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SPIN STRUCTURE OF THE INTERACTION OF TWO RELATIVISTIC PARTICLES IN ONE-MESON EXCHANGE MODEL

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Спиновая структура взаимодействия двух релятивистских частиц в модели одномезонного обмена

В квазипотенциальном уравнении для частиц со спином в качестве квазипотенциала рассмотрены фейнмановские матричные элементы, отвечающие диаграммам однобозонного обмена. Показано, что их можно параметризовать в терминах элементов пространства Лобачевского так, что спиновая структура взаимодействия принимает вид непосредственного геометрического обобщения спиновой структуры потенциала в квантовой механике.

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Skachkov N.B.

Spin Structure of the Interaction of Two Relativistic Particles in One-Meson Exchange Model

In quasipotential equation for particles with spin the Feynman matrix elements, corresponding to diagramms of one-boson exchange, are taken as quasipotential. It is shown that after separation of the Wigner rotation from this matrix elements the rest part is local in the sense of Lobachevsky geometry. With the help of the obtained representation the quasipotential equation is transformed to a from, in which the spin structure of the interaction looks like direct geometrical relativistic generalization (in sense of exchange of the Euclidean geometry for the Lobachevsky one) of the spin structure of the potential in quantum mechanics.

Communications of the Joint Institute for Nuclear Research.
Dubna, 1973

1. The eqtuations for the scattering amplitude and wave function are the main tool used for description of the system of two relativistic particles. In Kadyshevsky's paper [1] a covariant equation, which describes the interaction of two relativistic particles, was obtained. In contrast with the Bethe-Salpeter equation, based on the Feynman diagram techniques, the Kadyshevsky equation was derived in the framework of a covariant Hamiltonian formulation of the quantum field theory, developed in [2]. As a result the momentum space integration in this equation is performed over a three-dimensional manifold of the hyperboloid

$$p_0^2 - \vec{p}^{\,2} = m^2 \qquad (1)$$

while in the corresponding Bethe-Salpeter equation there is integration over the four-dimensional momenta of the virtual particles. This feature makes it close with the three-dimensional quasi-potential approach of Logunov-Tavkhelidze [3,4] which was successfully used for calculation of the relativistic corrections to the energy levels and the magnetic moments of the hydrogen-type systems [5,6] as well for description of strong interaction processes [7] +).

+) See also [8]

It was shown in the case of the spinless particles that in Kadyshevsky quasipotential approach the relativism appears, in the equations for the scattering amplitude and the wave function, in such a form, that they look like direct relativistic generalization of the Lippman-Schwinger and Schrödinger equations [9]. This generalization is equivalent to an exchange of the Euclidean geometry of the momentum space in the nonrelativistic equations, for Lobachevsky geometry, realized on the upper sheet of the hyperboloid (1) in the relativistic equation.

The quasi-potential equation for particles with spin, based on the covariant Hamiltonian formulation of the quantum field theory, was obtained in [10]. The aim of the present article is to consider this quasi-potential equation as geometrical relativistic generalization of the equations for the particles with spin in quantum mechanics. We shall restrict ourselves with the consideration of interaction in the one-boson exchange approximation. It will be demonstrated, that the introduction of quantities, that are elements of the Lobachevsky space, allows one to simplify the spin structure of the quasi-potential and it takes the form of direct geometrical relativistic generalization of the spin structure of the potential in quantum mechanics.

2. Let us consider first the case of interaction of a spinor particle with scalar one. The process of scattering of these particles is shown in Fig. 1, where the solid line corresponds to the spinor particle, the dotted one - to the scalar particle

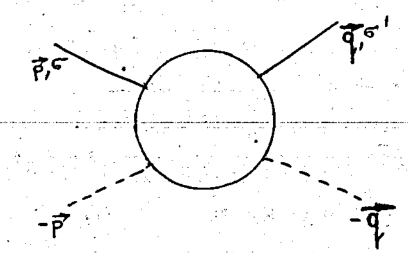


Fig. 1.

and the momenta of the initial and the final particles are in center-of-mass system (c.m.s.), i.e. $\vec{p}_1 = -\vec{p}_2 = \vec{p}$ and $\vec{q}_1 = -\vec{q}_2 = \vec{q}$ +).
Let us denote the initial state of two-particle system by $|\vec{p},\sigma\rangle$, where σ is spin projection on z-axis and the final state by $|\vec{q},\sigma'\rangle$.
The quasipotential equation for the relativistic scattering amplitude $\langle\vec{p},\sigma|T|\vec{q},\sigma'\rangle$ describing in c.m.s. the transition between the initial and final states, has

+) The masses of particles are taken to be equal. Generalization of the formalism to the case of particles with different masses was done in [11].

the form [10] .

$$\langle \vec{p},\sigma | T | \vec{q},\sigma' \rangle = \langle \vec{p},\sigma | V(E_q) | \vec{q},\sigma' \rangle + \tag{2}$$

$$+ \frac{1}{(4\pi)^3} \sum_{\sigma''=-\frac{1}{2}}^{\frac{1}{2}} \int \frac{d^3\vec{k}}{E_k} \cdot \frac{\langle \vec{p},\sigma | V(E_q) | \vec{k},\sigma'' \rangle \cdot \langle \vec{k},\sigma'' | T | \vec{q},\sigma' \rangle}{E_k \, (E_k - E_q - i\epsilon)} .$$

Here $E_k = \sqrt{\vec{k}^2 + m^2}$ and $\langle \vec{p},\sigma | V(E_q) | \vec{k},\sigma'' \rangle$ is the quasi-potential, which depends on the energy as a parameter and can have an imaginary part at $4E_q^2 = s > 4m^2$. The connection of the amplitude $\langle \vec{p},\sigma | T | \vec{q},\sigma' \rangle$ with the differential cross-section is given by

$$\frac{d\sigma}{d\omega} = \frac{1}{(8\pi)^2 \cdot s} \cdot \sum_{\sigma,\sigma'=-\frac{1}{2}}^{\frac{1}{2}} \left| \langle \vec{p},\sigma | T | \vec{q},\sigma' \rangle \right|^2 \tag{3}$$

Let us represent the scattering amplitude in the form

$$\langle \vec{p},\sigma | T | \vec{q};\sigma' \rangle = \sum_{\alpha,\beta} \bar{u}_\alpha^{\sigma}(\vec{p}) \, T_{\alpha\beta}(\vec{p},\vec{q}) \, u_\beta^{\sigma'}(\vec{q}) \tag{4}$$

$$(\alpha, \beta = 1,2,3,4),$$

where $u^{\sigma}(\vec{p})$ is a Dirac's bispinor.

Passing in (4) from bispinors to the two-component spinors ξ^{σ} we define the scattering matrix $t(\vec{p},\vec{q})$ as:

$$\sum_{\alpha,\beta} \bar{u}_\alpha^{\sigma}(\vec{p}) \, T_{\alpha\beta}(\vec{p},\vec{q}) \, u_\beta^{\sigma'}(\vec{q}) = \xi^{+\sigma} \, t_{\sigma\sigma'}(\vec{p},\vec{q}) \, \xi^{\sigma'} . \tag{5}$$

(In the right-hand side there is no summation over polarization indices). From (2) follows equation for the matrix $t_{\sigma\sigma'}(\vec{p},\vec{q})$,

$$t_{\sigma\sigma'}(\vec{p},\vec{q}) = V_{\sigma\sigma'}(\vec{p},\vec{q};E_q) +$$

$$+ \frac{1}{(4\pi)^3} \cdot \sum_{\sigma''=-\frac{1}{2}}^{\frac{1}{2}} \int \frac{d^3\vec{k}}{E_k} \cdot \frac{V_{\sigma\sigma''}(\vec{p},\vec{k};E_q) \, t_{\sigma''\sigma'}(\vec{k},\vec{q})}{E_k \, (E_k - E_q - i\varepsilon)} . \tag{6}$$

The wave function for the continuous spectrum can be defined as

$$\Psi_{q\sigma}(\vec{p}) = \frac{(2\pi)^3}{m} \delta(\vec{p}-\vec{q}) \sqrt{\vec{p}^{\,2}+m^2} \, \varphi_\sigma + \tag{7}$$

$$+ \frac{1}{8m} \cdot \frac{\sum_{\sigma'=-1/2}^{1/2} t_{\sigma\sigma'}(\vec{p},\vec{q}) \, \varphi_{\sigma'}}{E_p \, (E_p - E_q - i\varepsilon)} .$$

Substitution of (7) into (6) leads to equation for the wave function

$$\Psi_{q\sigma}(\vec{p}) = \frac{(2\pi)^3}{m} \cdot \delta(\vec{p}-\vec{q})\sqrt{\vec{p}^2+m^2}\,\varphi_\sigma + \tag{7'}$$

$$+\frac{1}{E_p(E_p-E_q-i\varepsilon)}\frac{1}{(4\pi)^3}\cdot\sum_{\sigma'=-\frac{1}{2}}^{\frac{1}{2}}\int\frac{d^3\vec{k}}{E_k}V_{\sigma\sigma'}(\vec{p},\vec{k};E_q)\,\Psi_{q\sigma'}(\vec{k})$$

or

$$E_p(E_p-E_q)\Psi_{q\sigma}(\vec{p})=\frac{1}{(4\pi)^3}\cdot\sum_{\sigma'=-\frac{1}{2}}^{\frac{1}{2}}\int\frac{d^3\vec{k}}{E_k}\,V_{\sigma\sigma'}(\vec{p},\vec{k};E_q)\,\Psi_{q\sigma'}(\vec{k})$$

It is clear from (6) and (7) that the "energy denominator" $\frac{1}{E_p(E_p-E_q-i\varepsilon)}$, which plays the role of a Green function in this equations, does not depend from spin indices and is the same as in spinless case. The quasi-potential equation for the case of particles with spin differs from the equation in the scalar case only in the spin dependence of quasi-potential. Thus, the formalism described here is the same as in nonrelativistic theory, where the free Hamiltonian is a scalar in spin space and only terms of interaction depend from spin.

Let us also note the fact that the integration in (6) and (7') is performed with volume element $d\Omega_{\vec{k}} = \frac{d^3\vec{k}}{E_k}$ which is the volume element in Lobachevsky space, realized on the upper sheet of hyperboloid (1). In equations (6) and (7') all the momenta of particles are on the mass-shell (1), i.e. they belong to the upper sheet of hyperboloid. Thus we can say that the momentum space in quasi-potential equations (6) and (7'), has geometrical properties of a Lobachevsky space.

Our aim is to show that it is possible in the case of particles with spin to describe an interaction by quasi-potential, which is local in Lobachevsky space, i.e. depends on the distance in this space. For this purpose let us consider some concrete quasi-potentials $V_{\sigma\sigma'}(\vec{p},\vec{k};E_q)$. In accordance with the rules of constructing of the quasi-potential from the matrix elements of the scattering amplitude [1,3,9] in the second order of the coupling constant g quasi-potential is proportional to the Feynman matrix element, that corresponds to the diagram, drown in Fig.2:

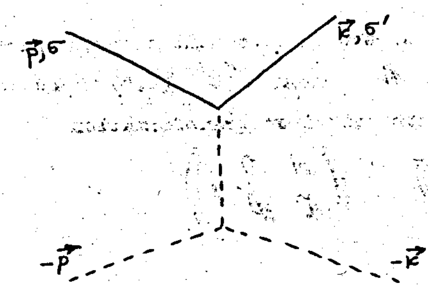


Fig.2.

and the equality takes place:+)

$$\langle \vec{p},\sigma | V^{(2)}(E_q) | \vec{k},\sigma' \rangle = - g^2 \cdot \frac{\bar{u}^{\sigma}(\vec{p})\, u^{\sigma'}(\vec{k})}{\mu^2 - (p-k)^2} \qquad (8)$$

In this expression all the momenta are on the mass-shell $p_0^2 - \vec{p}^{\,2} = m^2$, but are off the energy-shell, i.e. $p_0 \neq k_0$; when extrapolating the quasi-potential off the energy shell the terms are conserved that take into account the effect of relativistic retarding.

Now let us choose such a representation in which to any transformation A from SL (2.C) there corresponds diagonal matrix of the bispinor transformation

$$S(A) = \begin{pmatrix} A & 0 \\ 0 & A^{+-1} \end{pmatrix}$$

This representation is called spinor one and to it

+) This expression can be obtained from the analogous Kadyshevsky diagram, corresponding to the second order of g^2, by putting in it $\varkappa = \varkappa' = 0$ (see [16]), that leads to conservation of four-momentum.

there corresponds the choice of γ^{μ} -matrices in the form

$$\gamma^{\mu} = \begin{pmatrix} 0 & g^{\mu\mu}\sigma^{\mu} \\ \sigma^{\mu} & 0 \end{pmatrix} \;;\quad \gamma^5 = i \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

where $g^{\mu\mu} = (1,-1,-1,-1)$. The matrix $H_k \in SL(2.C)$

$$H_k = \frac{m + k_0 + \vec{\sigma}\vec{k}}{\sqrt{2m\cdot(k^0+m)}}$$

is pure Lorentz transformation $\Lambda_{\vec{k}}$, that transforms the four-vector with components $(m, 0)$ into the vector $(k_0, \vec{k})$. Then we shall pass in (8) with the help of the transformation $S(H_k)$ to bispinors, defined in the rest system (see Appendix)

$$\langle \vec{p},\sigma | V^{(2)}(E_q) | \vec{k},\sigma' \rangle = g^2 \cdot \frac{\bar{u}^{\sigma}(0)\, S^{-1}(H_p)\, S(H_k)\, u^{\sigma'}(0)}{(p-k)^2 - \mu^2} .$$

Pure Lorentz transformations do not form a group: their product is not pure Lorentz transformation, but containes also three-dimensional Wigner rotation, described by the matrix $V(A,k) \in SU(2)$ and defined by the equality+):

$$V(A,k) = H_k^{-1} \cdot A \cdot H_{\Lambda_A^{-1} \cdot k} \qquad ; \quad A \in SL\,(2.C)$$

+) With this definition the wave function $\Psi_{\sigma}(\vec{k})$ transforms under the Lorentz transformation $\Lambda_{\vec{p}}$, that corresponds to matrix $H_p \in SL$ (2.C), like

$$T_{H_p} \Psi_{\sigma}(\vec{k}) = \sum_{\sigma'=-s}^{s} D^{s}_{\sigma\sigma'}\{V(H_p,k)\}\, \Psi_{\sigma'}(\Lambda_p^{-1}\vec{k}) \quad (\sigma')$$

$$S^{-1}(H_p)\,S(H_k) = S(H_{\Lambda_p^{-1}k})\cdot\begin{pmatrix}1&0\\0&1\end{pmatrix}\otimes V^{-1}(H_p,k) \qquad (9)$$

Let us introduce now the next notation for the four-vector $\Lambda_{\vec{p}}^{-1}k$

$$(\Lambda_p^{-1}k)^0 \equiv (k(-)p)^0 = \frac{k^0p^0 - \vec{k}\vec{p}}{m} = \sqrt{m^2 + (\vec{k}(-)\vec{p})^2}$$

$$\overrightarrow{(\Lambda_p^{-1}k)} \equiv \vec{k}(-)\vec{p} = \vec{k} - \frac{\vec{p}}{m}\left(k^0 - \frac{\vec{k}\vec{p}}{p^0+m}\right). \qquad (10)$$

Three-vector $\vec{k}(-)\vec{p}$ is the "difference" of two vectors in Lobachevsky space. In the nonrelativistic limit it transforms into the usual difference $\vec{k}-\vec{p}$ of two vectors in Euclidean space. So it can be considered as relativistic generalization of the three-dimensional vector of momentum transfer. The denominator in (9) can be expressed with the help of this vector and as a result (8) takes the form:

$$\langle\vec{p},\sigma|V^{(2)}(E_q)|\vec{k},\sigma'\rangle = -g^2\cdot\frac{\bar{u}^{\sigma}(0)\cdot S(H_{k(-)p})\cdot D^{\frac{1}{2}}\{V^{-1}(H_p,k)\}\,u^{\sigma'}(0)}{\mu^2 - 2m^2 + 2m\sqrt{m^2+(\vec{k}(-)\vec{p})^2}} =$$

$$= \sum_{\sigma'_p=-\frac{1}{2}}^{\frac{1}{2}}\langle\vec{p},\sigma|V^{(2)}(E_q)|\vec{k},\sigma'_p\rangle\cdot D^{\frac{1}{2}}_{\sigma'_p\sigma'}\{V^{-1}(H_p,k)\}. \qquad (11)$$

Thus the quasi-potential (8) can be represented as product of a Wigner rotation and a new quasi-potential $V^{(2)}_{\sigma\sigma'_p}(E_q)$ parametrized by a vector $\vec{\Delta} = \vec{k}(-)\vec{p}$ belonging to the Lobachovsky space and the quantity $\vec{\sigma}\vec{\Delta}$ which is present in $H_{k(-)p}$. The quasi-potential $V^{(2)}_{\sigma\sigma'_p}(E_q)$ separated in such a way is local in Lobachevsky space.

Parametrization (11) is olosely conneoted with the obtained by Cheshkov and Shirokov [12] parametrization for the matrix elements of the local operators. In our notations their formula has the form:

$$\langle\vec{p},\sigma|j(0)|\vec{k},\sigma'\rangle =$$

$$= \frac{1}{(2\pi)^3}\sum_{\sigma_p=-s}^{s}\sum_{n=0}^{2s}\langle\sigma|\{ik^{\mu}\cdot W_{\mu}(\vec{p})\}^n|\sigma_p\rangle\cdot D^{s}_{\sigma_p\sigma'}\{V^{-1}(H_p,k)\}\cdot f_n(t) \qquad (12)$$

$$t = (p_0-k_0)^2 - (\vec{p}-\vec{k})^2,$$

where the invariant formfaotors $f_n(t)$ are obtained as a result of expansion of the matrix elements in linear independent scalars under rotations – the scalar products of momentum k^{μ} with relativistio spin four-vector $W_{\mu}(\vec{p})$:

$$W_0(\vec{p}) = \vec{p}\vec{\sigma}\;; \quad \vec{W}(\vec{p}) = m\vec{\sigma} + \frac{\vec{p}(\vec{p}\vec{\sigma})}{p^0+m}.$$

The vector $W_{\mu}(\vec{p})$ is obtained after a pure Lorentz transformation

$$W_{\mu}(\vec{p}) = (\Lambda_p)_{\mu}^{\nu}\,W_{\nu}(0). \qquad (13)$$

In the rest system it has components:

$$\vec{W}(0) = m\vec{\sigma} \quad ; \quad W_0(0) = 0 \,. \tag{14}$$

The following condition for $W_\mu(\vec{p})$ is valid

$$p^\mu W_\mu(\vec{p}) = 0 .$$

Due to the Lorentz invariance of the scalar product $k^\mu W_\mu(\vec{p})$ and taking into account (13) and (14) we can write:

$$k^\mu W_\mu(\vec{p}) = (\vec{k}(-)\vec{p})\,\vec{\sigma} .$$

That gives the parametrization of the matrix element $\langle \vec{p},\sigma | j(0) | \vec{k},\sigma' \rangle$ in three-dimensional form

$$\langle \vec{p},\sigma | j(0) | \vec{k},\sigma' \rangle =$$

$$= \frac{1}{(2\pi)^3} \sum_{\sigma'_p=-s}^{s} \sum_{n=0}^{2s} \langle \sigma | \{ i\,\vec{\sigma}(\vec{k}(-)\vec{p}) \}^n | \sigma'_p \rangle \, D^s_{\sigma'_p \sigma'} \{ V^{-1}(H_p, k) \} . \tag{12'}$$

The last equality makes clear the connection of the parametrization (12) with representation (11), obtained for the Feynman matrix element (8).

The presence in (12) of D^s -function, containing Wigner rotation, is connected with the following fact. The transformation law for the state vectors

$$\mathcal{U}(H_p^{-1}) | \vec{k},\sigma' \rangle = \sum_{\sigma''=-s}^{s} D^s_{\sigma'\sigma''} \{ V^{-1}(H_p,k) \} | \Lambda_p^{-1} k, \sigma'' \rangle$$

implies that matrices describing spin rotations under Lorentz transformations depend on the momentum of the state and so they are different for left and right spin indices in the matrix element (12). In the terminology of the authors of paper [12], the spin indices are "sitting" on different momenta. That is why the spin indices have to be "removed" on one momentum in order to obtain the invariant parametrization of the matrix element. This is just fullfilled+) in (11) and (12) by the function $D^s\{V^{-1}(H_p,k)\}$.

Let us note, than application of the equality (12)

$$k^\mu W_\mu(\vec{p})\, D^s\{ V^{-1}(H_p,k) \} = - D^s\{ V(H_k,p) \}\, p^\mu W_\mu(\vec{k})$$

allows one to write analogous to (12) parametrization for the element in which the spin indices are "sitting" on momentum $\vec{k}$. In quasi-potential (8) and (11) it is equivalent of using of the equality

$$S^{-1}(H_p)\, S(H_k) = V(H_k,p) \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot S^{-1}(H_{p(-)k}) \tag{15}$$

instead of (9)+).

+) In more detailes this question is discussed in [12 -14].

Quite naturally the question arises if it is possible to transform an equation (7) in such a way in order to extract from the Feynman matrix element (8) the function $\mathcal{D}^{S}\{V^{-1}(H_p,k)\}$ which has a pure kinematical origin. We shall use for this purpose the fact that in the right-hand side of equation (7) the summation is carried over the spinor index of the wave function $\Psi_{q\sigma'}(\vec{k})$. So, let us define under the sign of integration in (7) a new wave function

$$\Psi_{q\sigma'_p}(\vec{k}) = \sum_{\sigma'=-\frac{1}{2}}^{\frac{1}{2}} \mathcal{D}^{\frac{1}{2}}_{\sigma'_p\sigma'}\{V^{-1}(H_p,k)\}\,\Psi_{q\sigma'}(\vec{k}) \tag{16}$$

spin index of which shall "sit" on momentum $\vec{p}$.

Thus we have shown that equation (7) with the quasi-potential (8), can be transformed to the form

$$E_p(E_p-E_q)\Psi_{q\sigma}(\vec{p}) = \frac{1}{(4\pi)^3}\sum_{\sigma'_p=-\frac{1}{2}}^{\frac{1}{2}}\int\frac{d^3\vec{k}}{E_k}\,V^{(2)}_{\sigma\sigma'_p}(\vec{k}(-)\vec{p};E_q)\,\Psi_{q\sigma'_p}(\vec{k}), \tag{17}$$

+) Parametrization of quasi-potential, based on (15) was obtained in [15].

where all spin indices σ and σ'_p are "sitting" on one momentum $\vec{p}$. As a result of such a transformation the interaction is described by the quasi-potential $V^{(2)}_{\sigma\sigma'_p}(\vec{k}(-)\vec{p};E_q)$, which is local in Lobachevsky space. From (11) we get

$$V^{(2)}_{\sigma\sigma'_p}(\vec{k}(-)\vec{p};E_q) = -g^2\,\frac{\bar{u}^{\sigma}(0)\,S(H_{k(-)p})\,u^{\sigma'_p}(0)}{\mu^2-2m^2+\sqrt{m^2+(\vec{k}(-)\vec{p})^2}} = -2\sqrt{2m(\Delta_0+m)}\,\frac{\delta_{\sigma\sigma'_p}}{\mu^2-2m^2+2m\Delta_0}; \tag{18}$$

where in accordance with (10)

$$\Delta_0 = \sqrt{m^2+(\vec{k}(-)\vec{p})^2} = \sqrt{m^2+(\vec{p}(-)\vec{k})^2}. \tag{19}$$

3. Let us consider now the case of scattering of two spinor particles. Quasi-potential equation for this case was obtained in [10]. For the wave function of the system it has the form:

$$\Psi_q(\vec{p})_{\sigma_1\sigma_2} = \frac{(2\pi)^3}{m}\,\delta(\vec{p}(-)\vec{q})\sqrt{\vec{p}^2+m^2}\,\varphi_{\sigma_1}\varphi_{\sigma_2} + \frac{1}{E_p(E_p-E_q-i\varepsilon)}\,\frac{1}{(4\pi)^3}\sum_{\sigma'_1,\sigma'_2=-\frac{1}{2}}^{\frac{1}{2}}\int\frac{d^3\vec{k}}{E_k}\,V_{\sigma_1\sigma_2;\sigma'_1\sigma'_2}(\vec{p},\vec{k};E_q)\,\Psi_q(\vec{k})_{\sigma'_1\sigma'_2}, \tag{20}$$

or

$$E_p(E_p-E_q)\Psi_q(\vec{p})_{\sigma_1\sigma_2} = \frac{1}{(4\pi)^3}\sum_{\sigma'_1,\sigma'_2=-\frac{1}{2}}^{\frac{1}{2}}\int\frac{d^3\vec{k}}{E_k}\,V_{\sigma_1\sigma_2;\sigma'_1\sigma'_2}(\vec{p},\vec{k};E_q)\,\Psi_q(\vec{k})_{\sigma'_1\sigma'_2} \tag{20'}$$

In one-boson exchange approximation the quasi-potential corresponds to the Feynman graph:

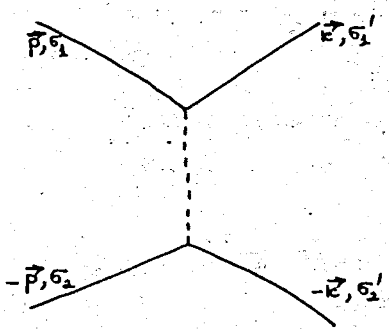


Fig.3.

In accordance with the general rules of construction of the quasi-potential from the matrix elements of the relativistic scattering amplitude [1,9], we have, for the case of exchange of scalar particle with mass μ, the next expression:

$$\langle \vec{p}, \sigma_1 \sigma_2 | V_S^{(2)}(E_q) | \vec{k}, \sigma_1' \sigma_2' \rangle =$$

$$= -g^2 \cdot \frac{\left[\bar{u}^{\sigma_1}(\vec{p}) u^{\sigma_1'}(\vec{k})\right] \cdot \left[\bar{u}^{\sigma_2}(-\vec{p}) u^{\sigma_2'}(-\vec{k})\right]}{\mu^2 - (p-k)^2} \qquad (21)$$

and for the case of exchange of pseudo-scalar (PS) particle

$$\langle \vec{p}, \sigma_1 \sigma_2 | V_{PS}^{(2)}(E_q) | \vec{k}, \sigma_1' \sigma_2' \rangle =$$

$$= -g^2 \frac{\left[\bar{u}^{\sigma_1}(\vec{p}) \gamma^5 u^{\sigma_1'}(\vec{k})\right] \cdot \left[\bar{u}^{\sigma_2}(-\vec{p}) \gamma^5 u^{\sigma_2'}(-\vec{k})\right]}{\mu^2 - (p-k)^2}. \qquad (22)$$

Let us perform now in the spinor part of expressions (21) and (22) the transformations (9)$^{+)}$. It leads to the representation of (21) and (22) in the form, analogous to (11):

$$\langle \vec{p}, \sigma_1 \sigma_2 | V^{(2)}(E_q) | \vec{k}, \sigma_1' \sigma_2' \rangle =$$

$$= \sum_{\sigma_{1p}', \sigma_{2p}' = -\frac{1}{2}}^{\frac{1}{2}} \langle \vec{p}, \sigma_1 \sigma_2 | V^{(2)}(E_q) | \vec{k}, \sigma_{1p}', \sigma_{2p}' \rangle D^{1/2}_{\sigma_{1p}' \sigma_1'} \left\{ V^{-1}(H_p, k) \right\} \cdot D^{1/2}_{\sigma_{2p}' \sigma_2'} \left\{ V^{-1}(H_p, k) \right\} \qquad (23)$$

Thus with account of (18) we get

$$\langle \vec{p}, \sigma_1 \sigma_2 | V_S^{(2)}(E_q) | \vec{k}, \sigma_{1p}' \sigma_{2p}' \rangle =$$

$$= -g^2 \cdot \frac{\left[\bar{u}^{\sigma_1}(0) S(H_{k(-)p}) u^{\sigma_{1p}'}(0)\right] \left[\bar{u}^{\sigma_2}(0) S(H_{k(+)p}) u^{\sigma_{2p}'}(0)\right]}{\mu^2 - 2m^2 + 2m\sqrt{m^2 + (\vec{k}(-)\vec{p})^2}} = \qquad (24)$$

$^{+)}$Matrix $S(H_k)$ depends on the Dirac matrix $\vec{\alpha} = \gamma^0 \vec{\gamma}$ (see Appendix) and so commute with γ^5.

$$= -g^2 \frac{\delta_{\sigma_1 \sigma'_{1p}} \cdot \delta_{\sigma_2 \sigma'_{2p}}}{\mu^2 - 2m^2 + 2m\Delta_0} \cdot 2m(m+\Delta_0)$$

and in the same way

$$\langle \vec{p}, \sigma_1 \sigma_2 | V^{(2)}_{PS}(E_q) | \vec{k}, \sigma'_{1p} \sigma'_{2p} \rangle =$$

$$= -g^2 \cdot \frac{\left[\bar{u}^{\sigma_1}(0)\gamma^5 S(H_{k(-)p}) u^{\sigma'_{1p}}(0)\right]\left[\bar{u}^{\sigma_2}(0)\gamma^5 S(H_{-k(+)p}) u^{\sigma'_{2p}}(0)\right]}{\mu^2 - 2m^2 + 2m\sqrt{m^2 + (\vec{k}(-)\vec{p})^2}} =$$

$$= g^2 \cdot \frac{(\vec{\sigma}_1 \vec{\Delta})_{\sigma_1 \sigma'_{1p}} \cdot (\vec{\sigma}_2 \vec{\Delta})_{\sigma_2 \sigma'_{2p}}}{\mu^2 - 2m^2 + 2m\Delta_0} \cdot \frac{2m}{(\Delta_0 + m)} \qquad (25)$$

where

$$\vec{\Delta} = \vec{k}(-)\vec{p} \quad ; \quad \Delta_0 = \sqrt{m^2 + \vec{\Delta}^2}$$

and

$$(\vec{\sigma}\vec{\Delta})_{\sigma_1 \sigma_{1p}} = \xi^{+\sigma_1} (\vec{\sigma}\vec{\Delta}) \xi^{\sigma_{1p}} .$$

In complete analogy with (16), we can perform a "change" of spin indices of the wave function $\Psi_q(\vec{k})_{\sigma'_1 \sigma'_2}$ from momentum $\vec{k}$ on momentum $\vec{p}$:

$$\Psi_q(\vec{k})_{\sigma'_{1p}\sigma'_{2p}} =$$

$$= \sum_{\sigma'_1 \sigma'_2 = -\frac{1}{2}}^{\frac{1}{2}} D^{1/2}_{\sigma'_{1p}\sigma'_1}\{V^{-1}(H_p, k)\} D^{1/2}_{\sigma'_{2p}\sigma'_2}\{V^{-1}(H_p, k)\} \Psi_q(\vec{k})_{\sigma'_1 \sigma'_2} . \qquad (26)$$

With the help of relations (23), (26) and taking into account (24) and (25) we can rewrite equation (20') in the form

$$E_p(E_p - E_q)\Psi_q(\vec{p})_{\sigma_1 \sigma_2} =$$

$$= \frac{1}{(4\pi)^3} \sum_{\sigma'_{1p}, \sigma'_{2p} = -\frac{1}{2}}^{\frac{1}{2}} \int \frac{d^3 \vec{k}}{E_k} \cdot V_{\sigma_1 \sigma_2 ; \sigma'_{1p} \sigma'_{2p}}(\vec{k}(-)\vec{p}; E_q) \Psi_q(\vec{k})_{\sigma'_{1p} \sigma'_{2p}} , \qquad (27)$$

where all spin indices are "sitting" on the same momentum $\vec{p}$.

Equations (20), (20') and (27) are written in the c.m.s. and the wave function $\Psi_q(\vec{p})_{\sigma_1 \sigma_2}$ describes the relative motion of two particles with spin $S = \frac{1}{2}$. That is why we can add particle spins with the use of usual rules of addition of two momenta in quantum mechanics. Let us define wave function with total spin $S = 0, 1$ with the help of the equality

$$\Psi_q(\vec{p})_{S,\sigma} = \sum_{\sigma_1, \sigma_2 = -\frac{1}{2}}^{\frac{1}{2}} \langle \tfrac{1}{2} \tfrac{1}{2} ; \sigma_1 \sigma_2 | S \sigma \rangle \Psi_q(\vec{p})_{\sigma_1 \sigma_2} . \qquad (28)$$

The relation for Clebsh-Gordan coefficients

$$\sum_{\sigma_1',\sigma_2'=-\frac{1}{2}}^{1/2} \langle \tfrac{1}{2}\tfrac{1}{2};\sigma_1'\sigma_2'|S\sigma\rangle \langle \tfrac{1}{2}\tfrac{1}{2};\sigma_1'\sigma_2'|S'\sigma'\rangle = \delta_{SS'}\cdot\delta_{\sigma\sigma'} \tag{29}$$

allows one to rewrite equation (27) in the form

$$E_p(E_p-E_q)\Psi_q(\vec{p})_{S\sigma} = \tag{30}$$

$$= \frac{1}{(4\pi)^3}\sum_{S'=0,1}\sum_{\sigma_p'=-S'}^{S'}\int\frac{d^3\vec{k}}{E_k}\, V^{(2)}_{S\sigma;S'\sigma_p'}(\vec{k}(-)\vec{p};E_q)\,\Psi_q(\vec{k})_{S'\sigma_p'},$$

where we have defined

$$\langle\vec{p},S\sigma|V^{(2)}(E_q)|\vec{k},S'\sigma_p'\rangle = \tag{31}$$

$$=\sum_{\sigma_1,\sigma_2=-\frac{1}{2}}^{1/2}\ \sum_{\sigma_{1p}',\sigma_{2p}'=-\frac{1}{2}}^{1/2}\langle\tfrac{1}{2}\tfrac{1}{2};\sigma_1\sigma_2|S\sigma\rangle\langle\vec{p},\sigma_1\sigma_2|V^{(2)}(E_q)|\vec{k},\sigma_{1p}'\sigma_{2p}'\rangle\langle\tfrac{1}{2}\tfrac{1}{2};\sigma_{1p}'\sigma_{2p}'|S'\sigma_p'\rangle.$$

It is known, that in some processes the symmetry properties of two-particle systems lead to a conservation of the total spin, i.e. that the total spin becomes integral of motion. Thus, for example, for positronium the conservation of the total spin comes from CP invariance of electromagnetic interactions. At the scattering of identical spinor-

particles singlet-triplet transitions are forbidden because of the Pauli principle, and in neutron-proton scattering the absence of singlet-triplet transitions (when the electromagnetic interactions are neglected) follows from "charge symmetry" principle,In what follows we shall restrict our consideration only with such processes, so we shall take the quasi-potential diagonal in total spin:

$$\langle\vec{p},S\sigma|V(E_q)|\vec{k},S'\sigma'\rangle = \langle\vec{p},S\sigma|V(E_q)|\vec{k},S\sigma'\rangle\cdot\delta_{SS'}$$

In this case the substitution of qusipotential (24) in (31) leads to a scalar in spin space .

$$\langle\vec{p},S\sigma|V(E_q)|\vec{k},S\sigma'\rangle = \tag{32}$$

$$= -g^2\frac{\delta_{\sigma\sigma'}}{\mu^2-2m^2+2m\Delta_0}\cdot 2m(m+\Delta_0)$$

and for quasi-potential (25) application of the formula

$$(\vec{\sigma}_1\vec{\Delta})(\vec{\sigma}_2\vec{\Delta}) = 2(\vec{S}\vec{\Delta})^2-\vec{\Delta}^2, \tag{33}$$

where

$$\vec{S} = \frac{1}{2}(\vec{\sigma}_1+\vec{\sigma}_2) \tag{34}$$

permits to divide it in parts, which give a contribution in singlet and triplet states:

$$V_{PS}^{(2)}(E_q) = + g^2 \frac{(\vec{S}\vec{\Delta})^2}{\mu^2 - 2m^2 + 2m\Delta_0} \cdot \frac{4m}{\Delta_0 + m} \tag{35}$$

$$- g^2 \frac{\Delta_0 - m}{\mu^2 - 2m^2 + 2m\Delta_0} \cdot 2m$$

Let us denote by the example of quasi-potential (22) one remarkable feature of formalism developed here. Usually when extracting information about interaction from the quantum field theory, one puts in correspondence with expression (22) potential

$$V(\vec{p}, \vec{k}) = - g^2 \frac{(\vec{\sigma}_1 \vec{\Delta}_э)(\vec{\sigma}_2 \vec{\Delta}_э)}{\mu^2 - \vec{\Delta}_э^2} \tag{36}$$

obtained from (22) in the nonrelativistic limit. In (36) the vector $\vec{\Delta}_э = \vec{k} - \vec{p}$ is the difference of two vectors in Euclidean space. In our approach, after exraction of the Wigner rotation, that has purely kinematical origin, we obtain quasi-potential (25). With this the spin structure of interaction is defined by the same expression $(\vec{\sigma}_1 \vec{\Delta})(\vec{\sigma}_2 \vec{\Delta})$ that in (36) with the only difference that in relativistic case $\vec{\Delta} = \vec{k} (-) \vec{p}$ is a difference of vectors in Lobachevsky space.

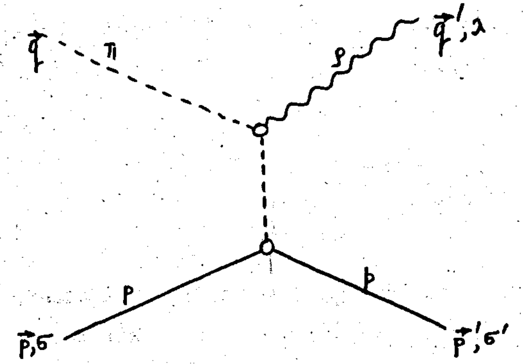
The form-factor $\frac{1}{\mu^2 + 2m(\Delta_0 - m)} \cdot \frac{}{(\Delta_0 + m)}$ at the spin structure in (25) depends on the square of the vector $\vec{\Delta} = \vec{k} (-) \vec{p}$, i.e. it is local in Lobachevsky space, while in (36) the nonrelativistic form factor $\frac{1}{\mu^2 - \vec{\Delta}_э^2}$ is local in Euclidean space.

The role of factor $\frac{1}{\Delta_0 + m}$ in (25) can be easily understood if one would consider this expression at $\mu = 0$, i.e. at scalar photon exchange. With the use of expression $\Delta_0^2 - \vec{\Delta}^2 = m^2$ in (25) we obtain for this case instead of (25) the expression

$$\frac{(\vec{\sigma}_1 \vec{\Delta}) \cdot (\vec{\sigma}_2 \vec{\Delta})}{\vec{\Delta}^2} \tag{37}$$

that has the same form as the nonrelativistic expression (36) at $\mu = 0$. Thus it is possible to say that in the case of exchange by massless particle the quasi-potential (25) after separtion of the Wigner rotation, has an "absolute" character in the transition from nonrelativistic theory to the relativistc one. The transition from one theory to another in obtained parametrization is equivalent to exchange of Euclidean geometry of the nonrelativistic momentum space by Lobachevsky geometry in relativistic case.

4. The obtained three-dimensional parametrization of the matrix elements of the scattering amplitude in terms of elements of Lobachevsky space can be applied to other processes with scalar or pseudo-scalar meson exchange. Let us consider, for example, the process with vector meson ρ : $\pi^+ + p \to \rho^+ + p$ [16]. In Born approximation the amplitude of this process that corresponds to Feynman diagramm



is defined by

$$T^{(2)}(p,p') = \frac{2i\, g \cdot f_{\rho\pi}}{(2\pi)^6} \cdot \frac{\bar u^{\sigma'}(\vec p')\, \gamma^5\, u^{\sigma}(\vec p)}{\mu_\pi^2 - (p-p')^2} \left(e^\mu(\lambda, q') \cdot q_\mu \right) \quad (38)$$

The four-vector $e^\mu(\lambda, q')$ that describes the polarization of vector meson (λ -polarization index , $\lambda = -1, 0, +1$) can be obtained from its value in the rest system by pure

Lorentz transformation $\Lambda_{q'}$:

$$e^\mu(\lambda, q') = (\Lambda_{q'})^\mu_\nu\, e^\nu(\lambda, 0) \, . \quad (39)$$

In the rest system the vector meson is characterized by the vectors:

$$e^\mu(1,0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ i \\ 0 \end{pmatrix} ; \qquad e^\mu(-1,0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -i \\ 0 \end{pmatrix} ;$$

$$e^\mu(0,0) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

so we have

$$e^0(\lambda, 0) = 0 \, .$$

Lorentz invariance of the scalar product $e^\mu(\lambda, q') \cdot q_\mu$ allows one to represent it with the help of (39) in three-dimentional form:

$$e^\mu(\lambda, \vec q') \cdot q_\mu = e^\nu(\lambda, 0) \left(\Lambda^{-1}_{q'} \right)^\mu_\nu \cdot q_\mu =$$

$$= \vec e(\lambda, 0) \cdot (\vec q \,(-)\, \vec q') = \vec e(\lambda, 0) \cdot \vec \Delta_q \, . \quad (40)$$

Taking into account the parametrization (18) and (25) for the current $\bar{u}^{\sigma'}(\vec{p}')\gamma^5 u^{\sigma}(\vec{p})$ we pass to the three-dimensional parametrization for the amplitude $T^{(2)}(\vec{p},\vec{p}')$ in terms of elements of Lobachevsky space:

$$T^{(2)}(\vec{p},\vec{p}') = \frac{2ig\cdot f_{\rho\pi\pi}}{(2\pi)^6}\cdot\frac{[\vec{\sigma}(\vec{p}(-)\vec{p}')]_{\sigma'\sigma}}{m+\Delta_p^0}\cdot\frac{\vec{e}(\lambda,0)\cdot(\vec{q}(-)\vec{q}')}{2m(m-\Delta_p^0)-\mu_\pi^2} =$$

$$= \frac{2ig\cdot f_{\rho\pi\pi}}{(2\pi)^6}\cdot\frac{[\vec{\sigma}\vec{\Delta}]_{\sigma'\sigma}}{m+\Delta_p^0}\cdot\frac{\vec{e}(\lambda,0)\cdot\vec{\Delta}_q}{2m(m-\Delta_p^0)-\mu_\pi^2}.$$

Analogous three-dimentional parametrization of the Feynman matrix elements in the case when the interaction is realized by vector particle will be given in the next paper.

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Appendix

Here we shall give an explicit expression for $\mathcal{D}^{\frac{1}{2}}\{V^{-1}(\Lambda_p,k)\}$ function, that contains the Wigner rotation, and an explicit form of the bispinors used in the text. Bispinors $u^{\sigma}(\vec{k})$ are normalized by the condition:

$$\bar{u}^{\sigma}(\vec{k})\, u^{\sigma'}(\vec{k}) = u^{+\sigma}(\vec{k})\gamma^0 u^{\sigma'}(\vec{k}) = 2m\,\delta_{\sigma\sigma'}\,.$$

The matrix of bispinor transformation $S(\Lambda_k)$, that corresponds to pure Lorentz transformation, has the form:

$$S(\Lambda_k) = \begin{pmatrix} \Lambda_k & 0 \\ 0 & \Lambda_k^{+-1}\end{pmatrix} = \sqrt{\frac{k_0+m}{2m}}\left(1+\frac{\vec{\alpha}\vec{k}}{k_0+m}\right) = \tag{I.1}$$

$$= ch\,\chi_k/2 + (\vec{\alpha}\,\vec{n}_k)\, sh\,\chi_k/2\,,$$

where $k_0 = m\, ch\,\chi_k$; $\vec{k} = m\, sh\,\chi_k\cdot\vec{n}_k$ $(\vec{n}_k^2=1)$.

In the spinor representation, used in the text, $\vec{\alpha}$ —matrix is:

$$\vec{\alpha} = \gamma^0\vec{\gamma} = \begin{pmatrix}\vec{\sigma} & 0\\ 0 & -\vec{\sigma}\end{pmatrix} \tag{I.2}$$

and the bispinor in the rest frame has the form

$$u^{\sigma}(0) = \sqrt{m}\begin{pmatrix}\xi^{\sigma}\\ \xi^{\sigma}\end{pmatrix}, \tag{I.3}$$

where two-component spinors are normalized by oondition $\xi^{+\sigma}\xi^{\sigma'} = \delta_{\sigma\sigma'}$, With the help of (I.3) and (I.2) we obtain an explicit form of bispinor in spinor representation:

$$u^{\sigma}(\vec{k}) = S(H_{\vec{k}})\,u^{\sigma}(0) = \sqrt{m}\begin{pmatrix} \dfrac{k_0+m+\vec{\sigma}\vec{k}}{\sqrt{2m(k_0+m)}} & \xi^{\sigma} \\ \dfrac{k_0+m-\vec{\sigma}\vec{k}}{\sqrt{2m(k_0+m)}} & \xi^{\sigma} \end{pmatrix} \qquad \text{(I.4)}$$

The transition from spinor to standard representation in which γ -matrixes are

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} ; \qquad \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} ;$$

oan be done with the help of the matrix $S_0 = S_0^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

As a result one has in the standard representation

$$u^{\sigma}(0) = \sqrt{2m}\begin{pmatrix} \xi^{\sigma} \\ 0 \end{pmatrix} \qquad \text{(I.5)}$$

and

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} . \qquad \text{(I.6)}$$

With the help of (I.1), where $\vec{\alpha}$ -matrix is defined by (I.6) we define the explicit form of the bispinor in standard representation

$$u^{\sigma}(\vec{k}) = S(H_k)\,u^{\sigma}(0) = \begin{pmatrix} \sqrt{k^0+m} & \xi^{\sigma} \\ \sqrt{k^0-m} & (\vec{\sigma}\vec{n}_k)\,\xi^{\sigma} \end{pmatrix} . \qquad \text{(I.7)}$$

The $D^{1/2}\{V^{-1}(H_{p,k})\}$-function, that contains Wigner rotation, has a simple form in the case of pure Lorentz transformation Λ_p :

$$D^{1/2}\{V^{-1}(H_{p,k})\} = H^{-1}_{k(-)p}\cdot H_p^{-1} H_k =$$

$$\text{(I.8)}$$

$$= \sqrt{\frac{(p_0+m)(k_0+m)}{2m(\Delta_0+m)}}\left\{ 1 - \frac{\vec{p}\vec{k} + i\,\vec{\sigma}\,[\vec{p}\,\vec{k}]}{(p_0+m)(k_0+m)} \right\} .$$

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