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HARMONIC ANALYSIS ON THE LORENTZ GROUP AND FREE HAMILTONIAN FOR PARTICLES OF ANY SPIN

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Introduction

The Logunov and Tavkhelidze quasipotential approach /1/ is a very effective method for the analysis of relativistic two-particle interactions. Kadyshevsky and collaborators /2/, /3/, /4/ proposed another version of this approach based on Hamiltonian formulation of quantum field theory. One of the distinctive features of the latter formalism is the natural possibility of exploiting the methods of the Poincare and Lorentz group representation theory and the harmonic analysis on the Lorentz group.

In the spirit of Kadyshevsky's quasipotential approach we assume that in the case of particles with arbitrary spin s, the quasipotential equation for the effective particle wave function $\psi^{(s)}_{q\mu}(p)$ in the center of mass system is:

$$(2E_p - 2E_q)\psi^{(s)}_{q\mu}(p) = -\frac{2}{(2\pi)^{3/2}} \sum_{\nu=-s}^{s} \int \frac{d^3k}{2k_0} V_{\mu\nu}(\underline{p},\underline{k};E_q)\psi^{(s)}_{q\nu}(k) . \qquad (1)$$

Note that for the case $s = 1/2$ eq.(1) is the same as that obtained in /3/ . The wave function $\psi^{(s)}_{q\mu}(p)$ is connected with the scattering amplitude $A_{\mu\nu}(\underline{p},\underline{q})$ by:

$$\psi^{(s)}_{q\mu}(p) = 2(2\pi)^{3/2} p_0 \delta(\underline{p}-\underline{q})\chi^{s,\nu}_{\mu} - 4\pi \frac{\sum A_{\mu\sigma}(\underline{p},\underline{q})\chi^{s,\nu}_{\sigma}}{2E_p-2E_q+i0} \qquad (2)$$

In ref./4/ a relativistic configuration representation for particles with $s=0$ was introduced in the framework of the unitary irreducible Poincare group representation. The modulus of the relativistic relative coordinate is expressed in terms of the Casimir operators of the principal series of unitary representations of the Lorentz $(SL(2,C))$ group. The connection between momentum and the new relativistic configuration spaces is realized by the Shapiro integral transform /5/ .

From the group-theoretical point of view the Shapiro transform is a reduction of the irreducible unitary Poincare group representation, contracted to the $SL(2,C)$ group representation, into the irreducible representations of the $SL(2,C)$ group. The kernels of this transformation are generating functions for the matrix elements of the Lorentz transformations (boosts), if the matrix elements of the $SU(2)$ group are used as basis functions (for the zero spin case see ref. /6/).

The reduction of unitary $SL(2,C)$ group representations of the principal series generalizing the Shapiro transform to the case of any spin was carried out in ref. /7/. In ref./8/ this reduction was given in a form convenient for the Kadyshevsky quasipotential approach, which gave the possibility of proving the "addition theorem" for the transformation kernels.

The Shapiro transformation kernels+)

$$\xi^{(0)}(\underline{p},\underline{r}) = (p_0 - \underline{p}\cdot\underline{n})^{-1-ir}$$

play the role of plane waves in the relativistic configuration space. The nonrelativistic limit of the plane waves (see ref./4/) is the usual exponent:

$$\xi^{e}_{nr}(\underline{p},\underline{r}) = e^{i\underline{p}\cdot\underline{r}}$$

The plane waves $\xi^{(0)}(\underline{p},\underline{r})$ are solutions of the "free Shrödinger equation":

$$H_0^{(0)}\,\xi^{(0)}(\underline{p},\underline{r}) = 2p_0\,\xi^{(0)}(\underline{p},\underline{r}) .$$

The free Hamiltonian $H_0^{(0)}$ is a "finite-difference" operator:

$$H_0^{(0)} = 2\,\mathrm{ch}\, i\partial_r + \frac{2i}{r}\,\mathrm{sh}\, i\partial_r + \frac{1}{r^2}\,\underline{L}^2 \exp i\partial_r , \qquad (3)$$

where $\underline{L}$ is the generator of the $SU(2)$ group.

In this paper, on the basis of harmonic analysis on the Lorentz group /10/, /11/ we generalize the finite-difference free Hamiltonian (3) for the case of particles of any spin. From the group-theoretical point of view this means that the algebra of the Poincaré group is to be constructed acting on the relativistic coordinate, rather than on the momentum.

+) Use is made of the system of units $m=\hbar=c=1$.

In § 1 the necessary results of the Poincare and Lorentz group representations and harmonic analysis on the Lorentz group are given (see for example refs. /8/, /10/, /11/, /12/, /13/).

In § 2 the plane wave for particles with any spin is considered. A series expansion of the plane wave components into $SU(2)$ group irreducible unitary representation matrix elements is given. The orthogonality and completeness conditions for the plane waves are derived explicitly.

In § 3 the operators of the free Hamiltonian $H_0^{(s)}$ and momentum $\underline{P}^{(s)}$ are constructed and their nonrelativistic limit is considered.

§ 1. Harmonic analysis on the Lorentz group.

Let an element of the Poincare group be represented by $\{g, a\} \in \mathcal{P}$, the wave function of a particle in momentum space be $\psi_{\mu}^{(s)}(p)$ where s is spin, μ is the spin projection, and p – the four-momentum $\left(p^2 \equiv p_0^2 - \underline{p}^2 = m^2\right)$.

The irreducible unitary representation of the Poincaré group is:

$$T_{\{g,a\}} \psi_{\mu}^{(s)}(p) = e^{ipa} \sum_{\nu=-s}^{s} D_{\mu\nu}^{(s)}\{V(g,p)\}\, \psi_{\nu}^{(s)}(g^{-1}p) . \quad (4)$$

The Casimir operators of this representation are the square of the momentum $p^2 = m^2$ and the square of the spin operator $W^2 = -m^2 s(s+1)$.

Let g be the "boost" matrix, transforming the vector $\overset{\circ}{q} \equiv (m, \underline{0})$ to the vector $q \equiv (q_0, \underline{q})$

$$g\, \overset{\circ}{q} = q ,$$

and the four-dimensional translations $a = 0$. Then eq.(4) takes the form+):

$$T_q \psi^{(s)}(p) = D^{(s)}\{V(q,p)\}\, \psi^{(s)}(p(-)q) , \quad (5)$$

+) If it is possible, the tensor indices are omitted.

where

$$p(-)q \equiv \Lambda_q^{-1} p = \left(pq, \underline{p} - \underline{q}\,\frac{p_0 + pq}{1+q_0}\right).$$

The unitary 2x2 matrix $V(q,p)$ is the so-called "Wigner's rotation":

$$V(q,p) = B_p^{-1} B_q B_{p(-)q} \; ,$$

where

$$B_q = \frac{1 + \underline{q}}{\sqrt{2(1+q_0)}}$$

is the boost:

$$B_q \, \hat{q} \, B_q^{+} = \underline{q} \; , \quad \underset{\sim}{q} = q_0 + \underline{q}\cdot\underline{\sigma}$$

($\underline{\sigma}$ are the Pauli matrices) .

Eq. (5) represents a reducible representation of the $SL(2,C)$ group. The integral connection with the irreducible representations of principal series is given by:

$$\psi_\mu^{(s)}(p) = \frac{1}{(2\pi)^{3/2}} \sum_{\nu=-s}^{s} \int (\nu^2 + r^2)\, dr\, d^2\underline{n}\; \xi_{\mu\nu}^{+(s)}(\underline{p},\underline{r})\, \psi_\nu^{(s)}(\underline{r}),$$

$$\psi_\mu^{(s)}(\underline{r}) = \frac{1}{(2\pi)^{3/2}} \sum_{\nu=-s}^{s} \int \frac{d^3 p}{2p_0}\, \xi_{\mu\nu}^{(s)}(\underline{p},\underline{r})\, \psi_\nu^{(s)}(p) \; . \qquad (6)$$

The wave function $\psi_\nu^{(s)}(\underline{r})$ is defined on the cone $n^2 \equiv 1 - \underline{n}^2 = 0$ and transforms under the irreducible unitary $SL(2,C)$ group representations :

$$T_p^{[\nu,r]}\, \psi_\nu^{(s)}(\underline{r}) = (pn)^{-1-ir}\, \psi_\nu^{(s)}(\underline{r}^p) \; , \qquad (7)$$

where

$$\underline{r}^p = r\, \underline{n}^p \; .$$

The vector $\underline{n}^p$ is defined by:

$$v_{p,n}\, \sigma_3\, v_{p,n}^{+} = \underline{n}^p \cdot \underline{\sigma} \; , \qquad (8)$$

where the unitary matrix $v_{p,n}$ satisfies the equation:

$$u_n^{+}\, B_p\, u_{n_p} = k_{p,n}\, v_{p,n} \; .$$

The unitary matrices u_n and u_{n_p} define rotations of the z -axis into direction defined by the unit vectors $\underline{n}$ and $\underline{n}_p$ correspondingly, $k_{p,n}$ is an upper triangular matrix with real diagonal elements (see /7/, /8/,/11/).

In spherical coordinates:

$$p = (ch\chi,\; sh\chi\, \underline{n}_p) \; , \qquad \chi \in [0,\infty],$$

$$\underline{n}_p = (\sin\vartheta\cos\phi,\; \sin\vartheta\sin\phi,\; \cos\vartheta), \quad \{\theta,\vartheta\} \in (0,\pi),$$

$$\underline{n} = (\sin\theta\cos\varphi,\; \sin\theta\sin\varphi,\; \cos\theta), \quad \{\varphi,\phi\} \in [0,2\pi],$$

the matrix elements of $v_{p,n}$ have the form:

$$(v_{p,n})_{11} = \frac{e^{-\chi/2}}{\sqrt{pn}}\left[\cos\frac{\theta}{2}\cos\frac{\vartheta}{2} + \sin\frac{\theta}{2}\sin\frac{\vartheta}{2}\, e^{i(\varphi-\phi)}\right],$$

$$(v_{p,n})_{12} = \frac{e^{\chi/2}}{\sqrt{pn}}\left[\cos\frac{\theta}{2}\sin\frac{\vartheta}{2} - \sin\frac{\theta}{2}\cos\frac{\vartheta}{2}\, e^{-i(\varphi-\phi)}\right]e^{-i\phi}. \qquad (8')$$

If in spherical coordinates the vector n^p is:

$$n^p = (\sin\theta_p\cos\varphi_p, \sin\theta_p\sin\varphi_p, \cos\theta_p)$$

from eqs.(8) and (8') it follows that:

$$\cos\theta_p = \frac{\operatorname{ch}\chi\, \mathbf{n}_p\cdot\mathbf{n} - \operatorname{sh}\chi}{pn}, \qquad (9)$$

$$\sin\theta_p = \frac{|\mathbf{n}_p\times\mathbf{n}|}{pn}$$

The Casimir operators of the $[\nu, r]$ representation have the values:

$$\mathbf{L}^2 - \mathbf{N}^2 = \nu^2 - r^2 - 1,$$

$$\mathbf{L}\cdot\mathbf{N} = \nu r$$

$$\{\nu\} = -s, -s+1, \ldots, s-1, s$$

$$r \in [0,\infty].$$

The kernels of the "transforms" in eqs.(6) have the form:

$$\xi^{(s)}_{\mu\nu}(\mathbf{p}, \mathbf{r}) = (pn)^{-1-ir} D^{(s)}_{\mu\nu}(\mathbf{n}^p) . \qquad (10)$$

The function $D^{(s)}_{\mu\nu}(\mathbf{n})$ is the matrix element of the $SU(2)$ irreducible unitary representation with weight $s\,(s=0,1/2,1,\ldots)$. Defining $L_\pm = L_1 \pm iL_2$ we have:

$$L_\pm D^{(s)}_{\mu\nu}(\mathbf{n}) = \sqrt{(s\mp\nu)(s\pm\nu+1)}\; D^{(s)}_{\mu\,\nu\pm1}(\mathbf{n}),$$

$$L_3 D^{(s)}_{\mu\nu}(\mathbf{n}) = \nu D^{(s)}_{\mu\nu}(\mathbf{n}) .$$

The forms of the invariant scalar products in momentum and relativistic configuration spaces follow from eqs.(6):

$$(\psi_1^{(s)}, \psi_2^{(s)}) = \sum_{\nu=-s}^{s}\int \frac{d^3p}{2p_0}\,\overline{\psi^{(s)}_{1\nu}(p)}\,\psi^{(s)}_{2\nu}(p)$$

and

$$(\psi_1^{(s)}, \psi_2^{(s)}) = \sum_{\nu=-s}^{s}\int(\nu^2+r^2)\,dr\,d^2\mathbf{n}\; \overline{\psi^{(s)}_{1\nu}(\mathbf{r})}\,\psi^{(s)}_{2\nu}(\mathbf{r}) .$$

The transformations (6) are made in such a manner that the projection of the spin is helicity.

(It must be noted, that the argument of $D^{(s)}$-functions is a three-dimensional unit vectors.)

§2. The Plane Waves

From the correspondence principle it follows that transformations (6) are relativistic generalization of the usual Fourier-expansion. From eqs. (2) and (6) it follows that the plane wave in relativistic configuration space of a particle with any spin s and helicity ν has the form:

$$\xi^{s,\nu}(\underline{p},\underline{r}) = (pn)^{-1-ir}\mathcal{D}^{(s)}(\underline{n}^{0})\,\chi^{s,\nu} \qquad (11)$$

The spinor $\chi^{s,\nu}$ is defined by the equations:

$$S_{\pm}\,\chi^{s,\nu} = \sqrt{(s\mp\nu)(s\pm\nu+1)}\;\chi^{s,\nu\pm1},$$

$$S_3\,\chi^{s,\nu} = \nu\,\chi^{s,\nu}.$$

The vector matrix operator $\underline{S}$ is defined from the equality:

$$\underline{S} = \mathcal{D}^{(s)}(\underline{n})\,\underline{L}\,\overset{+}{\mathcal{D}}^{(s)}(\underline{n}) \qquad (12)$$

where generator $\underline{L}$ operates on the $\mathcal{D}^{(s)}$ - function

only. Explicitly:

$$(S_{\pm})_{\mu\nu} = \sqrt{(s\mp\nu)(s\pm\nu+1)}\;\delta_{\mu,\nu\pm1},$$

$$(S_3)_{\mu\nu} = \nu\,\delta_{\mu\nu},$$

$$\chi^{s,\nu}_{\mu} = \delta_{\mu\nu}.$$

Any component of the plane wave $\xi^{s,\nu}_{\mu}$ is possible to expand in powers of the $\mathcal{D}^{(s)}$-functions:

$$\xi^{s,\nu}_{\mu}(\underline{p},\underline{r}) = \sum_{J=s}^{\infty}\sum_{M=-J}^{J}(2J+1)\,d^{[\nu,r]}_{Js\mu}(\chi)\,\overset{+}{\mathcal{D}}{}^{(J)}_{\mu M}(\underline{n}_p)\,\mathcal{D}^{(J)}_{M\nu}(\underline{n}). \qquad (13)$$

From eq.(13) it follows an integral representation for the function $d^{[\nu,r]}_{Js\mu}(\chi)$:

$$d^{[\nu,r]}_{Js\mu}(\chi) = \int_{-1}^{1}dx\,(\operatorname{ch}\chi - x\operatorname{sh}\chi)^{-1+ir}\,d^{(s)}_{\mu\nu}\left(\frac{x\operatorname{ch}\chi-\operatorname{sh}\chi}{\operatorname{ch}\chi-x\operatorname{sh}\chi}\right)d^{(J)}_{\mu\nu}(x), \qquad (14)$$

where $d^{(s)}_{\mu\nu}$ -functions are matrix elements of the operator rotating around the second axis:

$$\mathcal{D}^{(s)}_{\mu\nu}(\alpha,\theta,\varphi) = e^{i(\mu\alpha+\nu\varphi)}\,d^{(s)}_{\mu\nu}(\cos\theta)$$

Therefore from eq.(14) the expansion, of the function $d^{[\nu,r]}_{Js\mu}(\chi)$

in powers of hypergeometric functions is obtained as follows:

$$d^{[\nu,r]}_{JS\mu}(\eta) = 2\left\{\frac{(J+\nu)!(J-\nu)!(S+\nu)!(S-\nu)!}{(J+\mu)!(J-\mu)!(S+\mu)!(S-\mu)!}\right\}^{1/2} \times$$

$$\times(-1)^{J+S-2\nu}\sum_{n_1,n_2}(-1)^{n_1+n_2}\binom{S+\mu}{n_1}\binom{S-\mu}{S-\nu-n_1}\binom{J+\mu}{n_2}\binom{J-\mu}{J-\nu-n_2}\times$$

$$\times\frac{(J+S-\mu-\nu-n_1-n_2)!\,(\mu+\nu+n_1+n_2)!}{(J+S+1)!}\,e^{-\eta(\mu+\nu+ir+1+n_1)}\times \tag{15}$$

$$\times\,{}_2F_1\left(1+ir+S,\ \mu+\nu+n_1+n_2,\ J+S+2,\ 1-e^{-2\eta}\right).$$

From eq.(14) or eq.(15) and from (9), (11) (for example) it is obvious that the function $d^{[\nu,r]}_{JS\mu}(\eta)$ is a matrix element of the boost for the $[\nu,r]$ irreducible $SL(2,C)$ group unitary representation of the principal series. So the orthogonality and completeness conditions for the plane waves are simple consequences from orthogonality and completeness of the $SL(2,C)$ and $SU(2)$ groups matrix elements and from the forms of the invariant measures in momentum and relativistic configurational spaces:

$$\frac{1}{(2\pi)^3}\sum_{\nu=-S}^{S}\int\frac{d^3\underline{p}}{2p_0}\,\xi^{(S)}_{\mu\nu}(\underline{p},\underline{r})\,\overset{+}{\xi}{}^{(S)}_{\nu\mu'}(\underline{p},\underline{r}') = \frac{r^2}{\mu^2+r^2}\,\delta(\underline{r}-\underline{r}')\,\delta_{\mu\mu'},$$

$$\frac{1}{(2\pi)^3}\sum_{\nu=-S}^{S}\int(\nu^2+r^2)\,dr\,d^2\underline{n}\,\overset{+}{\xi}{}^{(S)}_{\mu\nu}(\underline{p},\underline{r})\,\xi^{(S)}_{\nu\mu'}(\underline{p}',\underline{r}) = 2p_0\,\delta(\underline{p}-\underline{p}')\,\delta_{\mu\mu'}. \tag{16}$$

3. Free Hamiltonian for a particle with any spin

Let us consider the operators:

$$\underline{J}^{(S)} = D^{(S)}(\underline{n}^p)\,\underline{L}\,\overset{+}{D}{}^{(S)}(\underline{n}^p), \tag{17}$$

$$H_0^{(S)} = D^{(S)}(\underline{n}^p)\,H_0^{(0)}\,\overset{+}{D}{}^{(S)}(\underline{n}^p), \tag{18}$$

$$\underline{P}^{(S)} = D^{(S)}(\underline{n}^p)\,\underline{P}^{(0)}\,\overset{+}{D}{}^{(S)}(\underline{n}^p), \tag{19}$$

where $H_0^{(0)}$ is free Hamiltonian (3) and $\underline{P}^{(0)}$ is the momentum operator in zero spin case. The explicit expression for the operator $\underline{P}^{(0)}$ in spherical coordinates was obtained in ref. 17 .

$$P_1^{(0)} = \sin\theta\cos\varphi\,\frac{H_0^{(0)}}{2} - \left[\sin\theta\cos\varphi + \frac{i}{r}\left(\cos\theta\cos\varphi\,\partial_\theta - \frac{\sin\varphi}{\sin\theta}\,\partial_\varphi\right)\right]\exp i\partial_r,$$

$$P_2^{(0)} = \sin\theta\sin\varphi\,\frac{H_0^{(0)}}{2} - \left[\sin\theta\sin\varphi - \frac{i}{r}\left(\cos\theta\sin\varphi\,\partial_\theta + \frac{\cos\varphi}{\sin\theta}\,\partial_\varphi\right)\right]\exp i\partial_r,$$

$$P_3^{(0)} = \cos\theta\,\frac{H_0^{(0)}}{2} - \left[\cos\theta - \frac{i}{r}\sin\theta\,\partial_\theta\right]\exp i\partial_r.$$

It is easy to verify that the operators $\underline{J}^{(S)}, H_0^{(S)}, \underline{P}^{(S)}$

satisfy the commutation relations of Poincare algebra for the angular momentum operators and the four-dimensional momentum operator:

$$[\mathbf{J}^{(s)}, \mathbf{J}^{(s)}] = i\, \mathbf{J}^{(s)} \qquad [\mathbf{J}^{(s)}, H_0^{(s)}] = 0,$$

$$[\mathbf{J}^{(s)}, \mathbf{P}^{(s)}] = i\, \mathbf{P}^{(s)}, \qquad [H_0^{(s)}, \mathbf{P}^{(s)}] = 0, \tag{20}$$

$$[\mathbf{P}^{(s)}, \mathbf{P}^{(s)}] = 0 .$$

From the expression for the plane wave (11) and from eq. (18) one obtains:

$$\left[2\,\mathrm{ch}\, i\partial_r + \frac{2i}{r}\,\mathrm{sh}\, i\partial_r + \frac{1}{r^2}\, \mathbf{J}^{(0)2} \exp i\partial_r\right] \xi^{s,v}(\mathbf{p},\mathbf{r}) = 2p_0\, \xi^{s,v}(\mathbf{p},\mathbf{r}). \tag{21}$$

The finite-difference matrix operator in the left-hand side of eq.(21) is a free Hamiltonian for spin S particle. It is interesting to note that, as it can be seen from eq.(17), the free Hamiltonian $H_0^{(s)}$ depends on the momentum.

The nonrelativistic limit of eq.(21) has the form:

$$\left[-\frac{1}{r^2}\partial_r r^2 \partial_r + \frac{1}{r^2}(\mathbf{L}+\mathbf{S})^2\right] e^{i\mathbf{p}\cdot\mathbf{r}} D^{(s)}(\mathbf{n})\, \chi^{s,v} = p^2 e^{i\mathbf{p}\cdot\mathbf{r}} D^{(s)}(\mathbf{n})\, \chi^{s,v} \tag{22}$$

If one uses the operator equality:

$$D^{(s)}(\mathbf{n})\, \mathbf{L}\, \overset{+}{D}{}^{(s)}(\mathbf{n}) = \mathbf{L} + \mathbf{S} ,$$

one gets, instead of eq.(22):

$$\hat{\mathbf{p}}^2 e^{-i\mathbf{p}\cdot\mathbf{r}} \chi^{s,v} = p^2 e^{i\mathbf{p}\cdot\mathbf{r}} \chi^{s,v},$$

where $\hat{\mathbf{p}}^2$ is the usual three-dimensional Laplacian.

So, it is trivial to get for the momentum operator:

$$\mathbf{P}^{(s)} \xi^{s,v}(\mathbf{p},\mathbf{r}) = \mathbf{p}\, \xi^{s,v}(\mathbf{p},\mathbf{r}) \tag{23}$$

The nonrelativistic limit for the momentum $\mathbf{P}^{(s)}$ is the operator:

$$P_{NR}^{(s)} = D^{(s)}(\mathbf{n})(-i\nabla_r)\, \overset{+}{D}{}^{(s)}(\mathbf{n}) .$$

Using eqs.(21) and (23) and commutation relations of the Poincare algebra it is possible to find the form of the pure Lorentz transformation generators.

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