \prod_{s=0}^{\omega_j} \{ 2\pi_j - s + \sum_{i \subset \Gamma_j} (t_i \frac{\partial}{\partial t_i} - t_{i-1} \frac{\partial}{\partial t_{i-1}}) \}, \quad (t_0 \equiv 0) \qquad (27)$$

Note, that if a divergent subgraph Γ_j is subordinate to the sector, i.e. is resulting from divergent structure containing all the lines $1,2,\dots,\ell$ (i.e. $L_j = \ell$), then the operation $\mathcal{L}'_j$ takes simpler form:

$$\mathcal{L}'_j = (2\pi_{L_j} + t_\ell \frac{\partial}{\partial t_\ell}) \dots (\ell + t_\ell \frac{\partial}{\partial t_\ell}) . \qquad (28)$$

Let us show, that the function $\chi(\underline{t})$ is holomorphic at $t_1 \geq 0, \dots, t_L \geq 0$. Due to the holomorphness of the function $W(\underline{t})$ and factorization of singularities in different t_i it is enough to convince oneself of the holomorphness of $\chi(\underline{t})$ in every t_ℓ, when the other $t > 0$ are fixed. If a subgraph constructed from lines $1,2,\dots,\ell$ is not divergent, the Laurent series of a function $\mathcal{L}'_1 \dots \mathcal{L}'_k \times W(\underline{t}) / \dots t_\ell^{2\pi_\ell} \dots t_\ell$ begins with a power $t_\ell^{-2\pi_\ell}$ and a pole is compensated by the factor $t_\ell^{\ell-1}$. Indeed $t^{\ell-1} / t^{2\pi} = t^{-\omega-1}$, and $\omega \leq -1$. Let now the lines $1,2,\dots,\ell$ define a divergent subgraph. Then the Laurent series of a function $W / \dots t_\ell^{2\pi_\ell} \dots$ again begins from a power $t_\ell^{-2\pi_\ell}$. But a series for $\mathcal{L}'_1 \dots \mathcal{L}'_k W / \dots t_\ell^{2\pi_\ell} \dots$ begins only from term proportional to $t_\ell^{-\ell+1}$. Indeed among operators $\mathcal{L}'_j$ entering into $\chi(\underline{t})$, an operator $\mathcal{L}'_j$ of the form (28) should be present. Thus

$$\mathcal{L}_j' \frac{W(\underline{t})}{\dots t_\ell^{2\eta_\ell}\dots} = (2\eta_\ell + t_\ell \frac{\partial}{\partial t_\ell}) \dots (\ell + t_\ell \frac{\partial}{\partial t_\ell}) \{ \frac{C_{-2\eta_\ell}}{t_\ell^{2\eta_\ell}} \cdot$$

$$\cdot \dots + \frac{C_{-\ell}}{t_\ell^{\ell}} + \frac{C_{-\ell+1}}{t_\ell^{\ell-1}} + \dots \} \tag{29}$$

It is manifest now that all negative powers of t_ℓ up to $t_\ell^{-\ell}$ vanish under the action of the operator $\mathcal{L}_j'$ so that after multiplying this function by $t_\ell^{\ell-1}$ the pole again disappears. Indeed, for any λ such that $2\eta_\ell \geq \lambda \geq \ell$ a factor $(\lambda + t_\ell \frac{\partial}{\partial t_\ell})$ may be found in the product (29) and

$$(\lambda + t_\ell \frac{\partial}{\partial t_\ell}) \frac{C_{-\lambda}}{t_\ell^{\lambda}} = 0 .$$

Thus we have shown that the function $\chi(t)$ is holomorphic in the region of integration (25) and the following estimate takes place:

$$|\chi(\underline{t})| \leq P \tag{30}$$

P- here is a constant, depending on $\tilde{\alpha}_L$ and rising with $\alpha_L \to \infty$ not faster then a polynomial. Therefore

$$|\Psi_{\underline{r}}^{s}| \leq P \int_{a_L}^{b_L} dt_L \dots \int_{a_1}^{b_1} dt_1 . \tag{31}$$

Returning in this integral to the previous variables we have

$$|\Psi_{\underline{r}}^{s}| \leq P \int_0^{\pi_L \tilde{\alpha}_L} d\beta_L \int_0^{\min\{\pi_{L-1}\tilde{\alpha}_{L-1}, \beta_L\}} d\beta_{L-1} \dots \int_0^{\min\{\pi_1 \tilde{\alpha}_1, \beta_2\}} d\beta_1 \frac{1}{\beta_2 \beta_3 \dots \beta_L} . \tag{32}$$

In the sector under consideration the parameters are subject to the condition $\beta_1 \leq \beta_2 \leq \dots \leq \beta_L$. Therefore

$$\frac{1}{\beta_2\beta_3\ldots\beta_L} \leq \frac{1}{\beta_1^{1-1/L}\ldots\beta_L^{1-1/L}} \qquad (33)$$

and

$$|\Psi_{\underline{r}}^{s}| \leq P \int_0^{\pi_L\tilde{a}_L} d\beta_L \ldots \int_0^{\pi_1\tilde{a}_1} d\beta_1 \beta_1^{-1+1/L} \ldots \beta_L^{-1+1/L} =$$

$$= P x_1^{\frac{L_1}{L}} \ldots x_k^{\frac{L_k}{L}} (\tilde{a}_1 \ldots \tilde{a}_L)^{1/L} . \qquad (34)$$

This is the statement of Lemma 4.

Using the estimate and reminding Lemma 2 we obtain

$$J_{\underline{r},\underline{\delta}} \leq P(\tilde{a}_1 \ldots \tilde{a}_L)^{1/L} \int_{\delta_1}^{1} dx_1 \ldots \int_{\delta_k}^{1} dx_k\, x_1^{-1+\frac{L_1}{L}} \ldots x_k^{-1+\frac{L_k}{L}} . \qquad (35)$$

Therefore the limit (14)

$$J_{0,0} = \lim_{\underline{r}\to 0, \underline{\delta}\to 0} J_{\underline{r},\underline{\delta}}$$

exists and is polynomially bounded.

Now the estimate (11) tells us that we are under conditions of Lemma 1. This completes the proof of the Theorem.

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References

1. N.N.Bogolubov, O.S.Parasiuk. Acta Math., 97, 227 (1957).
2. О.С.Парасюк. Укр.Мат.Журнал. 12, 287 /1960/.
3. K.Hepp. Comm.Math.Phys., 2, 301 (1966).
4. E.R.Speer. Journ.Math.Phys., 9, 1404 (1968).
5. W.Zimmermann. Lectures at Brandeis 1970, MIT 1971.
6. Б.М.Степанов. Теор. и мат. физ., 5, 356 /1970/.
7. В.А.Щербина. О вычитательном формализме в квантовой теории поля, 38-64. Каталог депонированных работ. ВИНИТИ 1964.
8. О.И.Завьялов, Б.М.Степанов. ЯФ, 1, 922 /1965/.
9. T.Appelquist. Annals of Physics (New York), 54, 27 (1969).

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