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VERTEX PART ASYMPTOTIC PROPERTIES AND CUTKOSKY RULES

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At the present time much attention is being paid to the study of high-energy behaviour of quantum field theory amplitudes. A significant contribution to this trend has come from the investigations of asymptotic properties of the Feynman amplitudes within the framework of perturbation theory. One of the most fruitful methods of analysis is the Feynman parametrization method, in which the rules of writing the Feynman integrals have been set down [1] (see eq. (3)), along with the recipes for obtaining the leading terms of the asymptotic expansion of individual graphs [2] (see the review of the earlier works in [3]). All these results are obtained, in one form or another, from the investigation of determinants which appear in the process of introducing the Feynman parameters.

In the present work, a new approach is suggested towards the study of the Feynman amplitudes and their asymptotic behaviour, based on the relationship between the asymptotic properties of the graphs and the unitarity condition or, more precisely, the Cutkosky rule [4]. This new approach allows us to find both simple rules for obtaining the form of the Feynman parametrized integrals and prescriptions for extracting the leading term of the asymptotic expansion. These prescriptions are much simpler than the rules set forth in ref. [2] and, what is more important, they are general and bring out some essential features which cannot be studied within the framework of the theory developed in ref. [2]. In particular, these prescriptions permit one to establish easily the fact of the failure of the eikonal approximation for the vertex part in the 5th order of perturbation theory, which has been discovered in ref. [5]. Furthermore, a procedure developed in the present work enables us to understand a somewhat unusual character of the 5th order graph and to extract relatively readily information about the vertex graph asymptotic behaviour in the higher orders.

This study is concerned with the boson theory with $\mathcal{L}_{int} = \lambda \psi\psi\phi$. However, the results obtained have a more general character and are applicable for a wide class of theories. The present work may serve as one more illustration of the value of the $\lambda\phi^3$ theory, in which a substantial simplification of calculations is no obstacle for obtaining general results applicable in many other cases (see, e.g.,[3]).

Let us consider a vertex diagram shown in Fig. 1. In what follows, we assume that the heavy bosons are on the mass shell, i.e. $p_1^2 = p_2^2 = m^2$ (m is the mass of a heavy boson), although the results will not, in principle, change if the momenta of particles p_1 and p_2 are taken beyond the mass shell.

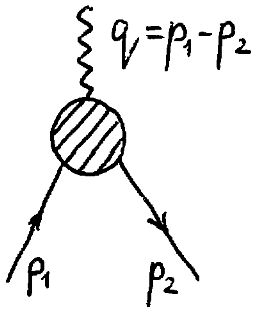

Fig. 1

For the vertex diagram of the $2n+1$ order, the corresponding Feynman integral is of the form

$$\Gamma_{2n+1}(t) = \left\{ \frac{i\lambda}{(2\pi)^4} \right\}^{2n+1} (3n-1)!\,(i\pi^2)^n\, g_{2n+1}(t), \tag{1}$$

$$g_{2n+1}(t) = \frac{1}{(i\pi^2)^n (3n-1)!} \int \prod_{j=1}^{n} d^4 k_j \prod_{i=1}^{3n} (q_i^2 - m_i^2 + i\epsilon)^{-1}.$$

Here $t = q^2 = (p_1 - p_2)^2$, k_j are the internal momenta of mesons of a mass $\mu\,(0 < \mu \leq m)$, q_i momenta corresponding to the remaining internal lines of the diagram and being linear combination of k_j and the external momenta p_1 and p_2, $m_i = m, \mu$ provided $m_i = \mu$, $q_i = k_i$.

Taking advantage of the Feynman identity, the function $g_{2n+1}(t)$ may be represented as follows

$$g(t) = \frac{1}{(i\pi^2)^n} \int_0^1 \prod_{i=1}^{3n} da_i \, \delta\Big(\sum_{i=1}^{3n} a_i - 1\Big) \int \prod_{j=1}^{n} d^4 k_j \Big\{\sum_{i=1}^{3n} a_i (q_i^2 - m_i^2)\Big\}^{-3n} \tag{2}$$

Integrating over the internal momenta k_j, this formula may be reduced to the following expression

$$g(t) = \int_0^1 \frac{\prod_{i=1}^{3n} da_i \, \delta\Big(\sum_{i=1}^{3n} a_i - 1\Big)[C(a)]^{n-2}}{[-A(a)t + B(a, m_i^2)]^n}, \tag{3}$$

where $C(a)$ is the principal minor of the quadratic form $\sum_{i=1}^{3n} a_i (q_i^2 - m_i^2)$ considered as a quadratic function of momenta k_j.

The character of the asymptotic behaviour of diagrams at $t \to \infty$ is determined by such values of one or several integration variables a_i for which $A(a)$ becomes zero (see, e.g., /3/). Due to zeroes of $A(a)$, the power of t may increase and there will appear terms such as $t^{-\ell} \ln^k t$, $\ell \leq n$, $k = 0, 1, \ldots$.

By analogy with the Landau equations /6/ (see also /3/) it may be stated that in the vertex diagrams there are only end-point type singularities, which are characterized by the fact that certain sets of variables a_i make the ratio $A(a) B^{-1}(a, m_i^2)$ turn to zero at the lower boundary of the integration range for the corresponding a_i (why we should not consider just the value of $A(a)$, but the ratio $A(a) B^{-1}(a, m_i^2)$ will be clarified later, see the condition (8)). In order to obtain the zeroes of the ratio $A(a) B^{-1}(a, m_i^2)$ in the explicit form we scale the variables

α_i with the result that the parameter set $\alpha_1, \ldots, \alpha_k$ is changed into the set of variables $\rho, \bar{\alpha}_1, \ldots, \bar{\alpha}_k$ and

$$\alpha_j \to \rho \bar{\alpha}_j \;, \quad d\alpha_1 \ldots d\alpha_k \to \rho^{k-1} d\rho \, d\bar{\alpha}_1 \ldots d\bar{\alpha}_k \, \delta\left(\sum_{j=1}^{k} \bar{\alpha}_j - 1\right). \tag{4}$$

Then we linearize the expression obtained in the scaling parameter ρ everywhere in the integrand, except for the coefficient ρ of the asymptotic variable t, which is essential for the calculation of the asymptotic behaviour. As has been shown in ref. /2/, it is important to find all possible minimal (i.e. containing the least number of parameters α_i) sets of variables α_i which make the ratio $A(\alpha)B^{-1}(\alpha, m_i^2)$ turn to zero. The number of such sets will determine the power of $\ln t$ and their length - the power of t in the asymptotic expression derived.

Proceeding from the unitarity considerations we may establish the validity of such an approach. Then, however, the insufficiency of the rules given in ref. /2/ becomes evident. Moreover, from the unitarity considerations it becomes clear that some other higher order scalings, including nonminimal number of parameters, must be taken into account. The knowledge of the set of all necessary scalings (the complete set) is essential for the calculation of the coefficient of the leading term of the asymptotic expansion. It is precisely the knowledge of the complete scaling set (which has been found from quite different considerations) that allowed the failure of the eikonal approximation in the fifth order of the perturbation theory (the $\lambda\phi^3$ model) to be discovered /5/.

It should be noted that the fact that the ranges of integration with small values of parameters of the set are important for the determination of the asymptotic behaviour of the vertex part, is essential in our arguments. Cases of the pinch-type singularities (not at the boundary of the integration range) call for a special investigation.

According to ref. /4/, $Im\, g(t)$ may be found from the condition of unitarity. To achieve this, the substitution

$$(q_i^2 - m_i^2 + i\epsilon)^{-1} \to 2\pi i \theta(q_{0i})\delta(q_i^2 - m_i^2) \tag{5}$$

should be made for the diagram under consideration (the Cutkosky rule). In other words, $Im\, g(t)$ is determined from the diagrams of fig. 2 where all the intersected lines lie on the mass shell and the integration is performed solely in the corresponding three-dimensional momentum. It will be noted that in this case there are several unitary intersections involving lines with a given momentum q_i for each propagator $(q_i^2 - m_i^2)^{-1}$ and the variable α_i associated with it.

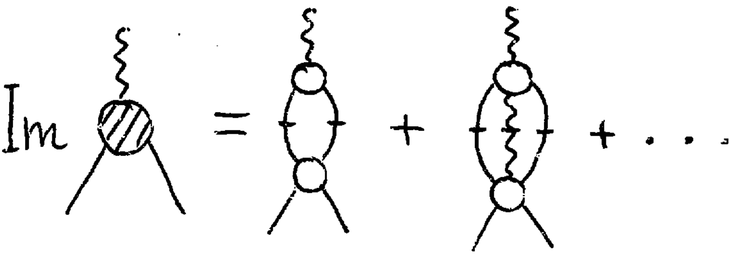

Fig. 2

Let us now turn our attention to the fact that when one of the variables α_i vanishes in the integrand of eq. (2) this leads to the elimination of the term $\alpha_i (q_i^2 - m_i^2)$. This implies that there is no Cutkosky rule for the given line. If we consider any of the sets of the scaled parameters, then the Cutkosky rule for all the propagators with the corresponding α_i are absent. It can be readily shown that all unitary intersections for the diagram must correspond to each set of the scaling parameters. If the latter vanish, then all unitary intersections which determine $Im\, g(t)$ for finite values are eliminated, i.e. $Im\, g(t)$ vanishes.

Indeed, consider $Im\, g(t)$ as obtained from eq. (3)

$$Im\, g(t) \sim \int_0^1 \prod_{i=1}^{3n} d\alpha_i\, \delta\left(\sum_{i=1}^{3n} \alpha_i - 1\right)[C(\alpha)]^{n-2}\, \delta^{(n-1)}(-At + B) \qquad (6)$$

$$(t_0 < t < \infty)$$

(the coefficient of the integral is neglected as inessential for our arguments). As has been already mentioned, at the boundary of the range of integration over some set of a_i $A(a)$ becomes zero. The scaling operation is as follows

$$A(a) = \rho^r A'(\rho, \bar{a}) \qquad (r \geq 1) \tag{7}$$

with $A'(\rho,\bar{a})$ being nonzero. It is clear that $Im\, g(t) = 0$ at $t_0 < t < \infty$ and $\rho \to 0$ if the condition

$$A(a) B^{-1} (a, m_i^2) = 0 \tag{8}$$

is satisfied, i.e., the scaling operation does correspond to $Im\, g(t) = 0$ at $t_0 < t < \infty$. On the other hand, it is understandable that at $t \to \infty$ and $A(a) \to 0$ the function $\delta^{(n-1)}(-At+B)$ may have singularities, i.e. $Im\, g(t)| \neq 0$. This conclusion is just in line with the general scheme ofarguments outlined above. It may be stated that the region of small values of the scaled parameters is essential only in the calculation of the asymptotic behaviour of $g(t)$ and tells but weakly at finite momentum transfer.

Thus, we shall define the scaling of parameters as an operation eliminating all the unitary intersections if $a_i = 0$. Then, the complete set of scaling is a set of all possible scalings each of which eliminates all the unitary intersections allowed for the given diagram.

An additional criterion for our choice of scalings is the condition (8) which has a profound physical meaning. The analysis of this condition is connected with the question of what is "the vertex function asymptotic behaviour". It is natural to understand this phrase as the behaviour of the $g(t)$ function at

$$|\ln t| >> \ln m_i^2 \qquad (m_i = m, \mu) \tag{9}$$

or $$|\ln t| >> \ln |p_1^2|, \ \ln |p_2^2|, \ \ln m_i^2 \tag{10}$$

when we consider the corresponding off-mass-shell diagram. As has been mentioned above, the character of the asymptotic behaviour of the Feynman integral is determined by ranges of integration of certain sets of variables a_i scaled, in which the coefficient at the asymptotic variable t vanishes. Taking into account that the function $B(a, m_i^2)$ may be written in the form

$$B(a, m_i^2) = m^2 B_1(a) + \mu^2 B_2(a), \quad B_2(a) = C(a) \sum_{j=1}^{n} \gamma_j \qquad (11)$$

(γ_j are the Feynman parameters corresponding to the meson lines), it can be readily understood that some other sets, for which $B_1(a)$ (or $B_2(a)$) vanish ($A(a)$ remaining finite), will determine the asymptotic properties of the function $g(t)$ with respect to m^2 (or μ^2) at $m^2 \to \infty$ (or $\mu^2 \to \infty$). That is, the condition (8) corresponds precisely to the consideration of the asymptotic behaviour with respect to the variable t. Bearing in mind that the complete set of scalings may consist not only of the simplest sets for which

$$A(a) \sim \rho, \quad B_1(a) \neq 0, \quad B_2(a) \neq 0, \qquad (12)$$

but also such sets for which, e.q.,

$$A(a) \sim \rho^2, \quad B_1(a) \sim \rho, \quad B_2(a) \sim \rho \qquad (13)$$

(in both cases the sets are determined by the Cutkosky rules and satisfy the condition (8)), we may generalize the condition (8) as follows: The conditions

$$\frac{A(a)}{B_1(a)} \sim \rho^k, \quad \frac{A(a)}{B_2(a)} \sim \rho^k \qquad (k \geq 1) \qquad (14)$$

must correspond to the asymptotic relations (9) or (10). The conditions (14) will be referred to as the linearuty condition. It is evident that this condition is satisfied, for example, with the scalings leading to expressions of the type

$$A(a) \sim \rho^2, \quad B_1(a) \neq 0, \quad B_2(a) \neq 0 \qquad (15)$$

which may be reduced to the case of eq. (12) by a suitable substitution of variables $\rho^2 = \delta$ .On the other hand, scalings leading to expressions such as

$$A(a) \sim \rho^2, \quad B_1(a) \sim \rho^2, \quad B_2(a) \sim \rho \text{ or } \rho^\circ \tag{16}$$

$$A(a) \sim \rho^2, \quad B_1(a) \sim \rho \text{ or } \rho^\circ, \quad B_2(a) \sim \rho^2, \tag{17}$$

$$A(a) \sim \rho^2, \quad B_1(a) \sim \rho, \quad B_2(a) \sim \rho^\circ \tag{18}$$

and so on, correspond to the cases

$$|\ln t| \sim \ln m^2 \gg \ln \mu^2, \tag{16a}$$

$$|\ln t| \sim \ln \mu^2 \gg \ln m^2, \tag{17a}$$

$$|\ln t| \gg \ln m^2 \gg \ln \mu^2. \tag{18a}$$

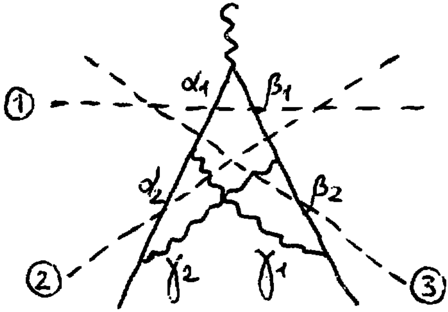

Fig. 3

These arguments become still more obvious in the analysis of asymptotic properties of the vertex part using its Mellin transform.

In conclusion, the arguments stated above can be illustrated with the example of the irreducible fifth-order vertex diagram (fig. 3). Three unitary intersections, shown in fig. 3 by dotted lines, correspond to this diagram. Reversing the above-described rules, which corresponds in the given case to the rules given in ref. /1/, we immediately find that

$$A(a) = a_1 \beta_1 (a_2 + \beta_2 + \gamma_1 + \gamma_2) + a_1 \gamma_2 \beta_2 + \beta_1 \gamma_1 a_2 \quad (19)$$

without considering the corresponding determinants. Unfortunately, the absence of relations analogous to those given in (5) with respect to mass does not allow us to write out immediately the expressions for $B(a, m_i^2)$ in a similar way. Nevertheless, it can be readily obtained that

$$B(a, m_i^2) = m^2 \{ (a_1 + \beta_1 + \beta_2)(a_1 + a_2 + \beta_1)(a_2 + \beta_2) +$$

$$+ \gamma_1 (a_1 + a_2 + \beta_1)^2 + \gamma_2 (a_1 + \beta_1 + \beta_2)^2 + \nu (\gamma_1 + \gamma_2) C(a) \},$$

(20)

where

$$C(a) = (a_1 + \beta_1)(a_2 + \beta_2 + \gamma_1 + \gamma_2) + (a_2 + \gamma_2)(\beta_2 + \gamma_1),$$

$$\nu = \frac{\mu^2}{m^2}. \quad (21)$$

In conformity with the above definition of the scaling operation, the minimal sets

$$a_1 a_2, \quad \beta_1 \gamma_2, \quad a_1 \gamma_1, \quad \beta_1 \beta_2, \quad a_1 \beta_1 \quad (22)$$

are allowed in the given case (by the way, the $a_1 \beta_1$ set is absent in the rules of ref. /2/), as well as the higher sets

$$a_1 a_2 \beta_1 \gamma_2, \quad \beta_1 \beta_2 a_1 \gamma_1. \quad (23)$$

The remaining possible sets (including, in particular the set $a_1 a_2 \beta_1 \beta_2$) do not satisfy the linearity condition (14).

Performing the scaling operation successively for all sets (23) and (22), substituting into eq. (3), linearizing the integral obtained and calculating it, we find the leading term of the asymptotic expansion

$$g_5^{(0)}(t) = 8 \frac{1}{(16\pi^2)^2} \frac{1}{t^2} \frac{1}{4!} \ln^4 \frac{t}{m^2} . \qquad (24)$$

Formula (24) is in line with the result obtained in ref. /5/.

It can be easily understood that among the diagrams of the $(2n+1)$ th order $(n > 2)$ there must be a diagram such as shown in fig. 4. It is clear that the $\alpha\gamma$ set is a minimal one and the leading behaviour of the vertex function has the form

$$g_{2n+1}(t) \sim \frac{1}{t^2} \qquad (25)$$

for diagrams of any order with respect to the interaction constant. Hence follows that the complete behaviour of the vertex part is essentially noneikonal.

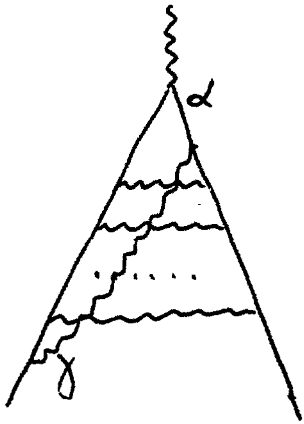

Fig. 4

It is seen from the above analysis that the method developed in the present work has a general character and is applicable for a broad class of theories. Such an approach is also valid when we consider diagrams of other types, in particular, for the processes of scattering and production of new particles.

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References

1. K.Symanzik. Progr.Theor.Phys. (Japan), 20, 690 (1958).
2. G.Tiktopoulos. Phys.Rev., 131, 480 (1963).
3. R.J.Eden, P.V.Landshoff, D.I.Olive, J.C.Polkinghorne. The Analytic S-Matrix, Cambridge Univ. Press, 1966.
4. R.Cutkosky. Journ.Math.Phys., 1, 429 (1960).
5. E.Eichten, R.Jackiw. Phys.Rev., D4, 439 (1971).
6. L.D.Landau. Nucl.Phys., 13, 181 (1959).

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