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AUTOMODEL BEHAVIOUR OF THE VIRTUAL COMPTON SCATTERING AMPLITUDE IN THE FRAMEWORK OF THE DYSON -JOST-LEHMANN REPRESENTATION

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AUTOMODEL BEHAVIOUR OF THE VIRTUAL COMPTON SCATTERING AMPLITUDE IN THE FRAMEWORK OF THE DYSON-JOST-LEHMANN REPRESENTATION

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Матвеев В.А., Робашик Д., Вицирек Э. **E2 - 7051**

Автомодельное поведение амплитуды виртуального комптон-эффекта в рамках представления Дайсона-Йоста-Лемана

Используя представления ДЙЛ для одночастичных матричных элементов T-произведения токов, исследуется автомодельное поведение амплитуды виртуального комптон-эффекта. Показано, что появляющийся в общем случае логарифмический член в асимптотике реальной части амплитуды исчезает благодаря кинематическим условиям. Найдено, что поведение реальной части амплитуды рассеяния содержит большую информацию о сингулярности на световом конусе, чем поведение мнимой части.

Препринт Объединенного института ядерных исследований.
Дубна, 1973

Matveev V.A., Robaschik D., Wiczorek E. **E2 - 7051**

Automodel Behaviour of the Virtual Compton Scattering Amplitude in the Framework of the Dyson-Jost-Lehmann Representation

Using a DJL representation for the one particle matrix elements of the T-product of currents the automodel behaviour of the virtual Compton scattering amplitude is studied. Logarithmic terms in the real part, which occur in general, vanish because of the kinematical constraints in the case of Compton scattering. It is observed that the real part of the scattering amplitude gives more information on the lead light cone singularity than the imaginary part.

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Dubna, 1973

1.

Recently the automodel properties of the structure functions for deep inelastic ep scattering have been studied on the basis of general principles of Quantum Field Theory by N.N.Bogolubov, A.N.Tavkhelidze, V.S.Vladimirov, /1/ . In that work causality properties of the invariant structure functions have been proved strictly. Furthermore sufficient conditions for the spectral function of the DJL representation are obtained which quarantee the automodel behaviour of the structure functions in the deep inelastic scattering region.

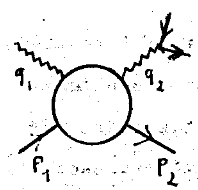
We remark that experimental and theoretical investigations in deep inelastic ep scattering are concerned essentially with the absorptive part of the virtual Compton amplitude in forward direction corresponding to a total crossection.

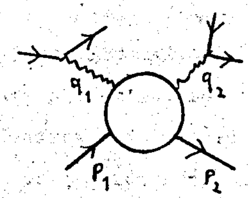
$$W_{\mu\nu}(q,p) = \frac{1}{8\pi}\sum_{\sigma}\int \langle p,\sigma | [j_\mu(x), j_\nu(0)] | p,\sigma\rangle e^{iqx} dx \quad (1.1)$$

Here using the notation of /1/ j_μ are the electromagnetic current components, q . is the four momentum of a virtual photon; the matrix elements are taken between identical one-nucleon states $|p,\sigma\rangle$ with the four momentum p of mass 1 $(p^2=1)$ and spin σ $(\sigma = \pm \frac{1}{2})$.

The experimental data of the above mentioned process, however, are not sufficient to determine the leading light cone singularity of the electromagnetic current commutator, if we do not impose additional restrictions on the Quantum Field Theory considered. We remember that causality and spectrality conditions are not sufficient for this purpose /2/ .

Therefore we want to study in the following virtual Compton scattering in non forward direction, because this process gives more information as compared with deep inelastic scattering. This has to include an investigation of the real part of the scattering amplitude in addition to the considerations of the imaginary part /3,4/ . Explicitly we have in mind the Compton scattering with Lepton pair production by a real or virtual incomming photon respectively





For these processes the interesting kinematical regions which are expected to give information on the light cone behaviour are

$$\nu = \frac{(P_1+P_2)(q_1+q_2)}{2} \to \infty \qquad -\frac{q_2^2}{\nu} = \xi_2 \text{ fix}, \quad q_1^2 = 0$$

$$q_2^2 \to \infty$$

or

$$\nu \to \infty$$

$$q_2^2 \to \infty \qquad -\frac{q_1^2}{\nu} = \xi_1 > 0 \text{ fix}, \quad -\frac{q_2^2}{\nu} = \xi_2 < 0 \text{ fix}$$

$$q_1^2 \to -\infty$$

respectively.

As theoretical framework we will use a DJL representation for time ordered or retarded products. If the weight functions belong to the classes considered in /1/ the automodel behaviour of real and imaginary parts of the scattering amplitude coincide. For integer values of the scaling dimension the real part may contain an additional logarithmic factor. For the anomalous cases /2/ the automodel behaviour of the real part both in space-like and time-like regions of q^2 is determined by the leading light cone singularity. This is in contrast with the behaviour of the imaginary part which shows different automodel properties for space-like and time-like q^2 .

To study the problems connected with the real part of the scattering amplitude in the first paper we restrict ourselves to the case of forward scattering. The generalization to the nonforward case may be done in quite analogy with the corresponding generalizations for the imaginary part /3/ .

2.

Let us consider at first the DJL representation for the complete scattering amplitude in forward direction. There are two possible expressions which define the amplitude (summed over nucleon spins)

$$T_{\mu\nu}(q,p) = \frac{i}{4\pi}\int dx\, e^{iqx} \sum_{\sigma} \langle p,\sigma | T\, j_{\mu}(x)\, j_{\nu}(0) \,|\, p,\sigma\rangle \tag{2.1}$$

and

$$R_{\mu\nu}(q,p) = \frac{i}{4\pi}\int dx\, e^{iqx} \sum_{\sigma} \langle p,\sigma | \theta(x_0)[j_{\mu}(x), j_{\nu}(0)] | p,\sigma\rangle \tag{2.2}$$

which coincide for positive values of q_0. In the following we prefer to use the first expression. Let us start from the DJL representation for the electromagnetic current commutator

$$\widetilde{W}_{\mu\nu}(x,p) = \frac{1}{8\pi}\sum_{\sigma} \langle p,\sigma | [j_{\mu}(x), j_{\nu}(0)] | p,\sigma\rangle$$

$$= O^{(1)}_{\mu\nu}\, \widetilde{V}_1(x,p) + O^{(2)}_{\mu\nu}\, \widetilde{V}_2(x,p) \tag{2.3}$$

with

$$\widetilde{V}_i(x,p) = \frac{1}{(2\pi)^4}\int dq\, e^{-iqx} \int d\vec{u} \int d\lambda^2\, \varepsilon(q_0)\, \delta(q_0^2 - (\vec{q}-\vec{u})^2 - \lambda^2)\, \psi_i(\vec{u},\lambda^2), \tag{2.4}$$

$$O^{(1)}_{\mu\nu} = g_{\mu\nu}\Box - \partial_{\mu}\partial_{\nu},$$

$$O^{(2)}_{\mu\nu} = -p_{\mu}p_{\nu}\Box + (p_{\mu}\partial_{\nu} + p_{\nu}\partial_{\mu})(p\partial) - g_{\mu\nu}(p\partial)^2. \tag{2.5}$$

Using the simplest possible definition, the T-product can be constructed from

$$\langle p,\sigma | T j_{\mu}(x) j_{\nu}(0) | p,\sigma\rangle = \theta(x_0)\langle p,\sigma | j_{\mu}(x) j_{\nu}(0) | p,\sigma\rangle + \theta(-x_0)\langle p,\sigma | j_{\nu}(0) j_{\mu}(x) | p,\sigma\rangle \tag{2.6}$$

where for the current products the positive and negative frequency parts of the DJL representation (2.3) (2.4) have to be inserted.

The Fourier transform gives finally the following expression for the T product (Appendix I)

$$\begin{aligned} T_{\mu\nu}(q,p) = {} & (-g_{\mu\nu}q^2 + q_{\mu}q_{\nu})T_1 + (p_{\mu}p_{\nu}q^2 - (p_{\mu}q_{\nu} + p_{\nu}q_{\mu})qp + g_{\mu\nu}(qp)^2)T_2 \\ & - \frac{1}{\pi}(g_{\mu\nu} - g_{\mu 0}g_{\nu 0})\int \psi_1\, d\vec{u}\, d\lambda^2 \\ & - \frac{1}{\pi}(-p_{\mu}p_{\nu} + (g_{\nu 0}p_{\mu} + p_{\nu}g_{\mu 0})p_0 - g_{\mu\nu}p_0^2)\int \psi_2\, d\vec{u}\, d\lambda^2, \end{aligned} \tag{2.7}$$

$$T_i = -\frac{1}{\pi}\int d\vec{u}\, d\lambda^2\, \frac{\psi_i(\vec{u},\lambda^2)}{q_0^2 - (\vec{q}-\vec{u})^2 - \lambda^2 + i\varepsilon}, \tag{2.8}$$

$$|\vec{u}| \leq 1, \quad \lambda^2 \geq (1 - \sqrt{1-\vec{u}^2})^2, \quad p = (1,0,0,0).$$

If we follow the usual description of the experimental results for deep inelastic ep scattering $V_1 \sim \frac{1}{\nu}$, $V_2 \sim \frac{1}{\nu^2}$ in terms of integrable spectral functions /1/ fulfilling

$$\int d\lambda^2\, \psi_1(\vec{u},\lambda^2) < \infty, \quad \int d\lambda^2\, \psi_2(\vec{u},\lambda^2) = 0$$

then the integrals (2.7), (2.8) converge. The other possible interpretation of the experiments $V_1 \sim \frac{1}{\nu^2}$, $V_2 \sim \frac{1}{\nu^2}$ gives the additional condition $\int d\lambda^2\, \psi_1(\vec{u},\lambda^2) = 0$. Then the non covariant polynomials vanish, i.e. our construction leads directly to the covariant T-product. In general one has to add some appropriate non covariant polynomials. The imaginary part i.e. the discontinuity coincides with the conventional representation.

For spectral functions increasing with respect to λ^2 we apply the usual subtraction procedure according to

$$\frac{1}{q_0^2-(\vec{q}-\vec{u})^2-\lambda^2+i\varepsilon} = \frac{(q_0^2-(\vec{q}-\vec{u})^2)^n}{(q_0^2-(\vec{q}-\vec{u})^2-\lambda^2+i\varepsilon)}\frac{1}{\lambda^{2n}} - \frac{(q_0^2-(\vec{q}-\vec{u})^2)^{n-1}}{\lambda^{2n}} - \dots \frac{1}{\lambda^2} \tag{2.9}$$

The first term defines a convergent integral which contains the essential properties of the amplitude. The remaining terms which formally lead to divergent integrals are interpreted as unknown polynomials in q^2 and qp.

In the following we apply only such number of subtractions which is really needed for a convergence of the integral. In other words the degree of the arbitrary polynomials is assumed to be minimal.

3.

Let us study at first the behaviour of the scattering amplitude in the automodel region. Following/1/ we consider separately two classes of spectral functions.

Increasing spectral functions (case I)

$$\psi(\vec{u},\lambda^2) \underset{\lambda^2\to\infty}{\sim} \lambda^{2K}\psi_0(\rho) \qquad , \quad |\vec{u}|=\rho \, , \tag{3.1}$$

require a subtracted DJL representation

$$T = \int d\vec{u}\, [q_0^2-(\vec{q}-\vec{u})^2]^n \int d\lambda^2 \frac{\psi(\vec{u},\lambda^2)}{(q_0^2-(\vec{q}-\vec{u})^2-\lambda^2+i\varepsilon)\lambda^{2n}} \tag{3.2}$$

$$+ P_{n-1}(q^2, qp) \, .$$

Restriction to a minimal number of subtractions means

$$-1 \le K-n < 0 \, . \tag{3.3}$$

Because we are interested in the leading term only we use for the spectral function its asymptotic form (2.1) and exclude in expression (2.2) the lower part of the spectrum for which an unsubtracted expression could be used.

To study the automodel behaviour in the limit

$$\nu = 2pq \to \infty \quad , \quad \xi = -\frac{q^2}{\nu} \ \text{fix} \, , \ (p^2=1,\ p=(1,0,0,0)) \tag{3.4}$$

we perform the integration over angles

$$T(\nu,\xi) \approx \frac{\pi}{q}\int_0^1 d\rho\,\rho\,\psi_0(\rho)\int_A^\infty d\lambda^2\,\lambda^{2(K-n)}$$

$$\Big\{ \sum_{\ell=0}^{n-1} (-1)^{n-\ell}\frac{\lambda^{2\ell}}{n-\ell}\Big((\nu(\xi-\rho)+\rho^2-2\rho\xi)^{n-\ell} - (\nu(\xi+\rho)+\rho^2+2\rho\xi)^{n-\ell} \Big) \tag{3.5}$$

$$+ \lambda^{2n} \log \frac{\nu(\xi-\rho)+\rho^2-2\rho\xi+\lambda^2-i\varepsilon}{\nu(\xi+\rho)+\rho^2+2\rho\xi+\lambda^2-i\varepsilon} \Big\} + P_{n-1}(\nu,\xi) \, ,$$

$q = |\vec{q}|$.

Note that from its derivation $P_{n-1}(\nu,\xi)$ is of degree n-1 in ν at most. Introducing the integration variable $\mu = \frac{\lambda^2}{\nu}$ we obtain

$$T(\nu,\xi) \approx 2\pi\nu^K \int_0^1 d\rho\,\rho\,\psi_0(\rho) \int_{A/\nu}^\infty d\mu\,\mu^{K-n} \, .$$

$$\Big\{ \sum_{\ell=0}^{n-1} (-1)^{n-\ell}\frac{\mu^{\ell}}{n-\ell}\big[(\xi-\rho)^{n-\ell} - (\xi+\rho)^{n-\ell} \big] \tag{3.6}$$

$$+ \mu^n \log\frac{\mu+\xi-\rho-i\varepsilon}{\mu+\xi+\rho-i\varepsilon} \Big\} + P_{n-1}(\nu,\xi) \, .$$

For noninteger κ the limit $\nu\to\infty$ can be performed at the lower boundary. Convergence at infinity is guaranteed by the subtraction procedure. The condition $-1<\kappa-n$ is responsible for that the integral in expression (3.6) gives the leading behaviour in the automodel region and dominates the contribution from the polynomial. Restricting expression (3.6) to the imaginary part the result of /1/ may be obtained. Obviously, the automodel behaviour of the real part and the imaginary part of the amplitude coincide. For integer κ we have $\kappa-n=-1$ and the leading terms coming from the lower boundary are

$$T(\nu,\xi) \simeq 2\pi\nu^{\kappa}\log\nu\,\frac{(-1)^{\kappa+1}}{\kappa+1}\int_0^1 d\rho\,\rho\,\psi_0(\rho)\left[(\xi-\rho)^{\kappa+1}-(\xi+\rho)^{\kappa+1}\right] + O(\nu^{\kappa}). \tag{3.7}$$

It should be remarked that this term does not contribute to the imaginary part which behaves as ν^{κ}.

Let us now apply this analysis to the invariant amplitudes t_i which correspond to the structure function F_1, F_2 introduced in /1/. From the condition

$$F_1(q_0,\vec{q}=0)=0 \tag{3.8}$$

(which means a kinematical zero of F_1) and the automodel behaviour of F_1

$$F_1\sim\nu^{0} \tag{3.9}$$

follows (see ref. /1/)

$$\int_0^1 d\rho\,\rho^2\psi_0(\rho)=0\,. \tag{3.10}$$

Consequently in the asymptotic expression for t_1

$$t_1(\nu,\xi)\simeq -4\pi\log\nu\int_0^1 d\rho\,\rho^2\psi_0(\rho) + O(\nu^{0}), \tag{3.11}$$

The leading logarithmic term drops out. The remaining terms are not determined because of the subtraction procedure. The same result is applied to t_2. This may be concluded from the asymptotic relation

$$T_1(\nu,\xi)\simeq\frac{2}{\nu^2}\left[3t_1(\nu,\xi)-t_2(\nu,\xi)\right] \tag{3.12}$$

and the fact that T_1 does not contain asymptotic logarithmic terms (see below).

Now we turn to spectral functions fulfilling

$$\int_{0-}^{\infty} d\lambda^2\,\psi_{(s)}(\rho,\lambda^2)=\psi_0(\rho) \tag{3.13}$$

(case II). Without difficulties we can use the original representation

$$T(q,p)=\int d\vec{u}\int d\lambda^2\,\frac{\psi(\rho,\lambda^2)}{q_0^2-(\vec{q}-\vec{u})^2-\lambda^2+i\varepsilon}\,. \tag{3.14}$$

Angle integrations lead to

$$T(q,p)=\frac{\pi}{q}\int_0^1 d\rho\,\rho\int d\lambda^2\,\psi(\rho,\lambda^2)\log\frac{\lambda^2-q^2-2q\rho+\rho^2-i\varepsilon}{\lambda^2-q^2+2q\rho+\rho^2-i\varepsilon} \tag{3.15}$$

and repeated partial integrations give

$$T(q,p)=-\frac{\Gamma(s)\,\pi}{q}\int_0^1 d\rho\,\rho\int d\lambda^2\,\psi_{(s)}(\rho,\lambda^2)\cdot\left\{\frac{1}{(\lambda^2-q^2-2q\rho+\rho^2-i\varepsilon)^{s}}-\frac{1}{(\lambda^2-q^2+2q\rho+\rho^2-i\varepsilon)^{s}}\right\}. \tag{3.16}$$

In the limit $\nu \to \infty$, ξ fix we get finally

$$T(\nu,\xi) \approx \frac{(-i)2\pi\Gamma(s)}{\nu^{s+1}} \int_c^1 d\rho\,\rho\, \psi_c(\rho) \left\{ \frac{1}{(\xi-\rho-i\varepsilon)^s} - \frac{1}{(\xi+\rho-i\varepsilon)^s} \right\} . \tag{3.17}$$

Again the complete amplitude shows the same automodel behaviour as its imaginary part. For $s=0$ formula (3.17) reads

$$T(\nu,\xi) \approx \frac{2\pi}{\nu} \int_0^1 d\rho\,\rho\, \psi_c(\rho) \left\{ \log(\xi-\rho-i\varepsilon) - \log(\xi+\rho-i\varepsilon) \right\} . \tag{3.18}$$

Again the complete amplitude shows the same automodel behaviour as its imaginary part. (For dispersion relation in ξ compare[8])

4.

For completeness we shall discuss the light cone singularities corresponding to the T product.

In the first case we have to study the Fourier transform of expression (3.2)

$$\tilde{T}(x,p) = \frac{1}{(2\pi)^4} \int d\vec{u} \int d\lambda^2 \frac{\psi(\vec{u},\lambda^2)}{\lambda^{2n}} \int dq\, e^{-iqx} \frac{[q_0^2-(\vec{q}-\vec{u})^2]^n}{q_0^2-(\vec{q}-\vec{u})^2-\lambda^2+i\varepsilon} \tag{4.1}$$

$$+ P_{n-1}(\Box, p\partial)\, \delta(x)$$

$$= \int d\lambda^2\, \tilde{\Delta}(\vec{r},\lambda^2) (-\Box)^n \mathcal{D}_c(x,\lambda^2) + P_{n-1}\, \delta(x) \tag{4.2}$$

with

$$\tilde{\Delta}(\vec{r},\lambda^2) = \frac{1}{\lambda^{2n}} \int d\vec{u}\, e^{i\vec{u}\vec{x}}\, \psi(\vec{u},\lambda^2) , \tag{4.3}$$

$$\mathcal{D}_c(x,\lambda^2) = \frac{1}{(2\pi)^4} \int dq\, e^{-iqx} (\lambda^2 - q^2 - i\varepsilon)^{-1} . \tag{4.4}$$

If we take for $\psi(\vec{u},\lambda^2)$ its asymptotic part $\lambda^{2K}\psi_0(\rho)$ only we have

$$\tilde{\Delta}(\vec{r},\lambda^2) \approx G(r^2)\, \lambda^{2(K-n)} \tag{4.5}$$

which leads to

$$\tilde{T}(x,p) = G(r^2)\, (-\Box)^n \int_A^\infty d\lambda^2 \lambda^{2(K-n)} \mathcal{D}_c(x,\lambda^2) . \tag{4.6}$$

For noninteger K, i.e. for $K-n > -1$ this integral may be evaluated as follows

$$\int_A^\infty d\lambda^2 \lambda^{2(K-n)} \mathcal{D}_c(x,\lambda^2) = \int_0^\infty d\lambda^2 \cdots - \int_0^A d\lambda^2 \cdots \tag{4.7}$$

The first integral gives

$$\int_0^\infty d\lambda^2 \lambda^{2(K-n)} \mathcal{D}_c(x,\lambda^2) = \frac{1}{(2\pi)^4} \int dq\, e^{-iqx} \int_0^\infty d\lambda^2 \frac{\lambda^{2(K-n)}}{\lambda^2-q^2-i\varepsilon}$$

$$= \frac{1}{(2\pi)^4} e^{-i(n-k)\pi}\, \Gamma(K-n+1)\, \Gamma(n-K) \int dq\, e^{-iqx} [q^2+i\varepsilon]^{K-n} \tag{4.8}$$

$$= [x^2-i\varepsilon]^{n-k-2} \frac{\pi^2}{(2\pi)^4} e^{-i\pi(n-K)-i\frac{3}{2}\pi}\, 2^{2(K-n+2)}\, \Gamma(K-n+1)\, \Gamma(K-n+2)$$

Application of the differential operator $(-\Box)^n$ results in

$$(-\Box)^n \int_0^\infty d\lambda^2 \lambda^{2(K-n)} \mathcal{D}_c(x,\lambda^2) = i \frac{4^n \pi^2}{(2\pi)^4} e^{i\pi K}\, 2^{2(K-n+2)} \cdot$$

$$\cdot\, \Gamma(K-n+1)\, \Gamma(K-n+2) [(-K-1)\cdots(n-K-2)][(-K)\cdots(n-K-1)] \cdot \tag{4.9}$$

$$\cdot\, (x^2-i\varepsilon)^{-K-2}$$

(for Fourier transforms and derivatives of causal distributions compare/5/). For the second term we have

$$(-\Box)^n \int_0^A d\lambda^2 \lambda^{2(\kappa-n)} \mathcal{D}_{c(x,\lambda^2)} = \int_0^A d\lambda^2 \lambda^{2\kappa} \mathcal{D}_{c(x,\lambda^2)}$$
$$+ \int_0^A d\lambda^2 \{ \lambda^{2(\kappa-n)} (-\Box)^{n-1} \delta_{(x)} \tag{4.10}$$
$$+ \lambda^{2(\kappa-n+1)} (-\Box)^{n-2} \delta_{(x)} + \cdots + \lambda^{2(\kappa-1)} \delta_{(x)} \},$$

It is obvious that the leading term (on the light cone) comes from the first integral i.e.

$$\tilde{T}_{(x,p)} \sim \frac{G_{(x_\varepsilon^2)}}{(x^2 - i\varepsilon)^{-\kappa-2}} \tag{4.11}$$

As it is well known the causal distributions are analytical functions with respect to the variable κ with single poles at $\kappa = 0,1,2, \ldots$. Therefore this result is meaningless for integer κ ($\kappa = 0,1,2, \ldots$). The generalized function $\tilde{T}_{(x,p)}$ is well defined also in that case. As a detailed investigation shows (Appendix II) one has to replace the singular expression by its regular part that means, one has to take the constant term of the Laurent series only. In this manner the usual subtraction procedure gives a receipt to handle the causal distributions at the positions of their poles.

For completeness we note that in the integrable case the analysis of/1/may be applied to the T-product without any complications.

5 .

In this section we turn to the so-called anomalous cases/2/. characterized by

$$\lim_{\lambda^2 \to \infty} \frac{\mathcal{H}_{(\rho,\lambda^2)}}{\lambda^{2n}} = \mathcal{H}_{0(\rho)} \tag{5.1}$$

or

$$\int_{0-}^{\infty} d\lambda^2 \, \mathcal{H}_{(s)}(\rho,\lambda^2) = \mathcal{H}_{0(\rho)} \tag{5.2}$$

and

$$\int_0^1 d\rho \, \rho^2 \mathcal{H}_{0(\rho)} \phi_{(\rho)} \text{ divergent}, \quad \phi_{(\rho)} \text{ test function}. \tag{5.3}$$

In/2/ it has been shown for two typical examples that the absorptive parts for space-like and timelike q^2 behave differently in the automodel region. Especially, the space-like behaviour does not correspond to the leading light cone singularity (i.e. measurements of the absorptive part in the space-like region cannot determine the leading light cone singularity).

As we shall see, the real part behaves quite differently. For this purpose we discuss the following example

$$\mathcal{H}_{(\vec{u},\lambda^2)} = \frac{\theta_{(1-\rho)} \, \theta_{(\lambda^2-1)}}{(\rho^\mu \lambda^2 + 1)^2} \quad , \mu > 3 . \tag{5.4}$$

A detailed investigation in Appendix III gives

$$\mathrm{Re}\, T_{(\nu,\rho)} \sim \nu^{-\frac{3}{\mu}} \qquad -\infty < \rho < 1 , \tag{5.5}$$

$$\mathrm{Im}\, T_{(\nu,\rho)} \sim \begin{cases} \nu^{-1} & 0 < \rho < 1 \\ \nu^{-3/\mu} & \rho < 0 \end{cases}$$

and correspondingly

$$\tilde{T}(\nu,p) \sim (x^2 - i\varepsilon)^{3/2 - 2} \, . \qquad (5.6)$$

We are quite sure that the foregoing example shows the typical features of the anomalous cases; i.e. for the class of spectral functions considered there is a one-to-one correspondence between the leading light cone singularity and the automodel behaviour of the real part of the scattering amplitude.

6.

In the cases discussed up to now it has been assumed that either for $\psi(\rho,\lambda^2)$ or for its α - derivative

$$\psi^{(\alpha)}(\rho,\lambda^2) = \frac{1}{\Gamma(-\alpha)} \int_0^{\lambda^2} d\lambda'^2 \, \psi(\rho,\lambda'^2)\,(\lambda^2-\lambda'^2)^{-\alpha-1} \equiv f_{-\alpha} * \psi(\rho,\lambda^2) \qquad (6.1)$$

the limit $\lambda^2 \to \infty$ exists in the classical sense.

Quite recently /6/ the class of spectral functions has been enlarged by demanding the existence of a so-called quasi-limit (q-limit) only

$$\text{q-}\lim \psi^{(\alpha)}(\rho,\lambda^2) = \lim_{t\to\infty} \psi^{(\alpha)}(\rho, t\lambda^2) = \psi_0(\rho)\,\theta(\lambda^2) \qquad (6.2)$$

(convergence in the sense of functionals).

The corresponding automodel behaviour of the absorptive part of the amplitude turns out to be ν^{α} .

Therefore the experimental automodel behaviour for V_1 and V_2 could be described in principle also by spectral

functions which have a q-limit of degree $\alpha = -2$.

Let us now consider the construction of the DJL representation for the T-product for this general class of test functions. For values $\alpha > 0$ obviously subtractions have to be taken into account. The cases more interesting for applications are $\alpha < 0$.

In this case we have at first to discuss the existence of an unsubtracted DJL representation. This means, the functional

$$\begin{aligned} \int d\xi \, f(\xi)\, T(\nu,\xi) &= 2\pi \int_0^1 d\rho\,\rho^2 \int_{-1}^{+1} dz \int_0^\infty d\lambda^2 \int_{-\infty}^{1} d\xi \, \frac{\psi(\rho,\lambda^2)\, f(\xi)}{\nu(\rho z-\xi)-\lambda^2-\rho^2+2\rho z\Delta+i\varepsilon} \\ &= 2\pi \int_0^1 d\rho\,\rho^2 \int_{-1}^{+1} dz \int_0^\infty d\tau \int_{-\infty}^{1} d\xi \, \frac{\psi(\rho,\nu\tau)\, f(\xi)}{\rho z-\xi-\tau+i\varepsilon+\frac{2\rho z\Delta-\rho^2}{\nu}} \, , \end{aligned} \qquad (6.3)$$

$\left(\Delta = \sqrt{1+\frac{4\xi}{\nu}}\right.$, $f(\xi)$ test function$\left.\right)$

has to be finite. This expression contains the convolution of distributions where for $\psi(\rho,\lambda^2)$ we allow a nonclassical behaviour for large λ^2 . A sufficient condition for convergence is the existence of some $\alpha < 0$ such that

$$\int d\rho\,\rho^2 \,\phi_1(\rho) \int d\lambda^2\, \psi^{(\alpha)}(\rho,\lambda^2+t)\,\phi_2(\lambda^2) < \infty \qquad (6.4)$$

ϕ_1, ϕ_2 test functions

for $t \to \infty$ (Appendix IV).

This condition may be more restrictive than the existence of the q-limit

$$\lim_{t\to\infty}\int d\xi\,\xi^2\phi_1(\xi)\int d\lambda^2\psi^{(\alpha)}(\xi,t\lambda^2)\,\phi_2(\lambda^2)=\int d\xi\,\xi^2\phi_1(\xi)\,\psi_0(\xi)\int d\lambda^2\phi_2(\lambda^2)\,\theta(\lambda^2). \tag{6.5}$$

On the other hand all known examples having a q-limit fulfil the condition (6.4). This is valid for examples of the type

$$\psi=e^{i\lambda^2}\,,\qquad \psi=\sum_{n=1}^{\infty}\delta(\lambda^2-n)\,.$$

Let us now turn to the question of the automodel behaviour of expression (6.3) for spectral functions fulfilling both conditions (6.2) and (6.4). Knowing the convergence of the functional we omit the test function $f(\xi)$. We apply the q-limit in the following form

$$\frac{\psi(\xi,\nu\tau)}{\nu^\alpha}\xrightarrow[\nu\to\infty]{}\psi_0(\xi)\,\frac{\tau_+^\alpha}{\Gamma(1+\alpha)} \tag{6.6}$$

which is equivalent to condition (6.2), [6] and obtain

$$\begin{aligned}T(\nu,\xi)&\approx 2\pi\nu^\alpha\int_0^1 d\rho\,\rho^2\int_{-1}^{+1}dz\int_0^\infty d\tau\,\frac{\psi(\rho,\nu\tau)}{\nu^\alpha}\,\frac{1}{\rho z-\xi-\tau+i\varepsilon}\\&\approx 2\pi\nu^\alpha\int_0^1 d\rho\,\rho^2\,\psi_0(\rho)\int_{-1}^{+1}dz\int_0^\infty d\tau\,\frac{\tau_+^\alpha}{\Gamma(1+\alpha)}\,\frac{1}{\rho z-\xi-\tau+i\varepsilon}\end{aligned} \tag{6.7}$$

Using

$$\frac{1}{\Gamma(1+\alpha)}\int_0^\infty d\tau\,\frac{\tau_+^\alpha}{\rho z-\xi-\tau+i\varepsilon}=(1+\alpha)\,\Gamma(-1-\alpha)\,(\xi-\rho z-i\varepsilon)^\alpha \tag{6.8}$$

and performing the z-integration we arrive at

$$T(\nu,\xi)\approx 2\pi\nu^\alpha\,\Gamma(1-\alpha)\int_0^1 d\rho\,\rho\,\psi_0(\rho)\left[(\xi+\rho-i\varepsilon)^{\alpha+1}-(\xi-\rho-i\varepsilon)^{\alpha+1}\right] \tag{6.9}$$

One should note that the former result (3.17) appears as a special case of eq. (6.9).

It is interesting to remark that the anomalous case discussed in the foregoing section belongs to the class of spectral functions having a q-limit. For our previous example

$$\psi=\frac{1}{(\xi^n\lambda^2+1)^2}\,\theta(\lambda^2-1)$$

we have

$$\frac{1}{t^\alpha}\,\frac{1}{(\xi^n t\lambda^2+1)^2}\xrightarrow[t\to\infty]{}c\,\delta(\vec{u})\,(\lambda_+^2)^\alpha\,,\quad\alpha=-\frac{3}{n}\,. \tag{6.10}$$

Obviously the introduction of the q-limit is very powerful mathematical tool. However, it seems to be not yet understood what a manifold of spectral functions has been allowed by this condition. Furthermore it would be very important to know if condition (6.4) is generally valid within this enlarged class of spectral functions. This should be clarified in order to know if the off-shell Compton amplitude would be given by an unsubtracted DJL representation for any spectral function which reproduces the observed behaviour in deep inelastic scattering.

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APPENDIX I

To get formula (2.7) (.28) we have to perform the Fourier integral of expression (2.6). Using equation

$$T_{\mu\nu}(q,p) = \frac{2i}{(2\pi)^4}\int dx\, e^{iqx}\int dq'\, e^{-iq'x}\int d\vec{u}\int d\lambda^2\, \delta(q_0'^2-(\vec{q}'-\vec{u})^2-\lambda^2)\cdot \qquad \text{(I.1)}$$
$$\cdot\left(\theta(x_0)\theta(q_0')+\theta(-x_0)\theta(-q_0')\right)\left(O^{(1)}_{\mu\nu}(-iq')\mathcal{H}_1 + O^{(2)}_{\mu\nu}(-iq')\mathcal{H}_2\right)$$

with

$$\int dx\, e^{i(q-q')x}\left\{\theta(x_0)\theta(q_0')+\theta(-x_0)\theta(-q_0')\right\} \qquad \text{(I.2)}$$
$$=(2\pi)^3 i\,\delta(\vec{q}-\vec{q}')\left\{\frac{\theta(q_0')}{q_0-q_0'+i\varepsilon}-\frac{\theta(-q_0')}{q_0-q_0'-i\varepsilon}\right\}$$

and

$$\int dq'\,\delta(q_0'^2-(\vec{q}'-\vec{u})^2-\lambda^2)\,\delta(\vec{q}-\vec{q}')\left\{\frac{\theta(q_0')}{q_0-q_0'+i\varepsilon}-\frac{\theta(-q_0')}{q_0-q_0'-i\varepsilon}\right\}\begin{pmatrix} q_i'q_j' \\ q_i'q_0' \\ q_0'q_0'\end{pmatrix} \qquad \text{(I.3)}$$
$$= \frac{1}{q_0^2-(\vec{q}-\vec{u})^2-\lambda^2+i\varepsilon}\begin{pmatrix} q_iq_j \\ q_iq_0 \\ q_0^2-[q_0^2-(\vec{q}-\vec{u})^2-\lambda^2]\end{pmatrix}$$

we get

$$T_{\mu\nu}(q,p) = -\frac{1}{\pi}\int d\vec{u}\int d\lambda^2\,\frac{1}{q_0^2-(\vec{q}-\vec{u})^2-\lambda^2+i\varepsilon}\cdot \qquad \text{(I.4)}$$
$$\cdot\left\{O^{(1)}_{\mu\nu}\mathcal{H}_1 + O^{(2)}_{\mu\nu}(-iq)\mathcal{H}_2\right\}$$
$$-\frac{1}{\pi}(g_{\mu\nu}-g_{\mu0}g_{\nu0})\int d\vec{u}\int d\lambda^2\,\mathcal{H}_1(\vec{u},\lambda^2)$$
$$-\frac{1}{\pi}(-p_\mu p_\nu+(p_\mu g_{\nu0}+p_\nu g_{\mu0})p_0-g_{\mu\nu}p_0^2\int d\vec{u}\int d\lambda^2\,\mathcal{H}_2\ .$$

APPENDIX II

For integer k we have $K = -n-1$. Using the Laurent series for $\Gamma(-n+k+1)$ and $(x^2-i\varepsilon)^\lambda$

$$(x^2-i\varepsilon)^\lambda = -i\,\frac{\pi^2}{4^m m!\,\Gamma(2+m)}\,\Box^m\delta(x)\,\frac{1}{\lambda+2+m} + (x^2-i\varepsilon)^{-2-m}_{F.P.} + \cdots \qquad \text{(II.1)}$$

we have from equation (4.9)

$$(-\Box)^n\int_0^\infty d\lambda^2\,\lambda^{2(k-n)}\,\mathcal{D}_c(x,\lambda^2) = \qquad \text{(II.2)}$$
$$= -\frac{(-\Box)^{n-1}\delta(x)}{-k+n-1} + C\,(-\Box)^{n-1}\delta(x) + (x^2-i\varepsilon)^{-2-k}_{F.P.} + \cdots$$

and from equation (4.10)

$$(-\Box)^n\int_0^A d\lambda^2\,\lambda^{2(k-n)}\,\mathcal{D}_c(x,\lambda^2) = -\frac{(-\Box)^{n-1}\delta(x)}{-k+n-1} + \frac{\alpha}{x^2-i\varepsilon} + \beta(-\Box)^{n-1}\delta(x) + \cdots \qquad \text{(II.3)}$$

Therefore the pole terms compensate each other. The essential new term is

$$\tilde{T}(x,p) \approx G(x_0^2)\,\frac{1}{(x^2-i\varepsilon)^{k+2}_{F.P.}} \qquad \text{(II.4)}$$

APPENDIX III

We have to evaluate

$$T(q,\rho) = \int d\vec{u} \int d\lambda^2 \, \frac{\psi(\vec{u},\lambda^2)}{q_0^2 - (\vec{q}-\vec{u})^2 - \lambda^2 + i\varepsilon} \tag{III.1}$$

with

$$\psi(\vec{u},\lambda^2) = \frac{\theta(1-\rho)\,\theta(\lambda^2-1)}{(\rho^\mu \lambda^2 + 1)^2} \quad , \mu > 3 \tag{III.2}$$

After having performed integrations over angles and γ_ρ we arrive at

$$\begin{aligned} T(v,\xi) = \frac{\pi}{q} \int_0^1 d\rho\, \rho^{1-2\mu} \Big[& \Big\{ \frac{1}{a_- - \rho^{-\mu}} - \frac{1}{a_+ - \rho^{-\mu}} \Big\} \log(1+\rho^{-\mu}) \\ & + \Big\{ \frac{1}{a_+ - \rho^{-\mu}} + \frac{1}{1+\rho^{-\mu}} \Big\} \log(1+a_+) \\ & - \Big\{ \frac{1}{a_- - \rho^{-\mu}} + \frac{1}{1+\rho^{-\mu}} \Big\} \log(1+a_-) \end{aligned} \tag{III.3}$$

where

$$a_\pm = -q^2 \pm 2q\xi + \rho^2 - i\varepsilon \approx 2q_0(\xi \mp \rho) + \xi^2 \mp 2\rho\xi - i\varepsilon , \tag{III.4}$$

$$q = |\vec{q}| \approx q_0 = \frac{v}{2} .$$

At first we remark that for all values of ξ the contribution from $\rho > c > 0$ is nonleading

$$\frac{1}{q} \int_c^1 d\rho \ldots \sim \frac{1}{v} . \tag{III.5}$$

The region $0 \le \xi \le c$ has to be studied in detail. Let us consider for example the case $0 \le \xi \le c$. After the substitution $x = q_0 \xi^\mu$

we have (neglecting terms of lower order in q_0)

$$\begin{aligned} T = \frac{\pi}{\mu} q_0^{-\frac{2}{\mu}} \int_0^{q_0 c^\mu} dx\, x^{\frac{2}{\mu}-2} \Big\{ & \Big(\frac{1}{2x(\xi-\rho)-1} - \frac{1}{2x(\xi+\rho)-1} \Big) \log\Big(1+\frac{q_0}{x}\Big) \\ & + \Big(\frac{1}{2x(\xi-\rho)-1} + \frac{1}{1+\frac{x}{q_0}} \Big) \log\big(1+2q_0(\xi-\rho)-i\varepsilon\big) \\ & - \Big(\frac{1}{2x(\xi+\rho)-1} + \frac{1}{1+\frac{x}{q_0}} \Big) \log\big(1+2q_0(\xi+\rho)-i\varepsilon\big) \Big\} . \end{aligned} \tag{III.6}$$

For the moment we restrict the integration from 0 to A.
Writing the logarithms as

$$\log\Big(1+\frac{q_0}{x}\Big) = \log q_0 + \log\Big(\frac{1}{q_0}+\frac{1}{x}\Big) \approx \log q_0 + \log\frac{1}{x} \tag{III.7}$$

$$\log\big(1+2q_0(\xi\pm\rho)\big) = \log q_0 + \log\Big(\frac{1}{q_0}+2(\xi\pm\rho)\Big) \approx \log q_0 + \log\big(2(\xi\pm\rho)\big)$$

and performing the limit on the integrand we get

$$T \approx \frac{4\pi}{\mu} q_0^{-\frac{3}{\mu}} \int_0^A dx\, x^{3/\mu - 1} \frac{\log 2x\xi - (2x\xi-1)}{(2x\xi-1)^2} \tag{III.8}$$

For a discussion of the integral $\int_A^{q_0 c^\mu}$ we change the variable by $y = \frac{1}{x}$ in expression and obtain terms of the type

$$q_0^{-\frac{3}{\mu}} \int_{\frac{1}{q_0 c^\mu}}^{\frac{1}{A}} dy\, f(y,q_0) \tag{III.9}$$

Here the integral is finite in the limit $q_0 \to \infty$. Consequently its contribution is negligible because A can be chosen arbitrarily large.

APPENDIX IV

We have to investigate

$$\int_{-\infty}^{1} d\xi\, T(v,\xi)\, f(\xi) = 2\pi \int_0^1 d\rho\, \rho^2 \int_{-1}^{+1} dz \int_0^{\infty} d\tau \int_{-1}^{\infty} d\xi\, \frac{\psi(\rho, v\tau)\, f(-\xi)}{\xi - \tau + \sigma + i\varepsilon}, \qquad \text{(IV.1)}$$

$$\sigma = \rho z + \frac{2\rho z \Delta - \rho^2}{v} \approx \rho z$$

In the following we discuss the convolution

$$\int d\tau\, \psi(\rho, v\tau)\, \frac{1}{\xi - \tau + \sigma + i\varepsilon} = \frac{1}{\xi + \sigma + i\varepsilon} * \psi(\rho, v\xi)$$

$$= \frac{1}{\xi + \sigma + i\varepsilon} * f_\alpha * f_{-\alpha} * \psi(\rho, v\xi) \qquad \text{(IV.2)}$$

$$= v^\alpha \frac{1}{\xi + \sigma + i\varepsilon} * f_\alpha * \psi^{(\alpha)}(\rho, v\xi)$$

leading to

$$\int d\tau \int d\xi \frac{\psi(\rho, v\tau)\, f(-\xi)}{\xi - \tau + \sigma + i\varepsilon} = -v^\alpha \int dw \int d\tau\, \psi^{(\alpha)}(\rho, v(\tau + w))\, f(-\tau)$$

$$\cdot \int dt\, \frac{t_+^{\alpha-1}}{\Gamma(\alpha)}\, \frac{1}{w - \sigma + t - i\varepsilon} \qquad \text{(IV.3)}$$

$$= \frac{\pi v^\alpha}{\Gamma(\alpha) \sin \pi(1+\alpha)} \int dw\, (w - \sigma - i\varepsilon)^{\alpha - 1} \int d\tau\, \psi^{(\alpha)}(\rho, v(\tau + w))\, f(-\tau).$$

The rules for handling convolutions of the generalized function f_α may be found in /7/.

If there exists some $\alpha < c$ such that

$$\int d\rho\, \rho^2\, \phi(\rho) \int d\tau\, \psi^{(\alpha)}(\rho, v(\tau + w))\, f(-\tau) < \infty$$

$$(\ \phi(\rho),\ f(-\tau) \text{ test function }) \qquad \text{(IV.4)}$$

for $w \to \infty$ then the w-integral converges.

So we write expr. (IV.1)

$$\int d\xi\, T(v,\xi)\, f(\xi) = \frac{2\pi v^\alpha}{\Gamma(\alpha) \sin \pi(1+\alpha)} \cdot \qquad \text{(IV.5)}$$

$$\cdot \int_{-1}^{+1} dz \int dw \int d\rho\, \rho^2 \int d\tau\, (w - \rho z - i\varepsilon)^{\alpha - 1}\, \psi^{(\alpha)}(\rho, v\tau + vw)\, f(-\tau)$$

Note that the ρ – convolution exists because of the compact support of $\psi^{(\alpha)}$ (with respect to ρ) and does not disturb the asymptotic behaviour for $w \to \infty$ (which is needed for the convergence of the w integration). Obviously a test function in ρ is not needed.

From this we conclude that the integral (IV.1) converges.

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