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EXTENSION OF THE S -MATRIX OFF THE MASS SHELL AND MOMENTUM SPACE OF CONSTANT CURVATURE

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Summary

In the framework of Bogolubov's axiomatic approach problems connected with the extension of the scattering matrix off the mass shell are considered. In the method of extended S-matrix the concepts of elementary particle interactions accepted in quantum field theory are reflected to most complete extent. A specific point for the standard extension procedure is the assumption that the 4-momentum space in which the extended objects (fields, currents, S-matrix coefficient functions, etc.) are defined is a flat Minkowski space. The hypothesis that such a choice of the geometry "fails" is put forward and this choice is actually responsible for the known difficulties of the theory connected with the problem of multiplying of generalized singular functions with coinciding singularities. As an alternative it is proposed to use in the extended S-matrix formalism a 4-momentum space of constant curvature (De-Sitter space) with curvature radius $\hbar/\ell_0$, where ℓ_0 is a fundamental length. The interaction laws of the elementary particles with De Broglie wave lengths $\leq \ell_0$ must be completely different in the new scheme, in comparison with those prescribed by the usual local field theory.

It is demonstrated that the off mass shell S-matrix extension in the spirit of De-Sitter p-space geometry can be made consistent with the requirements of relativistic invariance, unitarity, spectrality, completeness of the system of asymptotic states. With the help of a specific Fourier transform in momentum space of constant curvature a new configuration ξ-space is introduced, whose geometry for small distances $\leq \ell_0$ is essentially different from the pseudoeuclidean one. The causality condition on the S-matrix, which is direct generalization of Bogolubov's causality condition, going to it in the limit $\ell_0 \to 0$, is formulated in terms of this ξ-space. On several examples it is demonstrated that in the developed theory the problem of generalized singular function products loses its acuteness. In particular the commutation functions and propagators in the new scheme can be interpreted as usual (not generalized) functions and there is no arbitrariness in any their powers and products.

1. Extended Scattering Matrix in Bogolubov's Axiomatic Approach

Let S be the scattering matrix in a theory of neutral scalar field φ, describing particles of mass m. We shall consider S in the framework of Bogolubov's axiomatic approach to the quantum field theory [1–4]. Let us write down in p-representation the standard decomposition of this operator in terms of normal products of free out-fields:

$$S = \sum_n \int d^4p_1 \dots d^4p_n \, S_n(p_1, \dots, p_n) : \varphi(p_1) \dots \varphi(p_n) : . \tag{1.1}$$

By definition:

$$\varphi^{out}(x) = \frac{1}{(2\pi)^{3/2}} \int e^{ipx} \varphi(p) \, d^4p \tag{1.2a}$$

$$\varphi(p) = \frac{1}{(2\pi)^{3/2}} \int e^{-ipx} \varphi^{out}(x) \, d^4x \tag{1.2b}$$

$$(\varphi(p))^+ = \varphi(-p). \tag{1.2c}$$

From here

$$(m^2 - p^2)\varphi(p) = 0, \tag{1.3}$$

$$\varphi(p) = \delta(m^2 - p^2)\tilde{\varphi}(p). \tag{1.4}$$

Owing to eq.(1.4) the coefficient functions (c.f.) $S_n(p_1, \dots, p_n)$ in (1.1) are defined only on the mass-shell $p^2 = m^2$:

$$S_n(p_1, \dots, p_n) = S_n(p_1, \dots, p_n)\Big|_{p_1^2 = \dots = p_n^2 = m^2} . \tag{1.5}$$

However for formulation of a dynamical theory it is necessary to extend the S-matrix off the mass shell (see for instance

[4] and the report by Medvedev, Pavlov, Polivanov and Sukhanov submitted to this conference.)

When one extends the "scattering matrix" with respect to the field", i.e. when the quantized out-fields get classical additions and do not more satisfy the free equation (1.3), it is supposed that the extended S -matrix is still given by the decomposition (1.1). From here, taking into account the stability of the vacuum state, the following expression for the extended off the mass shell c.f. is obtained:

$$S_n(p_1,\dots,p_n) = \frac{1}{n!}\langle 0|\frac{\delta^n S}{\delta\varphi(p_1)\dots\delta\varphi(p_n)} S^+|0\rangle . \tag{1.6}$$

The quantity

$$S^{(n)}(p_1\dots p_n) = \frac{\delta^n S}{\delta\varphi(p_1)\dots\delta\varphi(p_n)} S^+ \tag{1.7}$$

is called n -th order radiation operator. The operator $i\,S^{(1)}(-p)$ is identified with the Heisenberg current operator in p -representation

$$i\,S^{(1)}(-p) = i\frac{\delta S}{\delta\varphi(-p)} S^+ = j(p) . \tag{1.8}$$

In coordinate space we have correspondingly:

$$j(x) = i\,S^{(1)}(x) = i\,\frac{\delta S}{\delta\varphi^{out}(x)} S^+ \tag{1.9}$$

$$j(x) = \frac{1}{(2\pi)^{5/2}} \int e^{-ipx} j(p)\, d^4p . \tag{1.10}$$

One of the conditions imposed on the extended objects (S - matrix, field operators, etc.) is conservation of the

former covariance properties with respect to Poincaré group transformations. For instance under translations:

$$x' = x + a \tag{1.11}$$

the extended operator transforms as

$$e^{i\hat{P}a}\varphi(p)e^{-i\hat{P}a} = e^{ipa}\varphi(p) \tag{1.12}$$

($\hat{P}$ is the energy-momentum of the system) and the extended scattering matrix remains invariant

$$e^{i\hat{P}a} S e^{-i\hat{P}a} = S . \tag{1.13}$$

From (1.8), (1.12) and (1.13) it follows that the transformation law of the current $j(p)$ coincides with (1.12):

$$e^{i\hat{P}a} j(p) e^{-i\hat{P}a} = e^{ipa} j(p) . \tag{1.14}$$

In terms of extended c.f. (1.6) the translation invariance condition (1.13) means that any such a function has to contain as a factor a four-dimensional δ -function:

$$S_n = \delta(p_1 + \dots + p_n) S_n' . \tag{1.15}$$

Let us express S_n' in terms of the original c.f. S_n . To do this let us consider in the decomposition (1.1) the term:

$$\int d^4p_1\dots d^4p_n\, S_n(p_1\dots p_n) : \varphi(p_1)\dots\varphi(p_n): . \tag{1.16}$$

Because of eq.(1.15) the expression (1.16) is identical to the following:

$$\int d^4p_1\dots d^4p_n\, S_n(p_1\dots p_n) : \varphi\left(p_1 - \frac{p_1+\dots+p_n}{n}\right)\dots\varphi\left(p_n - \frac{p_1+\dots+p_n}{n}\right): . \tag{1.17}$$

Let us make in (1.17) the substitution:

$$p_i = q_i + k ,$$

$$\sum q_i = 0 \ , \ k = \frac{p_1 + \dots + p_n}{n} \equiv U^{(p_1 \cdots p_n)} . \tag{1.18}$$

It is easy to see that:

$$d^4p_1 \dots d^4p_n = d^4q_1 \dots d^4q_n \, \delta\left(U^{(q_1 \cdots q_n)}\right) d^4k , \tag{1.19}$$

and therefore instead of (1.17) we have:

$$\int d^4q_1 \dots d^4q_n \, \delta\left(U^{(q_1 \cdots q_n)}\right) \int d^4k \, S_n(q_1+k \dots q_n+k) : \varphi(q_1) \dots \varphi(q_n) : . \tag{1.20}$$

Comparing (1.20) with (1.16) we get the final result [5,6]:

$$S_n(p_1 \cdots p_n) = \delta\left(\frac{p_1 + \dots + p_n}{n}\right) \int d^4k \, S_n(q_1+k \dots q_n+k) \equiv$$
$$\equiv \delta\left(U^{(p_1 \cdots p_n)}\right) \widetilde{S}_n(p_1, \dots p_n) . \tag{1.21}$$

Hence the function S'_n in (1.15) is given by the relation:

$$S'_n = n^4 \widetilde{S}_n (p_1 \cdots p_n) . \tag{1.22}$$

Let us note that the dependence of the right-hand side of the identity (1.21) on the 4-momenta $p_1 \dots p_n$ has a specific character with respect to translations

$$p' = p + b \tag{1.23}$$

in the p-space. Actually the integral:

$$\widetilde{S}_n(p_1 \cdots p_n) = \int S_n(p_1+k \dots p_n+k) \, d^4k \tag{1.24}$$

is invariant under displacements (1.23):

$$S_n(p_1 + b \dots p_n + b) = \widetilde{S}_n(p_1 \cdots p_n) \tag{1.25}$$

and the δ-function argument $\frac{p_1 + \dots + p_n}{n} = U^{(p_1 \cdots p_n)}$ transforms according to the same law (1.23) as any of the 4-momenta p_i:

$$U^{(p'_1 \cdots p'_n)} = U^{(p_1 \cdots p_n)} + b . \tag{1.26}$$

Owing to (1.25) the function $\widetilde{S}_n(p_1 \cdots p_n)$ depends only on the momenta differences $p_i - p_j \ (i,j = 1, \dots, n)$. Let us agree to call the difference $\frac{p_i - p_j}{2}$ <u>relative</u> 4-momentum. Evidently, similarly to the quantities p_i, they are vectors in the four dimensional pseudoeuclidean space or Minkowski space.

Identities of the type (1.21) which express the translation invariance of the S-matrix in momentum space, have a number of simple properties following directly from relations (1.25), (1.26). In particular if we apply the identity to the function $S_n(p_1 + k \dots p_n + k)$ under the integral sign in eq.(1.21) then we again come to (1.21) ("irreducibility" property).

Of special interest is the case when the o.f. $S_n(p_1 \cdots p_n)$ is disconnected, i.e. contains for instance additive terms of the form:

$$S_m(p_1 \cdots p_m) \, S_{n-m}(p_{m+1} \cdots p_n) , \tag{1.27}$$

where S_m and S_{n-m} are lower order c.f. satisfying by themselves identities of the type (1.21):

$$S_m(p_1 \cdots p_m) = \delta(U^{(m)}) \tilde{S}_m(p_1 \cdots p_m)$$
$$S_{n-m}(p_{m+1} \cdots p_n) = \delta(U^{(n-m)}) \tilde{S}_{n-m}(p_{m+1} \cdots p_m) \tag{1.28}$$

(for brevity we have put $U^{(p_1 \cdots p_m)} \equiv U^{(m)}$).

Let us demonstrate that if the relations (1.28) are fulfilled the product (1.27) automatically obeys the identity (1.21):

$$S_n(p_1 \cdots p_m) S_{n-m}(p_{m+1}, \dots, p_n) = \\ = \delta(U^{(n)}) \overline{S_n(p_1 \cdots p_m) S_{n-m}(p_{m+1} \cdots p_n)}. \tag{1.29}$$

We note that:

$$\delta(U^{(m)}) \delta(U^{(n-m)}) = \delta(U^{(n)}) \delta(U^{(n)} - U^{(n-m)}), \tag{1.30}$$

or

$$\delta(U^{(m)}) \delta(U^{(n-m)}) = \delta(U^{(n)}) \int \delta(U^{(n)} + k) \delta(U^{(n-m)} + k) d^4k. \tag{1.31}$$

Now multiplying both parts of the relation (1.31) by $\tilde{S}_m(p_1 \cdots p_m) \tilde{S}_{n-m}(p_{m+1} \cdots p_n)$ and taking into account the invariance of the functions with tilde with respect to displacements (1.23) we obtain:

$$S_m(p_1 \cdots p_m) S_{n-m}(p_{m+1} \cdots p_n) = \delta(U^{(n)}) \cdot \\ \cdot \int \delta(U^{(n)} + k) \tilde{S}_m(p_1 + k \cdots p_m + k) \delta(U^{(n-m)} + k) \tilde{S}_{n-m}(p_{m+1} + k \cdots p_n + k) d^4k.$$

From here, using property (1.26) of the U -vectors and relation (1.28) we get (1.29).

Let us emphasize that from the invariance of the functions S_n under translations (1.23) and their obvious relativistic invariance it follows that they are invariant under arbitrary transformations of the 10 -parametric motion group of the momentum space (Poincaré group):

$$p'_\mu = \Lambda_\mu{}^\nu p_\nu + b_\mu \quad (\mu = 0,1,2,3) \tag{1.32}$$

($\|\Lambda_\mu{}^\nu\|$ is the Lorentz transformation matrix). Therefore, functions $\tilde{S}_n$ actually depend on the squares of the "relative" 4 -momenta $(p_i - p_j)^2/4$ $(i, j = 1, \dots, n)$

$$\tilde{S}_n = \tilde{S}_n\left(\dots \frac{(p_i - p_j)^2}{4} \dots\right). \tag{1.33}$$

In view of the importance of this result for future constructions let us formulate it once more:

if one imposes the translation invariance condition on the extended off the mass shell scattering matrix written in p--representation, then from any connected c.f. $S_n(p_1 \dots p_n)$ it can be picked out a function $\tilde{S}_n(p_1 \cdots p_n)$ which is invariant under the Poincaré group (1.32) of the pseudoeuclidean momentum space.

In the extended S -matrix formalism the Bogolubov's causality condition plays the role of dynamical equation from which the c.f. $S_n(p_1 \dots p_n)$ are determined. This condition can be written either in differential form

$$\frac{\delta j(x_1)}{\delta \varphi^{out}(x_2)} = 0 , \qquad \text{if}^{+)} \ (x_1 - x_2) \equiv \xi \gtrsim 0, \tag{1.34}$$

or in integral form [1–4] :

$$\frac{\delta^2 S}{\delta\varphi^{out}_{(x_1)}\,\delta\varphi^{out}_{(x_2)}} S^+ = -j(x_1)j(x_2) + $$
$$+\theta(x_2^0 - x_1^0)\left[j(x_1), j(x_2)\right] \quad \text{+quasilocal terms} =$$
$$= -T\big(j(x_1)j(x_2)\big) \quad \text{+quasilocal terms} , \tag{1.35}$$

the current commutator satisfying the condition

$$[j(x_1), j(x_2)] = 0 , \qquad \text{if } (x_1 - x_2)^2 \equiv \xi^2 < 0 \tag{1.36}$$

Rel.(1.36) is called "locality condition" for the current operator $j(x)$.

+) The symbol $\xi \gtrsim 0$ means that either $\xi^2 > 0$ and $\xi_0 > 0$, or $\xi^2 < 0$

In the present quantum field theory commutators of the type (1.36) have always singularities on the surface+):

$$\xi^2 = 0 . \tag{1.37}$$

The product of such a commutator with step function $\theta(\xi^0) = \theta(x_1^0 - x_2^0)$, as it is well known [1–4] , can be defined only up to arbitrary quasilocal operators (in the causality condition (1.33) this is taken into account). One has to notice, that in all the formulations of the quantum field theory we confront with analogous difficulties, which from mathematical point of view reduce to the problem of multiplying singular generalized functions with coinciding singularities.

This originates, in particular, the famous ultraviolet divergences in the perturbation theory.

The existence of ultraviolet divergencies in quantum field theory have been exhibited at its earliest stages of development. Nowadays many physicists are convinced that this deffect is of principal character and testifies for the inapplicability of the theory to describe physical processes in small space-time regions, or, correspondingly at high energies and momenta.

+) This surface is the light cone in the pseudoeuclidean ξ-space $(\xi_\mu = (x_1 - x_2)_\mu, \ \mu = 0,1,2,3)$. Obviously this space is invariant under translations (1.11).
We shall call quantities of type ξ relative coordinates. They are canonically conjugated to the half-differences of 4-momenta, or in our terminology, to the relative momenta.

There exist a large amount of papers devoted to the so-called "nonlocal" quantum field theories, in which from different physical reasons and using different mathematical means the interaction of the elementary particles is modified in the region of small De Broglie wave lengths+). A common feature of these investigations is introduction of a new universal constant in the theory- the fundamental length ℓ_0 defining the space-time bounds of the region in which some of the "old" concepts about particles and their interactions are not more valid.

In the present report we would like to discuss one possible way of generalization of the quantum field theory which naturally leads to an appearance in its framework of the fundamental length ℓ_0 . From a mathematical point of view the formalizm we consider will recall Snyder's scheme of quantized space-time [8-13] . However the basic idea and physical interpretation of the theory we construct are essentially different from those of refs. [8 - 13] .

+) A comprehensive review of many attempts to construct a nonlocal field theories is given in the monograph [7] .

2. Transition to Constant Curvature Momentum Space in the off Mass Shell Extension.

In the previous section we considered a number of conditions which are satisfied by the extended scattering matrix in Bogolubov's axiomatic approach. In a more complete form the set of requirements, in accordance with which the extension of the S -matrix off the mass-shell is made looks as follows [1,2,3,4] :

I. Relativistic invariance.

II. Translation invariance.

III. Unitarity.

IV. Causality.

V. Completeness of the system of asymptotic states with positive energy and existence of unique vacuum state.

VI. Stability of the vacuum and one-particle states.

In the axiomatic construction of the scattering matrix the choice of a definite way of extension off the mass shell is essentially equivalent to acception of a definite way of description of quantized fields interactions. Therefore if we intend to modify the interaction laws of the elementary particles in the region of small De Broglie wave lenghts, comparable with some fundamental length ℓ_0 (see the end of Section 1.), then obligatory this must be reflected in the way of extension of the scattering matrix off the mass shell. It is evident that the new extended objects (fields, c.f., currents, etc.) in the region of energies and momenta $\gtrsim 1/\ell_0$ will be considerably different from their "classical"+) analogues. At the same time the difficulties of the old theory, connected with badly defined products of generalized singular functions with coinciding singularities have either to disappear or to be essentially reduced. In other words the extension of the scattering matrix off the mass-shell, effectively taking into account the existence of a fundamental length ℓ_0, has to be less singular, than the "classical" extension satisfying conditions I-VI. Then naturally arises the question: which of these conditions should be modified and in what direction ?

Presently we have no any arguments based on experimental grounds to drop the requirements of Lorentz and translation invariance (requirements I. and II.). The necessity of the unitarity condition on the S-matrix, do not evoke any doubt (requirement III). The requirements V and VI seem to be also well grounded.

Let us consider now requirement IV- the "classical" Bogolubov's causality condition written for instance in the form (1.35). As the quasilocal terms contribute in the point $\xi = x_1 - x_2 = 0$ then it is natural to suppose that the condition

+) Later on the terminus "classical" will be applied to quantities and relations in the limiting case $\ell_0 = 0$.

should be essentially changed in the region:

$$|\xi| \lesssim \ell_0 . \tag{2.1}$$

Further our reasoning unavoidably has the character of a search. First of all let us notice that one has necessarily to add to requirements I-VI in fact one more condition whose fulfilment in the extended off the mass shell S-matrix is considered usually like selfevident. We have in mind the pseudoeuclidean nature of the 4-momentum space in which the mass shell hyperboloid

$$p^2 - m^2 = 0 \tag{2.2}$$

is embedded.

In other words in the usual theory it is silently supposed that when the extrapolation off the shell (2.2) is done any of the 4-momenta p_μ, on which the extended operators $\varphi(p)$ and extended c.f. $S_n(p_1, \dots p_n)$ depend, becomes arbitrary vector in Minkowski space+). As a result because of (1.2a) the geometry of x-space and the geometry of ξ-space (see footnote on p. 11) are also pseudo-euclidean.

However the general principles of the theory do not

+) In perturbation theory 4-momenta p_μ off the mass shell (2.2) are usually called virtual.

uniquely imply that momentum space should be necessarily flat Minkowski space. In particular the relativistic invariance condition does not fix the choice of a definite geometry in this space, but only requires that the quantities (p_0, p_1, p_2, p_3) should be transformed under Lorentz transformations like 4-vector.

It could seem, if we recall about identity (1.21) and connected with it relations (1.25) and (1.26) that the pseudoeuclidean character of the p-space is a necessary corollary of the translation invariance of the S-matrix. However in order the translation invariance to be satisfied it is sufficient only the fulfilment of relations of type (1.15), and equations (1.21)-(1.22) are obtained from (1.15) and the accepted a priori pseudoeuclidean character of momentum space+).

Taking into account all that let us now formulate the hypothesis which in all our further constructions will be of fundamental importance:

The new extension of the S-matrix off the mass shell, which gives a consistent description of the elementary particle interaction with arbitrary De Broglie wave lengths, should be

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+) The latter is reflected in the explicit form of the volume element $d^4p = dp_0 d\vec{p}$, in the substitution (1.18) and relation (1.19).

based not on a pseudoeuclidean momentum space, but on momentum space with constant curvature. The mathematical realisation of this space is the hypersphere:

$$p_0^2 - p_1^2 - p_2^2 - p_3^2 + \frac{1}{\ell_0^2} p_4^2 = \frac{1}{\ell_0^2} \qquad (2.3)$$

in the pseudoeuclidean 5-space of the variables $(p_0, p_1, p_2, p_3, p_4)$. The constant ℓ_0 defining the curvature of the surface (2.3) plays the role of a fundamental length+).

We suppose that the new extension is conformed with the "classical" requirements I-III, V-VI and with causality condition modified in the spirit of the new p-space geometry.

The curved 4-space, described by eq.(2.3) is called De-Sitter space. It can be considered as the closest to Minkowski space in the hierarchy of metric spaces. The motion groups of these two spaces - Poincaré group (1.32) and De-Sitter group (I.6) - depend on 10 parameters.

Both contain the Lorentz group as a subgroup which realize homogeneous pseudoorthogonal transformations in the space (p_0, p_1, p_2, p_3) (see (1.32) when $b = 0$ and (I.7)):

+) Some necessary mathematical information about the constant curvature space (2.3) is collected in Appendix I.

$$p'_\mu = \Lambda_\mu{}^\nu p_\nu \quad (\mu,\nu = 0,1,2,3). \tag{2.4}$$

The presence or absence of fundamental length ℓ_0 in the theory does not affect at all equation (2.4). This means that both in the new scheme and in the "classical" theory, the requirement of relativistic invariance (req.I.) may be formulated in the same way and we shall not discuss this point anymore.

In the flat limit $\ell_0 \to 0$ the relations of De-Sitter geometry go into its pseudoeuclidean analogues+). In this case, evidently, all field-theoretical quantities extended off the mass shell in the spirit of the De-Sitter space geometry (2.3), have to obtain their "classical" form.

Later on it will be convenient to use a system of units in which:

$$\hbar = c = \ell_0 = 1 .$$

In these terms "classical" limit means that we consider region of momenta values:

$$|p| \ll 1 . \tag{2.5}$$

+) Let us note that there exists one more model of a space of constant curvature, which have a right pseudoeuclidean limit. It is connected with the surface [5] : $p_0^2 - \vec{p}^2 - p_4^2 = -\frac{1}{\ell_0^2}$. We shall not develop theory corresponding to this case, because of some physical reasons.

It can be easily seen that the mass shell (2.2) can be embedded in the space (2.3) only if the condition:

$$m^2 \leq 1 \tag{2.6}$$

is satisfied.

We shall suppose that the restriction (2.6) is always fulfilled for the masses of the objects, which are described by quantized fields. Then eq.(2.2) is equivalent to the relation:

$$(p_4 + m_4)(p_4 - m_4) = 0 , \tag{2.7}$$

where by definition, $m_4 \equiv \sqrt{1-m^2} \geqslant 0$. Since on the surface (2.3) to any fixed value of p there correspond two different by sign values of p_4 , then each of the brackets in (2.7) can vanish:

$$p_4 - m_4 = 0 \tag{2.8a}$$

$$p_4 + m_4 = 0 . \tag{2.8b}$$

Let us make now a <u>physical</u> assumption: for the <u>free field</u> $\varphi(p,p_4)$ defined in the De-Sitter p-space (2.3) only the condition (2.8a) is satisfied. In other words:

$$2(p_4 - m_4)\varphi(p,p_4) = 0 . \tag{2.9}$$

We introduced a factor of 2 in order eq.(2.9) to coincide exactly with (1.3) in the "classical" limit, $m, |p| \ll 1$ +).

From (2.9) it follows that:

$$\varphi(p,p_4) = \delta(2p_4 - 2m_4)\tilde{\varphi}(p,p_4), \tag{2.10}$$

where $\varphi(p,p_4)$ is operator which does not possess singularities on the mass shell (2.8a).

Later we shall consider decompositions of different quantities of the theory in terms of φ -field products. When doing that each operator $\varphi(p,p_4)$ will appear in the corresponding integrals accompanied by "its own" volume element (I.5):

$$\int \ldots d\Omega_p \varphi(p,p_4) \ldots \tag{2.11}$$

(the dots substitute the o.f., all other φ-operators and volume elements). On the mass shell taking into account (2.10) and (I.5), we can write eq.(2.11) in the form:

+) The equation based on relation (2.8b) has no formally correct "classical" limit. Let us note, however, that from an optimistical point of view on the theory developed here, we have not to exclude the possibility, that particle states with $p_4 < 0$ can have for the new theory such a fundamental meaning as, for instance, the states with negative energies in Dirac's theory of the electron.

$$\int \ldots d\Omega_p \varphi(p,p_4) \ldots = \int \ldots 2\delta(p_L^2 - 1) d^5p\, \delta(2p_4 - 2m_4)\tilde{\varphi}(p,m_4) \ldots =$$

$$= \int \ldots d^4p\, \delta(p^2 - m^2)\, \tilde{\varphi}(p,m_4) \ldots \tag{2.12}$$

In the "classical" case instead we would have taking into account (1.4):

$$\int \ldots d^4p\, \varphi(p) \ldots = \int \ldots d^4p\, \delta(p^2 - m^2)\tilde{\varphi}(p) \ldots \tag{2.13}$$

Comparing (2.12) and (2.13) we conclude that on the mass shell the equality should be satisfied:

$$\tilde{\varphi}(p,m_4) = \tilde{\varphi}(p). \tag{2.14}$$

Let us stress that between the extended off the mass shell operators $\varphi(p)$ and $\varphi(p,p_4)$ <u>there is no more any connection</u> because to each extension a different <u>geometry in the p-space</u> corresponds. In particular the classical field $\varphi(p)$ is defined for all values of p_μ, but the field $\varphi(p,p_4)$, because of (2.3), only in the domain

$$p^2 \leq 1 . \tag{2.15}$$

The relation (2.14) looks like a "correspondence principle". With its help the commutation relation which should be satisfied by the solutions of equation (2.9) can be determined.

Let us note first that directly from (2.14) it follows the definition of creation and annihilation operators (see [1])

$$\varphi^{(+)}(\vec{p}) = \frac{\tilde{\varphi}(p, m_4)}{\sqrt{2p_0}}\Bigg|_{p_0 = \sqrt{\vec{p}^2 + m^2}} \tag{2.16}$$

$$\varphi^{(-)}(\vec{p}) = \frac{\tilde{\varphi}(-p, m_4)}{\sqrt{2p_0}}\Bigg|_{p_0 = \sqrt{\vec{p}^2 + m^2}}$$

Further we evidently have:

$$[\varphi^{(-)}(\vec{p}_1), \varphi^{(+)}(\vec{p}_2)] = \delta^{(3)}(\vec{p}_1 - \vec{p}_2) . \tag{2.17}$$

From here, taking into account (2.16), (2.10) and (I.17) we obtain [6] :

$$[\varphi(p_1, p_{14}), \varphi(p_2, p_{24})] = \delta(p_1 - p_2)\,\varepsilon(p_2^0)\,\delta(2p_{24} - 2m_4). \tag{2.18}$$

Passing to coordinates $(\omega, \vec{p})$ (see I.13) and putting by definition+):

$$\varphi(p, p_4) \equiv \varphi(\omega, \vec{p}), \quad |\omega| \leq \frac{\pi}{2}$$

+)Reduction of the range of variation of ω here is connected with vanishing of the operator $\varphi(p, p_4)$ for $p_4 < 0$ (see (2.10)).

we shall have instead (2.18):

$$[\varphi(\omega_1, \vec{p}_1), \varphi(\omega_2, p_2)] = \delta(\omega_1 + \omega_2)\,\delta(\vec{p}_1 + \vec{p}_2)\,\varepsilon(\omega_2)\,\delta\left(2\cos\omega_2\sqrt{1 + \vec{p}_2^{\,2}} - 2m_4\right). \tag{2.19}$$

The neutrality condition of the free field $\varphi(p, p_4)$ in the new scheme because of (2.14) is written in form equivalent to (1.2c):

$$\varphi^+(p, p_4) = \varphi(-p, p_4) = \varphi(-\omega, -\vec{p}) . \tag{2.20}$$

We shall suppose that the relation (2.20) holds also for the extended φ -operators.

In De-Sitter p -space (2.3) the components of the 4 - vector p_μ , as in the flat space, evidently commute identically with each other:

$$[p_\mu, p_\nu] = 0, \; (\mu, \nu = 0, 1, 2, 3). \tag{2.21}$$

From here and (2.17) we conclude that the operator

$$\hat{P}_\mu = \int d\vec{k}\, k_\mu \varphi^{(+)}(\vec{k})\,\varphi^{(-)}(\vec{k}), \quad k_0 = \sqrt{\vec{k}^2 + m^2} \tag{2.22}$$

has all standard properties of the field energy-momentum operator. In particular:

$$[\hat{P}_\mu, \hat{P}_\nu] = 0 , \tag{2.23}$$

$$[\hat{P}_\mu, \varphi(p, p_4)] = p_\mu \varphi(p, p_4) . \tag{2.24}$$

In this way we have all which is needed to formulate condition V in the new scheme. We can introduce the vacuum state $|0\rangle$ with the condition $\varphi^{(-)}(\vec{k})|0\rangle = 0$, supposing that this state is unique. We can construct complete system of state vectors+)

$$|\varphi^{(+)}(\vec{k}_1)\dots\varphi^{(+)}(\vec{k}_n)|0\rangle, \tag{2.25}$$

and in each of them the spectrum of the operators $\hat{P}_0$ and $\hat{P}^2$ is positive. The only new feature in comparison with the usual theory is the limitation (2.6) on the mass of one-particle states.

It is easy to see that the notion of normal product of field operator's and the corresponding Wick's theorem can be introduced in the new scheme without any principal changes. The normal product and pairing of two operators are defined by the relations (see [1]):

$$\varphi(p_1,p_{14})\varphi(p_2,p_{24}) = \tag{2.26}$$

$$= :\varphi(p_1,p_{14})\varphi(p_2,p_{24}): + \underbrace{\varphi(p_1,p_{14})\varphi}(p_2,p_{24}),$$

+)By construction quantities $\varphi(p,p_4)$ are analogues of out-operators. Therefore, vectors (2.25) describe out-states of the free particles.

$$\underbrace{\varphi(p_1,p_{14})\varphi}(p_2,p_{24}) = \delta(p_1,-p_2)\,\theta(-p_1^0)\,\delta(2p_{14}-2m_4) \equiv$$

$$\equiv \delta(p_1,-p_2)\,\mathcal{D}^{(-)}(p_1). \tag{2.27}$$

Now in complete analogy with the "classical" decomposition (1.1), we can write the new S -matrix in the form of series in terms of normal products of $\varphi(p,p_4)$, defined in De-Sitter momentum space (2.3):

$$S = \tag{2.28}$$

$$= \sum_n \int d\Omega_{p_1}\dots d\Omega_{p_2}\, S_n(p_1,p_{14};\dots p_n,p_{n4}) :\varphi(p_1,p_{14})\dots\varphi(p_n,p_{n4}):$$

Decomposition (2.28), by assumption, remains valid also after extension off the mass shell (2.8a), i.e. also in that case, when the operator $\varphi(p,p_4)$ does not more satisfy equation (2.9) and the 4 -vector p_r becomes arbitrary vector of De-Sitter space (2.3).

Let us introduce into consideration the functional derivative of the S -matrix with respect to the φ -fields:

$$\frac{\delta^n S}{\delta\varphi(p_1,p_{14})\,\delta\varphi(p_2,p_{24})\dots\delta\varphi(p_n,p_{n4})}, \tag{2.29}$$

setting by definition that:

$$\frac{\delta\varphi(p,p_4)}{\delta\varphi(p',p'_4)} = \delta(p,p') \tag{2.30}$$

(see (I.17)). Recalling now, that the requirement VI is satisfied, we obtain from (2.28) with the help of (2.30) expression of the c.f. in terms of vacuum expectation values of the radiation operators (see (1.6))

$$S_n(p_1, p_{14}; \dots; p_n, p_{n4}) =$$

$$= \frac{1}{n!} \langle 0 | \frac{\delta^n S}{\delta\varphi(p_1,p_{14}) \dots \delta\varphi(p_n,p_{n4})} S^\dagger | 0 \rangle . \tag{2.31}$$

As the appropriate analysis shows the formulation of the present theory is simplified if extended off the mass shell S -matrix obeys the supplementary condition:

$$\frac{\delta S}{\delta\varphi(p,p_4)} = 0 \ , \ \text{if} \ p_4 < 0 . \tag{2.32}$$

This condition has dynamical character since it is imposed on the S -matrix. It is consistent with our definition of the mass-shell-eq.(2.8a) and of the choice of the free equation in the form (2.9).

Later we shall suppose condition (2.32) satisfied assuming that the theory obtained does not become too poor$^{+)}$. Then the extended off the mass shell c.f. (2.31) have to satisfy the relation:

$^{+)}$However confer the footnote on page 20.

$$S_n(p_1, p_{14}; \dots ; p_n, p_{n4}) = 0 , \tag{2.33a}$$

if even one of the forth components p_{i4} is negative:

$$p_{i4} < 0 \quad (i = 1,2,\dots,n) . \tag{2.33b}$$

Let us introduce the current operator (see (1.8))

$$j(p,p_4) = i \frac{\delta S}{\delta\varphi(-p,p_4)} S^\dagger . \tag{2.34}$$

From the unitarity of the extended S -matrix (requirement III) we have:

$$\frac{\delta S}{\delta\varphi(-p,p_4)} S^\dagger + S \frac{\delta S^\dagger}{\delta\varphi(-p,p_4)} = 0 . \tag{2.35}$$

From here and on base of (2.19) it can be concluded that the current operator (2.34) satisfies neutrality condition analogous to (2.20):

$$j(p,p_4)^\dagger = j(-p,p_4) . \tag{2.36}$$

The variational derivatives of the S -matrix in terms of φ -fields commute by definition and therefore the current operator should obey "solvability condition" [2,3,4] :

$$\frac{\delta j(p_1,p_{14})}{\delta\varphi(-p_2,p_{24})} - \frac{\delta j(p_2,p_{24})}{\delta\varphi(-p_1,p_{14})} = i \left[j(p_1,p_{14}), j(p_2,p_{24}) \right] . \tag{2.37}$$

Let us turn now to the problem of formulating the translation invariance condition of the theory (requirement II). Taking into account eq.(2.23), we have the right to conserve the former interpretation of the operator $\hat{P}_\mu$ as generator of the translation group (1.11). Then from (2.24) it follows that the free field operators $\varphi(p, p_4)$ transform under displacements (1.11) by the usual rule:

$$e^{i\hat{P}a}\varphi(p,p_4)e^{-i\hat{P}a} = e^{ipa}\varphi(p,p_4). \tag{2.38}$$

As in the "classical" theory we postulate this transformation law as well for the extended operators (see (1.12)).

Because of requirement II in the new scheme the translation invariance condition for extended S-matrix (1.13) have to be conserved. From here it follows immediately that the c.f. S_n in the decomposition (2.28) must be represented in the form (1.15):

$$S_n(p_1,p_{14};p_2,p_{24};\cdots;p_n,p_{n4}) =$$

$$= \delta(p_1+p_2+\cdots+p_n)\, S_n'(p_1,p_{14};\cdots;p_n,p_{n4}) \tag{2.39}$$

It is remarkable that in the new formalism, as in the "classical" theory, the quantities S_n' may be expressed in terms of the original c.f. In result new identities which are direct generalization of the "classical" ones (1.21) in the case of De-Sitter space appear [+)]

[+)] The proof is given in the next section.

From the translation invariance of the S-matrix and rel.(2.38) it follows also that the supplementary condition (2.32) is translation invariant and the current operator transforms in a standard way (see (1.14)):

$$e^{i\hat{P}a}j(p,p_4)e^{-i\hat{P}a} = e^{ipa}j(p,p_4). \tag{2.40}$$

3. Identities for the Extended S-Matrix Coefficient Functions.

The derivation of the above mentioned identities almost literally repeats the correspondent procedure in the "classical" theory (see Section 1, eqs. (1.16) - (1.21)).

First of all let us pick out the n'th order term from the decomposition (2.28) (see 1.16):

$$\int d\Omega_{p_1}\cdots d\Omega_{p_n} S(p_1,p_{14};\cdots;p_n,p_{n4}):\varphi(p_1,p_{14})\cdots\varphi(p_n,p_{n4}):. \tag{3.1}$$

Now consider the expression (see(1.17)):

$$\int d\Omega_{p_1}\cdots d\Omega_{p_n}\, S_n(p_1,p_{14};\dots;p_n,p_{n4})\cdot \tag{3.2}$$

$$\cdot :\varphi\left(p_1(-)U^{(p_1\cdots p_n)},(p_1(-)U^{(p_1\cdots p_n)})_4\right)\cdots$$

$$\cdots\varphi\left(p_n(-)U^{(p_1\cdots p_n)},(p_1(-)U^{(p_1\cdots p_n)})_4\right):,$$

where $U^{(p_1\cdots p_n)}$ is De-Sitter space vector, given by rel. (I.21a).

Because of (2.39) the integrand in (3.2) is defined on the surface:

$$p_1+p_2+\cdots+p_n=0. \tag{3.3}$$

Therefore we may put the U -vector in (3.2) equal to zero+). That proves the equivalence of the expressions (3.1) and (3.2).

Further, proceeding again in complete analogy with the "classical" case, (see (1.18)) we substitute in (3.2)

$$p_i=q_i\,(+)\,k \quad (i=1,2,\dots n)$$
$$k=U^{(p_1\cdots p_n)}, \tag{3.5}$$

where $U^{(p_1\cdots p_n)}$ is given by eq. (I.21a) and the vectors q_i satisfy the supplementary condition (I.20a). Taking into account

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+) In five dimensional form:

$$U_L=(0_\lambda,1). \tag{3.4}$$

(I.22), we obtain the following result (c.f.(1.20)):

$$\int d\Omega_{q_1}\cdots d\Omega_{q_n}\,\delta(U^{(q_1\cdots q_n)},0)\cdot$$

$$\cdot\int d\Omega_k\, S_n(q_1(+)k,(q_1(+)k)_4;\cdots q_n(+)k,(q_n(+)k)_4)\cdot \tag{3.6}$$

$$\cdot :\varphi(q_1,q_{14})\cdots\varphi(q_n,q_{n4}):.$$

As eq.(3.6) identically coincides with (3.1), the relation holds (see (1.21)):

$$S_n(p_1,p_{14};\dots;p_n,p_{n4})=$$

$$=\delta(U^{(p_1\cdots p_n)},0)\int d\Omega_k\, S_n(p_1(+)k,(p_1(+)k)_4;\dots;p_n+k,(p_n(+)k)_4)\equiv$$

$$\equiv\delta(U^{(p_1\cdots p_n)},0)\,\tilde{S}_n. \tag{3.7a}$$

From (I.17) and (I.21a) we get:

$$\delta(U^{(p_1\cdots p_n)},0)=\left(\sqrt{(p_1+\cdots+p_n)^2_M}\right)^4\delta(p_1+\cdots+p_n). \tag{3.8}$$

From here and from (2.39) and (3.7a) we obtain the analogue of eq. (1.22):

$$S'_n=\left(\sqrt{(p_1+\cdots+p_n)^2_M}\right)^4\tilde{S}_n \tag{3.9}$$

Therefore:

$$S_n(p_1, p_{14}; \cdots p_n, p_{n4}) =$$

$$= \delta(p_1 + \cdots + p_n)\left(\sqrt{(p_1 + \cdots + p_n)^2_M}\right)^4 \tilde{S}_n(p_1, p_{14}; \cdots; p_n, p_{n4}). \tag{3.7b}$$

It was our goal to prove the identity (3.7a)-(3.7b)+).

It should be clearly understood that validity of relations of the type (3.7) for the o.f. of the decomposition (2.28) garantees the translation invariance of the scattering matrix when it is extended off the mass shell in the spirit of the constant curvature p -space geometry.

Between identities (1.21) and (3.7) one may display a far going analogy if group theoretical considerations are involved. For instance under displacements (I.9) in curved p -space the 4 -dimensional vector $U^{(p_1 \cdots p_n)}$, which is argument of the δ -function in (3.7), transforms according to the law (I.24) (see (1.26)), and simultaneously the function $\tilde{S}_n(p_1, p_{14}; \cdots; p_n, p_{n4})$ remains invariant (see (1.25)):

+) It is evident that in the "classical" domain the identity (3.7) transforms into (1.21). In particular (see I.20b),

$$\sqrt{(p_1 + \cdots + p_n)^2_M} \longrightarrow \sqrt{n^2} = n \tag{3.10}$$

$$\tilde{S}_n(p_1 (+) b, (p_1 (+) b)_4; \cdots; p_n (+) b, (p_n (+) b)_4) =$$

$$= \tilde{S}_n(p_1, p_{14}; \cdots; p_n, p_{n4}). \tag{3.11}$$

We shall prove relation (3.11). Let us first note, that from (I.11), for arbitrary 4 -vectors p, k and b the equality takes place:

$$(p (+) b) (+) k = \Lambda_{b,k}(p (+) (k (+) b)), \tag{3.12}$$

where $\Lambda_{b,k}$ is a Lorentz transformation, which parameters depend on b and k.

Then taking into account the relativistic invariance of the extended function S_n (requirement I) we can write:

$$S_n(p_1, p_{14}; \cdots; p_n, p_{n4}) = S_n(\Lambda p_1, p_{14}; \cdots; \Lambda p_n, p_{n4}) \tag{3.13}$$

(here Λ is an arbitrary Lorentz transformation).

Now, using (3.12) and (3.13), the left-hand side of eq. (3.11) can be identically transformed:

$$\begin{aligned} &\tilde{S}_n(p_1 (+) b, (p_1 (+) b)_4; \cdots; p_n (+) b, (p_n (+) b)_4) = \\ &= \int d\Omega_k S_n((p_1 (+) b) (+) k, ((p_1 (+) b) (+) k)_4; \cdots; (p_n (+) b) (+) k, ((p_n (+) b) (+) k)_4) = \\ &= \int d\Omega_k S_n(\Lambda_{b,k}(p_1 (+) (k (+) b)), (p_1 (+) (k (+) b))_4; \cdots; \Lambda_{b,k}(p_n (+) (k (+) b)), (p_n (+) (k (+) b))_4) = \\ &= \int d\Omega_k S_n(p_1 (+) (k (+) b), (p_1 (+) (k (+) b))_4; \cdots; p_n (+) (k (+) b), (p_n (+) (k (+) b))_4). \end{aligned} \tag{3.14}$$

The substitution:

$$k_{(+)}b \to k \qquad (3.15)$$

in the last integral, owing to (I.10) and (3.15) leads us to an expression, coinciding with the right hand side of (3.11).

From the relativistic invariance of the functions $\widetilde{S}_n$ and rel.(3.11) which we just proved, follows that quantities $\widetilde{S}_n$ are invariant under <u>arbitrary</u> transformations (I.6) of De-Sitter group $SO(2,3)$:

$$\widetilde{S}_n(p_1, p_{14}; \dots; p_n, p_{n4}) = \widetilde{S}_n(\Lambda p_1, (\Lambda p_1)_4; \dots; (\Lambda p_n), (\Lambda p_n)_4). \qquad (3.16)$$

Therefore these functions depend on $SO(2,3)$ invariant scalar products of the type:

$$p_{i0}p_{j0} - \vec{p}_i\vec{p}_j + p_{i4}p_{j4} \equiv (p_i)_L (p_j)_L \quad (i,j = 1,2,\dots n):$$

$$\widetilde{S}_n = \widetilde{S}_n(\dots (p_i)_L (p_j)_L \dots). \qquad (3.17)$$

With the help of (I.9) it is easy to show that (c.f.I.20b):

$$(p_i)_L (p_j)_L = \sqrt{1 - (p_i (-) p_j)^2}. \qquad (3.18)$$

Therefore

$$\widetilde{S}_n = \widetilde{S}_n(\dots (p_i (-) p_j)^2 \dots) \qquad (3.19)$$

Substituting (3.19) and (I.20b) in the identity (3.7b), we have:

$$S_n(p_1, p_{14}; \dots; p_n, p_{n4}) =$$

$$= \delta(p_1 + \dots + p_n)\Big[n + \sum_{\substack{k=l=1 \\ k \neq l}}^{n} \sqrt{1 - (p_k (-) p_l)^2}\,\Big]^2 \widetilde{S}_n(\dots (p_i (-) p_j)^2 \dots). \qquad (3.20)$$

Let us recall (see Section I), that the "classical" functions $\widetilde{S}_n$ are invariant with respect to Poincaré group (1.32) of Minkowski p -space and this fact is reflected in eq. (1.33). Comparing (1.33) with (3.19) and taking into account (3.20), we can interprete the extension accepted in the new scheme, as a transition+) to "curved" relative momenta with the condition that the conservation law of the total 4 -momentum has usual "classical" form [5,6] . As we shall be convinced later in Section 4 in such an approach

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+) If the "curved" relative momenta are defined in accordance with eq.(I.25a) - (I.25b), putting

$$q_{ij} = \frac{\mu_j p_i - \mu_i p_j}{\mu_i + \mu_j}$$

$$\Big(\mu_i = \frac{1}{2}\big(p_{i4} + \frac{1}{2}\sqrt{(p_i + p_j)^2_L}\big), \quad \mu_j = \frac{1}{2}\big(p_j + \frac{1}{2}\sqrt{(p_i + p_j)^2_L}\big)\Big),$$

then it is not difficult to show that:

$$(p_i (-) p_j)^2 = (q_{ij} (+) q_{ij})^2.$$

In "classical" limit this relation, evidently, goes into equation $(p_i - p_j)^2 = 4 q^2_{ij}$.

also the relative coordinates ξ (see the footnote on p 45) which are canonically conjugated to the relative momenta change in essential manner. Let us now continue discussing the properties of identity (3.7a).

First of all we notice that this relation is "irreducible" in the same sense as it is "irreducible" identity (1.21) (see Section 1. p.7.). That is if the integrand $S_n(p_1(+)k,(p_1(+)k)_4;\dots;p_n(+)k,(p_n+k)_4)$ in (3.7a) is itself written in a form of an integral multiplied by a δ -function using (3.7a) then we again obtain (3.7a). Indeed, because of (3.7a), (I.24) and (3.11) we have:

$$S_n(p_1(+)k,(p_1(+)k)_4;\dots;(p_n+k),(p_n+k)_4)= \\ = \delta\left(U^{(p_1\dots p_n)}(+)k,0\right)\tilde{S}_n(p_1,p_{14};\dots;p_n,p_{n4}). \tag{3.21}$$

Therefore,

$$\int d\Omega_k\, S_n(p_1(+)k,(p_1(+)k)_4;\dots;p_n(+)k,(p_n+k)_4)= \\ = \tilde{S}_n \int d\Omega_k\, \delta\left(U^{(p_1\dots p_n)}(+)k,0\right). \tag{3.22}$$

But from (I.18) and (I.17)

$$\int\delta\left(U^{(p_1\dots p_n)}(+)k,0\right)d\Omega_k = \int\delta\left(U^{(p_1\dots p_n)},-k\right)d\Omega_k = 1 \tag{3.23}$$

which proves the "irreducibility" of (3.7a).

Let us now consider the case, when the c.f. $S_n(p_1,p_{14};\dots;p_n,p_{n4})$ contains disconnected components of the type (c.f.(1.27)):

$$S_m(p_1,p_{14};\dots;p_m,p_{m4})\,S_{n-m}(p_{m+1},p_{m+1\,4};\dots;p_n,p_{n4}), \tag{3.24}$$

and each of the c.f. S_m and S_{n-m} satisfies identity of the type (3.7a):

$$S_m(p_1,p_{14};\dots;p_m,p_{m4})= \\ = \delta\left(U^{(p_1\dots p_m)},0\right)\tilde{S}_m(p_1,p_{14};\dots;p_m,p_{m4}) \tag{3.25}$$

$$S_{n-m}(p_{m+1},p_{m+1\,4};\dots;p_n,p_{n4})= \\ = \delta\left(U^{(p_{m+1},\dots,p_n)},0\right)\tilde{S}_{n-m}(p_{m+1},p_{m+1\,4};\dots;p_n,p_{n4}).$$

It happens, that it is sufficient the relations (3.25) to hold in order the disconnected component (3.24) also to obey the identity (3.7a):

$$S_m(p_1,p_{14};\dots;p_m,p_{m4})\,S_{n-m}(p_{m+1},p_{m+1\,4};\dots;p_n,p_{n4}) = \\ = \delta\left(U^{(p_1\dots p_n)},0\right)\widetilde{S_m(p_1,p_{14};\dots;p_m,p_{m4})S_{n-m}(p_{m+1},p_{m+1\,4};\dots;p_n,p_{n4})}. \tag{3.26}$$

It is clear that eq.(3.26) is a direct generalization to the case of De-Sitter p -space of the "classical" formula (1.24). The derivation of (3.26) goes over the same pattern as the correspondent reasoning in Section 1.

First from (3.8) and (I.17) it is easy to obtain that (see (1.30))

$$\delta\left(U^{(p_1\cdots p_m)},0\right)\delta\left(U^{(p_{m+1},\cdots,p_n)},0\right)=$$

$$=\delta\left(U^{(p_1\cdots p_n)},0\right)\delta\left(U^{(p_1\cdots p_m)},U^{(p_{m+1},\cdots,p_n)}\right). \tag{3.27}$$

Further, using (I.18) we write (3.27) in a form analogous to (1.31):

$$\delta\left(U^{(p_1\cdots p_m)},0\right)\delta\left(U^{(p_{m+1},\cdots,p_n)},0\right)=$$

$$=\delta\left(U^{(p_1\cdots p_n)},0\right)\int d\Omega_k\,\delta\left(U^{(p_1\cdots p_m)}(+)k,0\right)\delta\left(U^{(p_{m+1}\cdots p_n)}(+)k,0\right). \tag{3.28}$$

Finally, multiplying both sides of (3.28) by

$\tilde{S}_m(p_1,p_{1\gamma};\cdots;p_m,p_{m\gamma})\,\tilde{S}_{n-m}(p_{m+1},p_{m+1\gamma};\cdots;p_n,p_{n\gamma})$

and taking into account (3.21) and (3.25) we obtain eq.(3.26).

4. Configuration Representation. Locality and Causality Condition.

Let us make a Fourier transformation of the field $\varphi(p,p_4)$ defined in De-Sitter space (2.3), using the basis function (I.33)-(I.34). If we introduce $\langle\xi|p\rangle$ as universal notation for these functions, where ξ is one of the sets (L,n,ℓ,m), (Λ,n,ℓ,m), (L,N) and (Λ,N) we shall have:

$$\frac{1}{(2\pi)^{3/2}}\int\langle\xi|p\rangle\,d\Omega_p\,\varphi(p,p_4)\equiv\varphi(\xi). \tag{4.1}$$

If $\varphi(p,p_4)$ satisfies the free equation (2.9) then in the right-hand side of (4.1) we shall write $\varphi^{out}(\xi)$. Therefore, taking into account (2.10) and (I.5),

$$\varphi^{out}(\xi)=\frac{1}{(2\pi)^{3/2}}\int\langle\xi|p\rangle\,\delta(p_L^2-1)\,\delta(p_4-m_4)\,\tilde{\varphi}(p,m_4)\,d^5p. \tag{4.2}$$

In the case when as basis functions $\langle\xi|p\rangle$ the "plane waves" (I.34) are chosen the operator $\varphi^{out}(\xi)$ owing to (I.35)-(I.36) satisfies differential-difference equation of "Klein-Gordon" form:

$$2(K_L-m_4)\,\varphi(\xi_L)=0\quad(L\text{-series},\ \xi=(L,N))$$

$$2(K_L-m_4)\,\varphi(\xi_\Lambda)=0\quad(\Lambda\text{-series},\ \xi=(\Lambda,N)). \tag{4.3}$$

Now applying to the free operator $\varphi(p,p_4)$ simultaneously two operations - Fourier transform (4.1) with "plane-waves" (I.34) and translation transformation with parameter x we have:

$$\frac{1}{(2\pi)^{3/2}}\int \langle\xi|p\rangle e^{i\hat{P}_x}\varphi(p,p_4)e^{-i\hat{P}_x}d\Omega_p = \tag{4.4}$$

$$= \frac{1}{(2\pi)^{3/2}}\int \langle\xi|p\rangle e^{ipx}\varphi(p,p_4)\,d\Omega_p \equiv \varphi_x^{out}(\xi)$$

Here the operator $\varphi_x^{out}(\xi)$ depends essentially on two variables x and ξ as far as the functions $\langle\xi|p\rangle$ and e^{ipx} are different mathematical objects. This phenomenon has no analogue in the "classical" theory. The point is that in the usual formalism both momentum space and the space of the parameters of the translation group (1.11) are pseudoeuclidean. For this reason the plane waves in p-space, the quantities $\langle\xi|p\rangle$ and the plane waves e^{ipa} which realize representation of the group (1.11) have the same form:

$$\langle\xi|p\rangle = e^{ip\xi}. \tag{4.5}$$

As a result, instead of eq. (4.4) we get[+]:

$$\varphi_x^{out}(\xi) = \frac{1}{(2\pi)^{3/2}}\int e^{i\xi p}e^{ipx}\varphi(p)\,d^4p = \varphi^{out}(\xi+x) \tag{4.6}$$

It is useful to keep in mind that the plane wave in "classical" p-space, can be considered either like simultaneous eigenfunction of the generators $\xi_\mu = -i\frac{\partial}{\partial p_\mu}$ (4.7)

[+] The operator (4.6) is, of course, "classical" limit of (4.4) (see (I.37)).

of the transformation (1.23), or as solution of the eigenvalue problem for the operator of the " 4 -interval squared":

$$\xi^2 = -\left(\frac{\partial}{\partial p}\right)^2, \tag{4.8}$$

which plays the role of Casimir's operator of the motion group of the flat p-space. In the latter case

$$-\left(\frac{\partial}{\partial p}\right)^2\langle\xi|p\rangle = \lambda\langle\xi|p\rangle, \tag{4.9}$$

where the necessary type of the spectrum:

$$\lambda = \begin{cases} \xi^2 > 0 & \text{-timelike region} \\ \xi^2 = 0 & \text{-light cone} \\ \xi^2 < 0 & \text{-spacelike region} \end{cases} \tag{4.10}$$

is obtained if unitarity of the considered representation of the Poincaré group (1.32) is required.

Let us also recall, that the representations of the group (1.32) which correspond to intervals $\xi^2 \geqslant 0$, are labelled by one more invariant eigenvalue of the time operator $\xi_0 = -i\frac{\partial}{\partial p_0}$:

$$\frac{\xi_0}{|\xi_0|} = invar. \tag{4.11}$$

When one goes to De-Sitter p-space the "degeneration" of the plane waves, fixed in rel.(4.5), is removed. As a result, there appear two Fourier transformations: one of them like before is connected with decomposition in terms of matrix elements of the translation group (1.11) – usual plane waves, and the other uses as basic functions the quantities $\langle \xi | p \rangle$ (see I.34) closely connected with the matrix elements of the unitary irreducible representations of De-Sitter group $SO(2,3)$. If we apply a translation with parameter a, to the operator (4.4), which is obtained as a result of simultaneous action of the two mentioned Fourier transformations, then the obtained result can be represented as "displacement" to a quantity a of the "index" x, keeping ξ constant:

$$e^{i\hat{P}a} \varphi_x^{out}(\xi) e^{-i\hat{P}a} = \varphi_{x+a}^{out}(\xi). \tag{4.12}$$

The invariance of ξ under displacement transformation in rel. (4.12) gives a hint that this variable can be used in the new apparatus as analogue of the "classical" relative coordinate $\xi = x_1 - x_2$.

Let us consider like an example commutation relations of the type+):

$$\begin{aligned} &a) \quad [\varphi_x^{out}(\xi), \varphi_x^{out}(0)] \\ &b) \quad [\varphi_x^{out}(\xi), \varphi_x^{out}(-\xi)], \end{aligned} \tag{4.14}$$

+) The quantity $-\xi$ is determined by the equality

$$-\xi_L = (L, -N), \quad -\xi_\Lambda = (\Lambda, -N). \tag{4.13}$$

where by definition+),

$$\begin{aligned} &\varphi_x^{out}(0) = e^{i\hat{P}x} \varphi^{out}(0) e^{-i\hat{P}x} \\ &\varphi^{out}(0) = \frac{1}{(2\pi)^{3/2}} \int \varphi(p, p_4) \, d\Omega_p . \end{aligned} \tag{4.15}$$

Simple calculations using (4.4), (4.15) and (2.18a) demonstrate that both commutators do not depend on x, i.e. because of (4.12) are translation invariant. They can be expressed by the following integrals++):

$$\begin{aligned} &[\varphi_x^{out}(\xi), \varphi_x^{out}(0)] \equiv \\ &\equiv \frac{1}{i} \mathcal{D}(\xi, 0) = \frac{-1}{(2\pi)^3} \int \langle \xi | p \rangle \, \varepsilon(p_0) \, \delta(2p_4 - 2m_4) \, d\Omega_p \end{aligned} \tag{4.16}$$

$$\begin{aligned} &[\varphi_x^{out}(\xi), \varphi_x^{out}(-\xi)] \equiv \\ &\equiv \frac{1}{i} \mathcal{D}(\xi, -\xi) = \frac{-1}{(2\pi)^3} \int (\langle \xi | p \rangle)^2 \, \varepsilon(p_0) \, \delta(2p_4 - 2m_4) \, d\Omega_p . \end{aligned} \tag{4.17}$$

Comparing (4.16) with the "classical" commutator relation

$$[\varphi^{out}(x_1), \varphi^{out}(x_2)] = \frac{1}{i} \mathcal{D}(x_1 - x_2), \tag{4.18a}$$

+) In "classical" limit $\varphi^{out}(0) = \varphi^{out}(\xi)|_{\xi=0}$

++) Let us notice that the function $\mathcal{D}(\xi, 0)$ obeys the differential-difference Klein-Gordon equation (4.3).

it can be concluded, that quantity ξ in (4.16) plays the same role as the relative coordinate $x_1 - x_2$ in eq.(4.18a). Actually, substituting in (4.18a), $x_2 = x$, $x_1 - x_2 = \xi$ and taking into account (4.6) we get:

$$[\varphi_x^{out}(\xi), \varphi_x^{out}(0)] = \frac{1}{i} \mathcal{D}(\xi). \qquad (4.18b)$$

Obviously the "classical" analogue of the relation (4.17) is the equality:

$$[\varphi_x^{out}(\xi), \varphi_x^{out}(-\xi)] = [\varphi^{out}(x+\xi), \varphi^{out}(x-\xi)] = \frac{1}{i} \mathcal{D}(2\xi). \qquad (4.19)$$

Therefore in (4.17) we have to interprete ξ as "half" of the relative coordinate.

What is the nature of the new coordinate ξ ? The appearance of ξ in our formalism is directly connected with the solution of the eigenvalue problem for the Casimir's operator of the group SO (2,3) (see (I.30) - (I.34))+). If one compares (I.30) and (4.8) it is easy to notice that this operator is a direct geometrical generalization of the operator of the 4-interval squared $(-\frac{\partial}{\partial p})^2$. Moreover, as the quantity $(-\frac{\partial}{\partial p})^2$ is Casimir's operator of the motion group of Minkowski's p-space, then its substitution with the Casimir's operator (I.30) in the transition to De-Sitter

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+) Similar mathematical origin has the three dimensional relativistic coordinate [14] , introduced in the framework of the quasipotential approch. (See also [15]).

p -space is a natural step from group-theoretical point of view too. In the "classical" limit evidently:

$$-\frac{1}{\sqrt{g}} \frac{\partial}{\partial p_\mu} \left(g_{\mu\nu}^{-1} \sqrt{|g|} \frac{\partial}{\partial p_\nu} \right) \longrightarrow -\left(\frac{\partial}{\partial p}\right)^2 .$$

Comparing spectra (I.32) and (4.10) we see that L -series (I.32a) goes into the timelike region $\xi^2 > 0$, and Λ-series (I.32b) into the spacelike region $\xi^2 < 0$ of the pseudoeuclidean ξ -space (see also (I.37)).

We would like to emphasize that for "distances" ~ 1 (in normal units $\sim \ell_0$) the structure of the new ξ -space is essentially different from the geometry of the ξ-space in the usual theory. In particular, as it is seen from (I.32), the boundary between the "timelike" L -series and the "spacelike" Λ -series+) can not be more described by equation of type (I.36). So the light cone is "smeared".

A remarkable property of the representations, corresponding to L-series, is the existence in these representations of a

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+) We have the right to call the discrete L -series "timelike" not only for reasons of "classical" correspondence, but also because the spectrum of the time coordinate ξ_0 in our case, is always discrete (see (I.29b)). In this context the word "spacelike" applied to the continuous Λ -series can be connected with the continuity of an arbitrary component of the coordinate operator ξ_λ (I.26).

supplementary $SO(2,3)$-invariant – the sign of the discrete eigenvalue of the time operator ξ_0 (see (I.28) and (I.29)) [16] :

$$\frac{n}{|n|} = \text{invar.} \tag{4.20a}$$

Relation (4.20a) is a direct generalization of eq.(4.11) and evidently have to be taken into account when the <u>causality principle</u> is formulated in the new scheme.

A direct calculation using eqs.(I.33a) and (I.34a) demonstrates that in L -series

$$\frac{N_0}{|N_0|} = \frac{n}{|n|} .$$

Therefore, eq.(4.20a) is equivalent to the following:

$$\frac{N_0}{|N_0|} = \text{invar.} \tag{4.20b}$$

Let us prove that the commutation functions $\mathcal{D}(\xi, 0)$ and $\mathcal{D}(\xi, -\xi)$, defined by (4.16) – (4.17), vanish in the "<u>space-like</u>" region $\xi = \xi_\Lambda$.

Let us first calculate $\mathcal{D}(\xi_\Lambda, 0)$. Because of (I.34b) and (I.5):

$$\mathcal{D}(\xi_\Lambda, 0) =$$

$$= \frac{1}{(2\pi)^3 i} \int \varepsilon(p_0)\delta(2p_4 - 2m_4) 2\delta(p_L^2 - 1)(p_4 + pN)^{-3/2 + i\Lambda} d^5 p , \tag{4.21}$$

where $N^2 = N_0^2 - \vec{N}^2 = 1$. Taking into account the relativistic invariance of (4.21) and the fact that $\varepsilon(p_0)$ is odd function of p_0 we get the desired result:

$$\mathcal{D}(\xi_\Lambda, 0) =$$

$$= \frac{1}{(2\pi)^3 i} \int \varepsilon(p_0)\delta(2p_4 - 2m_4) 2\delta(p_L^2 - 1) dp_0 d\vec{p} dp_4 (p_4 - p_3)^{-3/2 + i\Lambda} = 0. \tag{4.22}$$

Similarly it is proved that:

$$\mathcal{D}(\xi_\Lambda, -\xi_\Lambda) = 0. \tag{4.23}$$

Let us now demonstrate that there is another way to obtain the commutation function $\mathcal{D}(\xi, 0)$, defined by (4.16). Namely one can introduce the relative coordinate ξ as a variable canonically conjugated to the "curved" relative momenta:

$$q = p_1 (-) U^{(p_1 p_2)} = \frac{\mu_2 p_1 - \mu_1 p_2}{\mu_1 + \mu_2} \tag{4.24}$$

$$\left[\mu_1 = \frac{1}{2}\left(p_{14} + \frac{1}{2}\sqrt{(p_1 + p_2)_L^2}\right), \mu_2 = \frac{1}{2}\left(p_{24} + \frac{1}{2}\sqrt{(p_1 + p_2)_L^2}\right) \right]$$

(see (I.25a) – (I.25b)), where p_1 and p_2 are the arguments of the φ -fields in the commutator (2.18). Let us consider in connection with this the integral:

$$\frac{1}{(2\pi)^3} \int e^{i(p_1 + p_2)x} \langle \xi | q \rangle d\Omega_{p_1} d\Omega_{p_2} \left[\varphi(p_1, p_{14}), \varphi(p_2, p_{24}) \right] . \tag{4.25}$$

It is clear that in "classical" limit when $\langle \xi | q \rangle \to e^{i\xi \frac{p_1 - p_2}{2}}$

this expression goes into the commutation relation:

$$[\varphi(x+\tfrac{\xi}{2}),\varphi(x-\tfrac{\xi}{2})] = \tfrac{1}{i}\mathcal{D}(\xi). \qquad (4.26)$$

From the other hand, if we substitute in (4.25) the commutator (2.18) and integrate with the help of the δ - function over p_2, and then taking also into account (4.24) and (4.16) we obtain:

$$\frac{1}{(2\pi)^3}\int e^{i(p_1+p_2)x}\langle\xi|q\rangle d\Omega_{p_1}d\Omega_{p_2}[\varphi(p_1,p_{1v}),\varphi(p_2,p_{2v})] = \qquad (4.27)$$
$$= -\frac{1}{(2\pi)^3}\int\langle\xi|p_1\rangle d\Omega_{p_1}\varepsilon(p_1^0)\,\delta(2p_{1v}-2m_v) = \frac{1}{i}\mathcal{D}(\xi,0).$$

Thus the same commutation relation (2.18) for the free field operators in p -representation originates different <u>in form</u> but equivalent in their content commutation relations in configuration space: (4.16), (4.17) and (4.27).

All these commutators possess a specific <u>locality</u> property - they vanish for ξ from the "spacelike" Λ - series:

$$[\varphi_x^{out}(\xi),\varphi_x^{out}(0)] = 0, \qquad (4.28)$$

$$[\varphi_x^{out}(\xi),\varphi_x^{out}(-\xi)] = 0, \qquad (4.29)$$

$$\frac{1}{(2\pi)^3}\int e^{i(p_1+p_2)x}\langle\xi|\frac{\mu_2p_1-\mu_1p_2}{\mu_1+\mu_2}\rangle d\Omega_{p_1}d\Omega_{p_2}[\varphi(p_1,p_{1v}),\varphi(p_2,p_{2v})] = \qquad (4.30)$$
$$= 0,$$

if $\xi=\xi_\Lambda$ and x is arbitrary.

Let us recall that in the "classical" theory, both for the free operators and for operators, describing interacting systems, in particular for the current operator (1.9), the locality condition <u>has the same form</u> and is reduced to vanishing of the correspondent commutator out of the light cone (see for instance (1.36)). It is tempting to suppose that also in the new scheme <u>one+) of the equalities (4.28)-(4.30)</u> can be taken as a pattern when the locality condition on the current operator (2.34) is formulated. If we prefer relation (4.30)++), then the locality condition on the current operator in our formalism is written in the form:

$$\int e^{i(p_1+p_2)x}\langle\xi|\frac{\mu_2p_1-\mu_1p_2}{\mu_1+\mu_2}\rangle d\Omega_{p_1}d\Omega_{p_2}[j(p_1,p_{1v}),j(p_2,p_{2v})] = 0, \qquad (4.31)$$

if $\xi=\xi_\Lambda$ and x is arbitrary.

+) There is no any guarantee that the equivalence of the relations (4.28)-(4.30), which holds in the theory of the free φ -fields, is conserved after transition to more general operators.

++) The reasons for such a choice are mainly technical: there is in (4.30) a complete separation of the variables to "relative" and "absolute", both in configuration and momentum space, which is very convenient for calculations, taking into account the translation invariance of the theory.

Relation (4.31) is translation invariant since only the arbitrary parameter x varies in it under transformation (2.40).

In "classical" limit we obtain from (4.31), in complete analogy with (4.26), the standard locality condition for the current operator (1.10), equivalent to (1.36):

$$\left[j\left(x+\frac{\xi}{2}\right), j\left(x-\frac{\xi}{2}\right)\right] = 0 , \qquad (4.32)$$

if $\xi^2 < 0$ and x is arbitrary.

As it is well known relation (4.32) in the usual theory is a corollary of the Bogolubov's causality condition (1.34) and the "solvability condition" [2,3,4] . As far as in this scheme we already postulated the locality condition in the form (4.31) and the "solvability condition" has the form (2.37) then naturally the question arises: how the new causality condition has to look like in order that from it the locality condition (4.31) to follow, when eq.(2.37) is taken into account? Recalling that in the new ξ -space, for "timelike" values $\xi_L = (L, N)$, the sign of the component N_0 (which coincides with the sign of the discrete time) is relativistic invariant (see (4.20)) we can put in complete analogy with (1.34)+):

+) It is easy to see that in "classical" limit eq.(4.33) goes in the causality condition (1.34), written in terms of "relative" and "absolute" coordinates:

$$\frac{\delta j\left(x+\frac{\xi}{2}\right)}{\delta \varphi\left(x-\frac{\xi}{2}\right)} = 0 , \quad \text{if } \xi \gtrsim 0 , \quad x \text{ is arbitrary.}$$

$$\left(x = \frac{x_1+x_2}{2} , \ \xi = x_1 - x_2 \right).$$

$$\int e^{i(p_1+p_2)x} \left\langle \xi \left| \frac{\mu_2 p_1 - \mu_1 p_2}{\mu_1+\mu_2} \right.\right\rangle \frac{\delta j(p_1, p_{14})}{\delta \varphi(-p_2, p_{24})} d\Omega_{p_1} d\Omega_{p_2} = 0 , \qquad (4.33)$$

if $\xi \gtrsim 0$, x – arbitrary.

The symbol $\xi \gtrsim 0$ has the following sense in our context:

1) either $\xi = \xi_L = (L, N)$ and $\operatorname{sign} N_0 = \operatorname{sign} n > 0$

2) or $\xi = \xi_\Lambda = (\Lambda, N)$.

Later we shall consider relation (4.33) as <u>causality condition in the developed field theory</u>. We shall suppose that the extension of the S -matrix off the mass shell, based on De-Sitter momentum space, should be consistent with (4.33) (the analogue of the "classical" requirement IV, see Section 1).

Similarly to (4.31), equation (4.33) is translation invariant. If we substitute in (4.33) ξ with $-\xi$ (see (4.13)), put $\xi = \xi_\Lambda$ in the original and thus obtained relations and subtract them from each other then because of "solvability condition" (2.37) we obtain the locality condition (4.31). With the help of a similar procedure it is not difficult to demonstrate that:

$$\int e^{i(p_1+p_2)x} \left\langle \xi \left| \frac{\mu_2 p_1 - \mu_1 p_2}{\mu_1+\mu_2} \right.\right\rangle d\Omega_{p_1} d\Omega_{p_2} \frac{\delta^2 S}{\delta\varphi(-p_1, p_{14})\,\delta\varphi(-p_2, p_{24})} S^{+} =$$

$$= -\int e^{i(p_1+p_2)x} \left\langle \xi \left| \frac{\mu_2 p_1 - \mu_1 p_2}{\mu_1+\mu_2} \right.\right\rangle d\Omega_{p_1} d\Omega_{p_2}\, j(p_1, p_{14})\, j(p_2, p_{24}) + \qquad (4.34)$$

$$+ \theta(-N_0) \int e^{i(p_1+p_2)x} \left\langle \xi \left| \frac{\mu_2 p_1 - \mu_1 p_2}{\mu_1+\mu_2} \right.\right\rangle d\Omega_{p_1} d\Omega_{p_2} \left[j(p_1, p_{14}), j(p_2, p_{24}) \right] + \cdots ,$$

where

$$\theta(N_0) = \theta(n) = \begin{cases} 1, & n>0 \\ 0, & n<0 \end{cases} \tag{4.35}$$

and the dots at the right hand side indicate that here in principle could appear supplementary additive terms, which could be caused by a possible ununiqueness of the product of $\theta(N_0)$ and the current commutator (confer with the quasilocal terms in (1.35)). It is clear that investigation of products of this type will allow to judge how singular is the new off-mass-shell extension, based on the introduction of fundamental length in the theory. We shall come back to discussion of this problem in Section 5.

Obviously, relation (4.34) generalizes the "integral" causality condition (1.35) of the usual theory. Commuting explicitly the currents and performing identical transformation we can write the right-hand side of (4.34) in a form of a specific chronological product:

$$\begin{aligned}
&\int e^{i(p_1+p_2)x}\left\langle \xi \left| \frac{\mu_2 p_1 - \mu_1 p_2}{\mu_1+\mu_2}\right.\right\rangle d\Omega_{p_1} d\Omega_{p_2} \frac{\delta^2 S}{\delta\varphi(-p_2,p_{24})\,\delta\varphi(-p_1,p_{14})} S^+ = \\
&= -\theta(N_0)\int e^{i(p_1+p_2)x}\left\langle \xi \left| \frac{\mu_2 p_1 - \mu_1 p_2}{\mu_1+\mu_2}\right.\right\rangle d\Omega_{p_1} d\Omega_{p_2}\, j(p_1,p_{14})\, j(p_2,p_{24}) - \\
&-\theta(-N_0)\int e^{i(p_1+p_2)x}\left\langle \xi \left| \frac{\mu_2 p_1 - \mu_1 p_2}{\mu_1+\mu_2}\right.\right\rangle d\Omega_{p_1} d\Omega_{p_2}\, j(p_2,p_{24})\, j(p_1,p_{14}) + \cdots \equiv \\
&\equiv -T_\xi \int e^{i(p_1+p_2)x}\left\langle \xi \left| \frac{\mu_2 p_1 - \mu_1 p_2}{\mu_1+\mu_2}\right.\right\rangle d\Omega_{p_1} d\Omega_{p_2}\, j(p_1,p_{14})\, j(p_2,p_{24}) + \cdots
\end{aligned} \tag{4.36}$$

Let us now consider as an illustration, the application of the T_ξ-operation, defined in (4.36), to the free φ-fields. Notice first that the step θ-function (4.35), which appears in relation (4.36), has the following Fourier decomposition:

$$\theta(n) = \frac{1}{4\pi i}\int_{-\pi}^{\pi} e^{i\omega n} \frac{d\omega}{\operatorname{tg}\frac{\omega}{2} - i\varepsilon}. \tag{4.37}$$

Further, taking into account (2.26)-(2.27) we obtain:

$$\begin{aligned}
&T_\xi\left\{\frac{1}{(2\pi)^3}\int e^{i(p_1+p_2)x}\left\langle \xi \left| \frac{\mu_2 p_1 - \mu_1 p_2}{\mu_1+\mu_2}\right.\right\rangle d\Omega_{p_1} d\Omega_{p_2}\, \varphi(p_1,p_{14})\varphi(p_2,p_{24})\right\} = \\
&= \frac{1}{(2\pi)^3}\int e^{i(p_1+p_2)x}\left\langle \xi \left| \frac{\mu_2 p_1 - \mu_1 p_2}{\mu_1+\mu_2}\right.\right\rangle d\Omega_{p_1} d\Omega_{p_2} :\varphi(p_1,p_{14})\varphi(p_2,p_{24}): + \\
&+ \frac{1}{(2\pi)^3}\int e^{i(p_1+p_2)x}\, d\Omega_{p_1} d\Omega_{p_2}\, \delta(p_1,-p_2)\,\theta(p_2^0)\,\delta(2p_{24}-2m_4)\cdot \\
&\cdot\left[\theta(n)\left\langle \xi \left| \frac{\mu_2 p_1 - \mu_1 p_2}{\mu_1+\mu_2}\right.\right\rangle + \theta(-n)\left\langle -\xi \left| \frac{\mu_2 p_1 - \mu_1 p_2}{\mu_1+\mu_2}\right.\right\rangle\right].
\end{aligned} \tag{4.38}$$

The last integral in the right-hand side of (4.38) can be written in the form:

$$\begin{aligned}
&\frac{1}{(2\pi)^3}\int d\Omega_p\, \theta(-p_0)\,\delta(2p_4-2m_4)\left[\theta(n)\langle\xi|p\rangle + \theta(-n)\langle\xi|p\rangle\right] = \\
&= \frac{1}{i}\left[\theta(n)\mathcal{D}^{(-)}(\xi,0) - \theta(-n)\mathcal{D}^{(+)}(\xi,0)\right] \equiv \frac{1}{i}\mathcal{D}^{(c)}(\xi,0),
\end{aligned} \tag{4.39}$$

where $\mathcal{D}^{(\pm)}(\xi,0)$ are the positive and negative frequency parts of the commutation function (4.16) which define the normal pairings of fields in configuration space:

$$\mathcal{D}^{(-)}(\xi,0)=\frac{i}{(2\pi)^3}\int\langle\xi|p\rangle\,\theta(-p)\,\delta(2p_4-2m_4)\,d\Omega_p = \qquad (4.40a)$$

$$= i\,\underbrace{\varphi_x^{out}(\xi)\,\varphi_x^{out}}(0)\,,$$

$$\mathcal{D}^{(+)}(\xi,0)=-\mathcal{D}^{(-)}(-\xi,0)=\frac{1}{(2\pi)^3 i}\int\langle\xi|p\rangle\,\theta(p_0)\,\delta(2p_4-2m_4)\,d\Omega_p = \qquad (4.40b)$$

$$= -i\,\underbrace{\varphi_x^{out}(0)\,\varphi_x^{out}}(\xi)\,.$$

As the quantity $\mathcal{D}^c(\xi,0)$ in (4.39) is even funotion of ξ, we can use the "spherical" basis (I.33) for its calculation. As a result we shall have:

$$\mathcal{D}^c(\xi,0)=\frac{1}{(2\pi)^4}\int\langle\lambda,n,\ell,m|p\rangle\,d\Omega_p\,. \qquad (4.41)$$

$$\left\{\tilde{\theta}(\omega)*\mathcal{D}^{(-)}(\omega,\vec{p})+\left[\tilde{\theta}(\omega)*\mathcal{D}^{(-)}(\omega,\vec{p})\right]\Big|_{\omega=-\omega}\right\},$$

where $\lambda=\left\{\begin{matrix}L\\ \Lambda\end{matrix}\right.$, and with $*$ we denote the convolution operation of the function $\theta(\omega)=\frac{1}{2}\,\frac{1}{\mathrm{tg}\frac{\omega}{2}-i\varepsilon}$ and $\mathcal{D}^{(-)}(\omega,\vec{p})=\theta(-\omega)\,\delta\!\left(2\cos\omega\sqrt{1+\vec{p}^{\,2}}-2m_4\right)$ on the circle $|\omega|\le\pi$:

$$\tilde{\theta}(\omega)*\mathcal{D}^{(-)}(\omega,\vec{p})=\frac{1}{2}\int_{-\pi}^{\pi}\frac{d\omega'}{\mathrm{tg}\frac{\omega-\omega'}{2}-i\varepsilon}\,\mathcal{D}^{(-)}(\omega',\vec{p})\,d\omega'= \qquad (4.42)$$

$$=\frac{1}{2\sqrt{\vec{p}^{\,2}+m^2}}\,\frac{1}{\mathrm{tg}\frac{\Omega+\omega-i\varepsilon}{2}}\;;\quad \cos\Omega\equiv\cos\sqrt{\frac{1-m^2}{1+\vec{p}^{\,2}}}\,.$$

Now, with the help of (4.42) we finally get:

$$\mathcal{D}^{(c)}(\xi,0)=\frac{1}{(2\pi)^4}\int\langle\xi|p\rangle\,\frac{d\Omega_p}{2\cos\omega\sqrt{1+\vec{p}^{\,2}}-2m_4-i\varepsilon}= \qquad (4.43)$$

$$=\frac{1}{(2\pi)^4}\int\langle\xi|p\rangle\,\frac{d\Omega_p}{2(p_4-m_4-i\varepsilon)}\,,$$

where $\langle\xi|p\rangle$, because of the already mentioned symmetry property $\mathcal{D}^{(c)}(\xi,0)=\mathcal{D}^{(c)}(-\xi,0)$ can be considered again as the "plane wave" (I.34).

Coming back to the initial relation (4.38), we can write:

$$T_\xi\left\{\frac{1}{(2\pi)^3}\int e^{i(p_1+p_2)x}\left\langle\xi\left|\frac{\mu_2p_1-\mu_1p_2}{\mu_1+\mu_2}\right.\right\rangle d\Omega_{p_1}d\Omega_{p_2}\,\varphi(p_1,p_{14})\,\varphi(p_2,p_{24})\right\}=$$

$$=:\left\{\frac{1}{(2\pi)^3}\int e^{i(p_1+p_2)x}\left\langle\xi\left|\frac{\mu_2p_1-\mu_1p_2}{\mu_1+\mu_2}\right.\right\rangle d\Omega_{p_1}d\Omega_{p_2}\,\varphi(p_1,p_{14})\,\varphi(p_2,p_{24})\right\}:+ \qquad (4.44)$$

$$+\frac{1}{i}\,\mathcal{D}^{(c)}(\xi,0)\,.$$

For comparison let us write the corresponding "olassical" formula:

$$T_\xi\left[\varphi^{out}\!\left(x+\frac{\xi}{2}\right)\varphi^{out}\!\left(x-\frac{\xi}{2}\right)\right]=:\varphi^{out}\!\left(x+\frac{\xi}{2}\right)\varphi^{out}\!\left(x-\frac{\xi}{2}\right):+\frac{1}{i}\,\mathcal{D}^{(c)}(\xi)\,, \qquad (4.45)$$

where

$$\mathcal{D}^c(\xi)=\theta(\xi_0)\mathcal{D}^{(-)}(\xi)-\theta(-\xi_0)\mathcal{D}^{(+)}(\xi)=\frac{1}{(2\pi)^4}\int e^{ip\xi}\,\frac{d^4p}{m^2-p^2-i\varepsilon}\,. \qquad (4.46)$$

Therefore we are convinced that the quantity we obtained:

$$\mathcal{D}^{(c)}(\xi,0)=\theta(N_0)\mathcal{D}^{(-)}(\xi,0)-\theta(-N_0)\mathcal{D}^{(+)}(\xi,0)=\frac{1}{(2\pi)^4}\int\frac{\langle\xi|p\rangle\, d\Omega_p}{2(p_4-m_4-i\varepsilon)} \tag{4.47}$$

is closely analogous to the causal Green function (propagator) of the "classical" theory.

From (4.39) and (4.40) it is easy to demonstrate that:

$$T_\xi\{\varphi_x(\xi)\varphi_x(0)\}=\theta(N_0)\varphi_x(\xi)\varphi_x(0)+\theta(-N_0)\varphi_x(0)\varphi_x(\xi)= \tag{4.48}$$

$$=:\varphi_x(\xi)\varphi_x(0):+\frac{1}{i}\mathcal{D}^{(c)}(\xi,0).$$

A slight modification of the calculations which leads us to (4.39) and (4.41) gives one more formula with the T_ξ -product (see (4.17)):

$$T_\xi\{\varphi_x(\xi)\varphi_x(-\xi)\}=\theta(N_0)\varphi_x(\xi)\varphi_x(-\xi)-\theta(N_0)\varphi_x(-\xi)\varphi_x(\xi)= \tag{4.49}$$

$$=:\varphi_x(\xi)\varphi_x(-\xi):+\frac{1}{i}\mathcal{D}^{(c)}(\xi,-\xi),$$

where

$$\mathcal{D}^{(c)}(\xi,-\xi)=\frac{1}{(2\pi)^4}\int(\langle\xi|p\rangle)^2\frac{d\Omega_p}{2(p_4-m_4-i\varepsilon)}\,. \tag{4.50}$$

We would like to emphasize that the relativistic invariance

of all considered T_ξ -products is guaranteed, first by eqs.(4.20a)-(4.20b) (in the "timelike" L -region) and second by the locality condition (4.28)-(4.30) (in the space-like Λ -region).

The "integral" causality condition together with the recurrent relation for the radiation operators+)

$$\frac{\delta^{n+1}S}{\delta\varphi(-p_1,p_{14})\ldots\delta\varphi(-p_n,p_{n4})\delta\varphi(-p_{n+1},p_{n+1,4})}S^+= \tag{4.51}$$

$$=\left[\frac{\delta^n S}{\delta\varphi(-p_1,p_{14})\ldots\delta\varphi(-p_n,p_{n4})}S^+\right]\left(\overleftarrow{\frac{\delta}{\delta\varphi(-p_{n+1},p_{n+1,4})}}-ij(p_{n+1},p_{n+1,4})\right)$$

and the identities of type (3.7) may be used as a base to obtain a chain of connected integral equations for the c.f. (2.31). Doing that it is convenient to write (4.34) in p - representation:

+) Eq.(4.51) (see eq.(2.3) in [4]) is a corollary of the unitarity condition for the extended S -matrix and is obtained after n -fold variation of eq.(2.35). The arrow under the symbol $\frac{\delta}{\delta\varphi}$ shows that the operation is performed from right to left.

$$\frac{\delta^2 S}{\delta\varphi(-p_1,p_{1\nu})\,\delta\varphi(-p_2,p_{2\nu})}\,S^+ = -j(p_1,p_{14})\,j(p_2,p_{24}) +$$

$$+\frac{p_{14}+p_{24}}{\sqrt{(p_1+p_2)^2_L}}\,\frac{1}{(2\pi)^4}\int \delta^{(4)}(p_1+p_2-p_1'-p_2')\,\overline{\langle\xi|\frac{\mu_2p_1-\mu_1p_2}{\mu_1+\mu_2}\rangle}\,\theta(-N_0)\,d\Omega_\xi\,\cdot \qquad (4.52)$$

$$\cdot\langle\xi|\frac{\mu_2'p_1'-\mu_1'p_2'}{\mu_1'+\mu_2'}\rangle\,d\Omega_{p_1'}\,d\Omega_{p_2'}\,\left[j(p_1',p_{14}'),\,j(p_2',p_{24}')\right]+\cdots,$$

where μ_1, μ_2, μ_1' and μ_2' are given by eq. (I.25b), the volume element $d\Omega_\xi$ in ξ – space has the form:+)

$$d\Omega_\xi=\begin{cases}2(L+1)(L+2)(L+\frac{3}{2})\,\delta(N^2-1)\,d^4N, & \text{if } \xi=\xi_L=(L,N)\\ 2\Lambda(\Lambda^2+\frac{1}{4})\,\mathrm{th}\,\pi\Lambda\;\delta(N^2+1)\,d^4N, & \text{if } \xi=\xi_\Lambda=(\Lambda,N)\end{cases} \qquad (4.53)$$

and by definition:

$$\frac{1}{(2\pi)^4}\int\overline{\langle\xi|q'\rangle}\,d\Omega_\xi\,\langle\xi|q\rangle=\delta(q',q) \qquad (4.54)$$

+) See the footnote on p. 45.

(the integral in (4.54) denotes summation over L -series and integration over Λ -series).

Let us calculate with the help of (4.52) as example the c.f. $S_2(p_1,p_{14};p_2,p_{24})$. Taking into account (2.31) and using the completeness of the system of states (2.52), we shall have+):

$$S_2(-p_1,p_{14};-p_2,p_{24})=$$

$$=-\frac{1}{2!}\,\frac{p_{14}+p_{24}}{\sqrt{(p_1+p_2)^2_L}}\,\frac{1}{(2\pi)^4}\int\delta^4(p_1+p_2-p_1'-p_2')\,d\Omega_\xi\,d\Omega_{p_1'}\,d\Omega_{p_2'}\,\overline{\langle\xi|\frac{\mu_2p_1-\mu_1p_2}{\mu_1+\mu_2}\rangle}\cdot$$

$$\cdot\Big\{\theta(N_0)\langle\xi|\frac{\mu_2'p_1'-\mu_1'p_2'}{\mu_1'+\mu_2'}\rangle\sum_n\langle 0|j(p_1',p_{14}')|n\rangle\langle n|j(p_2',p_{24}')|0\rangle+$$

$$+\;\theta(-N_0)\langle\xi|\frac{\mu_2'p_1'-\mu_1'p_2'}{\mu_1'+\mu_2'}\rangle\sum_n\langle 0|j(p_2',p_{24}')|n\rangle\langle n|j(p_1',p_{14}')|0\rangle\Big\}, \qquad (4.55)$$

where

$$\langle 0|j(p_1',p_{14}')|n\rangle=\langle 0|j(p_1',p_{14}')|\vec{k}_1,\dots,\vec{k}_n\rangle \qquad (4.56)$$

and $\sum_n$ denotes summation and integration over all intermediate states. Then, using relations (2.40) and (2.32) it is easy to obtain (compare with [6]) that:

+) For simplicity here we do not take into account the hypothetical additive terms, denoted by dots in the causality condition (4.52).

$$\langle 0|j(p,p_4)|\vec{k}_1,\dots,\vec{k}_n\rangle = |p_4|\,\delta(p+P_{(n)})\langle 0|\int d\Omega_q\, j(q,q_4)|\vec{k}_1\dots\vec{k}_n\rangle,$$

$$\langle \vec{k}_1,\dots\vec{k}_n|j(p,p_4)|0\rangle = |p_4|\,\delta(p+P_{(n)})\langle \vec{k}_1,\dots,\vec{k}_n|\int d\Omega_q\, j(q,q_4)|0\rangle, \tag{4.57}$$

where

$$k_1+\dots+k_n = P_{(n)}\,. \tag{4.58}$$

Since p_μ is a 4 -vector in De-Sitter space (2.3), its 4 -square is obligatory bounded: $p^2\leq 1$. Therefore, owing to (4.57) and (4.58) only such intermediate states contribute to the right-hand side of (4.55), which invariant mass obeys analogous limitation:

$$P_{(n)}^2 \leq 1\,. \tag{4.59}$$

Substituting the matrix elements (4.57) in (4.56), taking into account (2.36), the spectral condition, the requirement (4.59) and integrating over p_1' and p_2' with the help of the δ -functions we get:

$$S_2(-p_1,p_{14};-p_2,p_{24}) = -\frac{1}{2!}\frac{\delta(p_1+p_2)}{(2\pi)^4}\int \overline{\langle\xi|p_1\rangle}\, d\Omega_\xi$$
$$\Big\{\theta(N_0)\langle\xi|-p\rangle\,\theta(1-p^2)\,\theta(p_0)\,\theta(p^2) + \tag{4.60}$$
$$+\theta(-N_0)\langle\xi|p\rangle\,\theta(1-p^2)\,\theta(p_0)\,\theta(p^2)\Big\}\varrho(p^2)\,d^4p,$$

where the spectral function is introduced in a standard way:

$$\varrho(p^2) = \sum{}' \Big|\int\langle 0|\, j(q,q_4)\,d\Omega_q|\vec{k}_1,\dots\vec{k}_n\rangle\Big|^2 \tag{4.61}$$

(the summation in this formula is performed only over intermediate states with total 4-momentum p).

If we notice now that:

$$\theta(1-p^2)\,\theta(p^2)\,d^4p = \int_0^1 d\mu^2\,\delta\big(2\sqrt{1-p^2}-2\mu_4\big)\frac{d^4p}{\sqrt{1-p^2}} =$$
$$= \int_0^1 d\mu\,\delta(2p_4-2\mu_4)\,d\Omega_p\,, \quad \Big(\mu_4 = \sqrt{1-\mu^2}\Big),$$

and using (4.47) and (4.54) we can write equation (4.60) in the form:

$$S_2(-p_1,p_{14};-p_2,p_{24}) = \frac{-1}{2!}\frac{\delta(p_1+p_2)}{2\pi i}\int_0^1 d\mu^2\varrho(\mu^2)\,\overline{\langle\xi|p_1\rangle}\,d\Omega_\xi\,\cdot$$
$$\cdot\frac{1}{(2\pi)^4}\int\frac{\langle\xi|p\rangle\,d\Omega_p}{2(p_4-\mu_4-i\varepsilon)} = -\frac{1}{2!}\frac{\delta(p_1+p_2)}{2\pi i}\int_0^1\varrho(\mu^2)\frac{d\mu^2}{2(p_{14}-\mu_4-i\varepsilon)} \tag{4.62}$$

Hence we proved, that in the new scheme there exist an analogue of the Kälen-Lehmannn spectral representation for the o.f. S_n . The most interesting feature of the new representation is the cut-off of the spectral integral or the finiteness of the spectral function. As far as in the "classical" theory the integration over μ^2 is carried up to infinity and usually the integral is divergent on its upper limit+), the result we obtained should be considered as an indication for a possible softening of the difficulties connected with the ultraviolet divergencies in the present approach (see also Section 5.).

5. The Problem of Generalized Singular Function Product in the New Scheme.

It was mentioned in Section 2 that one of the goals of our approach is to obtain a satisfactory solution of the problem of multiplying generalized singular functions with coinciding singularities. We shall report some of the results obtained in attempts to investigate this problem. A detailed discussion of this question will be given in a separate paper.

Let us begin with one-dimensional example from the usual theory – the product of the step function $\theta(\xi^\circ)$ and $\delta(\xi^\circ)$ which is a good illustration of the discussed difficulties [3] .

As far as:

+) As it is well known this is directly connected with the ultraviolet divergencies.

$$\theta(\xi^\circ)=\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{e^{iE\xi^\circ}}{E-i\varepsilon}\,dE, \tag{5.1}$$

then formally we have

$$\theta(\xi^\circ)\delta(\xi^\circ)=\left\{\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{dE}{E-i\varepsilon}\right\}\delta(\xi^\circ)=\infty\;\delta(\xi^\circ). \tag{5.2}$$

A more rigorous approach based on generalized function theory arguments gives:

$$\theta(\xi^\circ)\delta(\xi^\circ)=c\,\delta(\xi^\circ), \tag{5.3}$$

where c is an arbitrary constant.

The analogue of $\theta(\xi^\circ)\,\delta(\xi^\circ)$ in the new scheme is the expression $\theta(n)\,\delta_{n,o}$ where $\theta(n)$ is the step function (4.35) with Fourier decomposition (4.37), and $\delta_{n,m}$ is the Kronecker symbol. Therefore

$$\theta(n)\delta_{n,o}=\left\{\frac{1}{4\pi i}\int_{-\pi}^{\pi}\frac{d\omega}{tg\frac{\omega}{2}-i\varepsilon}\right\}\delta_{n,o}=$$

$$=\left\{\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{dE}{E-i\varepsilon}\,\frac{1}{1+E^2}\right\}\delta_{n,o}=\frac{1}{2}\delta_{n,o} \tag{5.4}$$

($tg\frac{\omega}{2}\equiv E$ is the new integration variable).

The conclusion which can be drawn by considered example is that the functions $\theta(n)$ and $\delta_{n,0}$ are, contrary to their continuous analogues, ordinary (not generalized) functions and their product is defined uniquely.

It turns out that a similar situation holds in more general case. For instance the commutator (4.16) for zero mass particles is given by the expression:

$$\mathcal{D}(\xi,0)\Big|_{m=0} = \frac{1}{2\pi}\,\varepsilon(n)\,\frac{1}{L+2}\,\delta_{L,-1}, \qquad |n| \geqslant L+3, \qquad L=-1,0,1,\dots, \tag{5.5}$$

where

$$\varepsilon(n) = \theta(n)-\theta(-n) = \begin{cases} 1, & \text{if } n>0 \\ -1, & \text{if } n<0 \end{cases}. \tag{5.6}$$

In the "classical" case we could have correspondingly+):

$$\mathcal{D}(\xi)\Big|_{m=0} = \frac{1}{2\pi}\,\varepsilon(\xi^0)\,\delta(\xi^2) \tag{5.7}$$

+)The condition $m=0$ picks out the most singular part of the "classical" $\mathcal{D}$ -function (4.18).

A comparison of formulae (5.5) and (5.7) demonstrates that the first one has a completely well defined mathematical sense and can be interpreted as an ordinary product of ordinary functions+) and in the same time the second formula is a typical for the "classical" field theory example of multiplying of singular generalized functions with coinciding singularities.

It should be clearly understood that the appearance in our formalism of discrete (quantized) variables n and L is directly connected with the boundedness of the new p -space in timelike direction in the sense of De-Sitter metrics. Owing to the same reason the "plane waves" (I.34a), corresponding to the timelike L -series are square integrable functions (see the footnote on p. 79). The last circumstance will play an important role in the example++), which we consider below.

+)A similar statement is true for the function $\mathcal{D}(\xi,0)\Big|_{m\neq 0}$, eq.(4.16), for all above considered commutation functions and propagators and also for arbitrary powers and products of these quantities.

++)In the present report this example has only methodical meaning.

Let

$$j_x(\xi) = :\varphi_x^{out}(\xi)^n: \tag{5.8}$$

is a "bilocal" operator, constructed of the fields (4.4). From (4.29) it is obvious that:

$$[j_x(\xi), j_x(-\xi)] = 0 \quad , \quad \text{if } \xi = \xi_\Lambda \text{ and } \tag{5.9}$$

x arbitrary .

It is clear also that:

$$\langle 0|[j_x(\xi), j_x(-\xi)]|0\rangle = \langle 0|[j(\xi), j(-\xi)]|0\rangle , \tag{5.10}$$

where

$$j(\xi) = j_0(\xi) = :\varphi^n(\xi): .$$

Now let us consider the integral:

$$g(p) =$$

$$= i\int \langle\xi|p\rangle\theta(N_0)\langle 0|[j(\xi), j(-\xi)]|0\rangle\langle\xi|p\rangle\, d\Omega_\xi , \tag{5.11}$$

where $d\Omega_\xi$ is defined in (4.53).

In the "classical" limit this quantity coincides up to a constant factor with the real part of the one particle propagator, calculated in second order of the perturbation theory, in a model with $:\varphi^{n+1}(x):$ interaction $(n \geqslant 2)$. As it is well known in this case

the correspondent integral is divergent. The reason is that the product of generalized functions $\theta(\xi^0)$ and

$$\langle 0|[j(\xi), j(-\xi)]|0\rangle$$

is not integrable because of coincidence of their singularities in the point $\xi = 0$.

Let us investigate the convergence of the integral (5.11). Notice first that owing to the locality condition (5.9) the continuous Λ -series do not contribute to the integral:

$$g(p) = i\sum_{L=-1}^{\infty}\int 2(L+1)(L+2)(L+\tfrac{3}{2})\delta(N^2-1)d^4N \cdot \tag{5.12}$$

$$\cdot \langle\xi_L|p\rangle\theta(N_0)\langle 0|[j(\xi_L), j(-\xi_L)]|0\rangle\langle\xi_L|p\rangle .$$

Using the Wick's theorem we easily obtain:

$$\langle 0|[j(\xi_L), j(-\xi_L)]|0\rangle = \langle 0|[\varphi^{out\,n}(\xi_L), \varphi^{out\,n}(-\xi_L)]|0\rangle =$$

$$= n!\left(\frac{1}{i}\right)^n\left\{\left(\mathcal{D}^{(-)}(\xi_L, -\xi_L)\right)^n - \left(\mathcal{D}^{(-)}(-\xi_L, \xi)\right)^n\right\} , \tag{5.13}$$

where

$$\frac{1}{i}\mathcal{D}^{(-)}(\xi, -\xi) = \underbrace{\varphi^{out}(\xi)\,\varphi^{out}}(-\xi) = \tag{5.14}$$

$$= \frac{1}{(2\pi)^3}\int (\langle\xi|k\rangle)^2\theta(-k_0)\delta(2k_4 - 2m_4)\, d\Omega_k ,$$

and (cf. (4.17))

$$\mathcal{D}^{(-)}(\xi,-\xi)-\mathcal{D}^{(-)}(-\xi,\xi)=\mathcal{D}(\xi,-\xi). \qquad (5.15)$$

Let us now estimate the modulus of the function (5.12), assuming for definiteness that $p^2>0$ and making obvious majorizations in the integrand (compare with (II.2)):

$$|g(p)|\leq$$

$$\leq\sum_{L=-1}^{\infty}(L+1)(L+2)(L+\tfrac{3}{2})|\langle 0|[j(\xi_L),j(-\xi_L)]|0\rangle|\int\frac{d\vec{N}}{N_0}(p_4^2+(pN)^2)^{-L-3}. \qquad (5.16)$$

The integral over $d\vec{N}$ can be calculated explicitly:

$$\int\frac{d\vec{N}}{N_0}(p_4^2+(p\cdot N)^2)^{-L-3}\equiv-\frac{4\pi}{p^2}\left[J_{L+3}(p^2)-J_{L+2}(p^2)\right]\equiv$$

$$\equiv-\frac{4\pi}{p^2}\Delta_L J_{L+2}(p^2), \qquad (5.17)$$

where [17]

$$J_{L+2}(p^2)=\int_0^{\infty}\frac{dX}{(p_4^2+p^2ch^2X)^{L+2}}= \qquad (5.18)$$

$$=\frac{\sqrt{\pi}}{2}\frac{\Gamma(L+2)}{\Gamma(L+\frac{5}{2})}\,{}_2F_1\left(L+2,\frac{1}{2};L+\frac{5}{2};p_4^2\right)$$

and Δ_L is the finite difference symbol. It is easy to see that expression (5.17) as a function of L has no singularities for $L=-1,0,1,\ldots$. In the region $L\gg 1$

$$|J_{L+2}(p^2)|\leq\frac{\sqrt{\pi}}{2}L^{-1/2}(p^2)^{-1/2}. \qquad (5.19)$$

Taking into account (II.7) and (5.17) we obtain from (5.16) the following inequality:

$$|g(p)|\leq\frac{C}{p^2}\sum_{L=-1}^{\infty}(L+1)(L+2)(L+\tfrac{3}{2})\,|\Delta_L J_{L+2}(p^2)|\cdot$$

$$\cdot\left(\frac{\Gamma(L+2)}{\Gamma(L+\frac{3}{2})}\right)^n\left[{}_2F_1\left(L+3,\frac{3}{2};L+\frac{7}{2};m_4^2\right)\right]^n, \qquad (5.20)$$

where C collects all constant factors. It is easy to see that series (5.20) are absolutely convergent. Indeed, let us choose sufficiently large number $\tilde{L}$, such that when $L>\tilde{L}$ we can substitute the expression under the sign of summation in (5.20) by its asymptotic value. Then, introducing the notation $M(p^2,\tilde{L})$ for the <u>finite sum</u> from $L=-1$ to $L=\tilde{L}$ and taking into account (5.19) and (II.8) we obtain:

$$|g(p)|\leq\frac{C}{p^2}M(p^2,\tilde{L})+\frac{const.}{(p^2)^{3/2}}\sum_{L>\tilde{L}}^{\infty}L^{-3(\frac{n}{2}-1)}|\Delta_L L^{-1/2}|. \qquad (5.21)$$

The infinite number series in the right-hand side of (5.21) are convergent for $n \geqslant 2$ because:

$$\int_{\tilde{L}}^{\infty} dx\, x^{-3\left(\frac{n}{2}-1\right)} \left|\frac{d}{dx}\left(\frac{1}{\sqrt{x}}\right)\right| = \qquad (5.22)$$

$$= \frac{1}{2}\int_{\tilde{L}}^{\infty} x^{-\frac{3}{2}(n-1)} dx < \infty$$

(Cauchy criterion). Therefore, the function $|g(p)|$ is bounded and the initial integral (5.11) is absolutely convergent.

The considered examples testify, apparently, that the constant curvature p -space extension of the S - matrix off the mass shell is really "less singular" than the "classical" extension, using flat Minkowski momentum space (compare Section 2). It can be expected also that in the causality condition (4.34) the number of hypothetical additive terms, denoted by dots in the right-hand side, will be reduced to a minimum+). A great success of the theory would be a unique, selfconsistent determination of the mentioned terms because this would allow to answer the question: which kind of interactions among the quantized fields is realized in the Nature?

+) This additions cannot disappear completely, for this would imply a trivial unit S -matrix [4] .

6. Conclusion

In this Section we would like to formulate a program for future investigations in the framework of the approach we proposed here.

1) A detailed analysis of the generalized causality condition (4.34) (including the question of the arbitrary additional terms) should be carried and on its base the macrocausality condition on the scattering matrix should be obtained.

2) Construction of perturbation theory and developing appropriate diagram techniques.

3) Three-dimensional formulation of the two-body problem in the spirit of the quasipotential approach [18,19,20] and development on its base a phenomenological theory of interaction of hadrons with De Broglie wave lengths $\lesssim l_0$.

4) Obtaining of different qualitative predictions in the given scheme, based on the fact, that the 4-momentum of arbitrary virtual particle obeys the restriction $p^2 \leq 1$. In particular this is related to the one-photon pair (e^+e^-) annihilation, lepton pair production, deep inelastic processes, etc.

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Appendix I.

Some Relations Used in the Constant Curvature p-Space Formalism

1) The hypersphere equation (2.3) in units $\hbar = c = \ell_0 = 1$:

$$p_0^2 - p_1^2 - p_2^2 - p_3^2 + p_4^2 = 1 \tag{I.1}$$

In five-dimensional form:

$$g^{LM} p_L p_M = 1 \tag{I.2}$$

$$(g^{00} = g^{44} = -g^{11} = -g^{22} = -g^{33} = 1, \; g^{LM} = 0 \text{ for } L \neq M),$$

or simply

$$p_L^2 = 1 \, . \tag{I.3}$$

The line element:

$$ds^2 = dp^2 + \frac{(p\,dp)^2}{1-p^2} \tag{I.4}$$

$$(dp^2 = dp_0 - d\vec{p}^{\,2} \, , \; p\,dp = p_0\,dp_0 - \vec{p}\cdot d\vec{p}) \, .$$

The volume element:

$$d\Omega_p = 2\,\delta(p_L^2 - 1)\,d^5p \, . \tag{I.5}$$

2) The motion group+) (De-Sitter group $SO(2,3)$):

$$p_L' = \Lambda_L^M p_M \qquad (L, M = 0,1,2,3,4)$$

$$g^{LN} = g^{MK} \Lambda_M^L \Lambda_K^N \tag{I.6}$$

Lorentz transformations (5 -rotations around the p_4-axis):

$$p_4' = p_4$$
$$p_\mu' = \Lambda_\mu{}^\nu p_\nu \quad (\mu,\nu = 0,1,2,3). \qquad \text{(I.7)}$$

"Translations" to a 4 -vector b_μ (5 -rotations in the plane $(\mu 4)$):

$$p_\mu' = \Lambda_\mu{}^\nu(b)\, p_\nu + \Lambda_\mu{}^4(b)\, p_4$$
$$p_4' = \Lambda_4{}^\nu(b)\, p_\nu + \Lambda_4{}^4(b)\, p_4 . \qquad \text{(I.8)}$$

In explicit form:

$$p_\mu' \equiv (p(+)b)_\mu = p_\mu + b_\mu \left(p_4 - \frac{p.b}{1+b_4}\right),$$
$$p_4' \equiv (p(+)b)_4 = -\, p.b + p_4 b_4 . \qquad \text{(I.9)}$$

By definition:

$$p(-)b \equiv p(+)(-b).$$

Obviously,

$$d\Omega_p = d\Omega_{p(+)b} . \qquad \text{(I.10)}$$

Properties of the "translation" operation (I.8)-(I.9):

$$p(\pm)0 = p ,$$
$$p(-)p = 0 . \qquad \text{(I.11)}$$

$\Lambda(b_1)\Lambda(b_2)\Lambda^{-1}(b_1(+)b_2)$=Lorentz rotation.

+) The representations of reflections are not considered.

3) "Spherical" coordinates $(\omega,\chi,\theta,\varphi)$:

$$p_0 = \sin\omega\, \mathrm{ch}\chi$$
$$p_4 = \cos\omega\, \mathrm{ch}\chi$$
$$p_1 = \mathrm{sh}\chi \sin\theta\cos\varphi ,\ p_2 = \mathrm{sh}\chi\sin\theta\sin\varphi ,\ p_3 = \mathrm{sh}\chi\cos\theta \qquad \text{(I.12)}$$

$$(|\omega| \le \pi,\ 0 \le \chi < \infty,\ 0 \le \theta \le \pi,\ 0 \le \varphi \le 2\pi)$$

"Semispherical" coordinates $(\omega, \vec{p})$:

$$p_0 = \sqrt{1+\vec{p}^{\,2}}\ \sin\omega$$
$$p_4 = \sqrt{1+\vec{p}^{\,2}}\ \cos\omega \qquad \text{(I.13)}$$
$$\vec{p} = \vec{p}$$
$$(|\omega| \le \pi ,\ 0 \le p < \infty).$$

The volume element in coordinates (I.13):

$$d\Omega_p = d\omega\, d\vec{p} . \qquad \text{(I.14)}$$

Group-theoretical sense of the coordinate (I.13):

$$p = q^{\perp} (+) k^{\parallel} , \qquad \text{(I.15)}$$

where

$$p_L = (p_0, \vec{p}, p_4) = (\sqrt{1+\vec{p}^{\,2}}\sin\omega , \vec{p}, \sqrt{1+\vec{p}^{\,2}}\cos\omega)$$
$$q_L^{\perp} = (0, \vec{p}, \sqrt{1+\vec{p}^{\,2}}) \qquad \text{(I.16)}$$
$$k_L^{\parallel} = (\sin\omega , \vec{0}, \cos\omega)$$

4) The function $\delta(p',p)$:

$$f(p) = \int f(p)\,\delta(p',p)\, d\Omega_p , \qquad \text{(I.17)}$$
$$\delta(p',p) = |p_4|\,\theta(p_4' p_4)\,\delta^{(4)}(p'-p) = \delta(\omega'-\omega)\,\delta^{(3)}(\vec{p}\,' - \vec{p})$$
$$\delta(p'(+)k , p(+)k) = \delta(p',p) \qquad \text{(I.18)}$$

5) Transition to the variables $q_i\ (i=1,\dots,n)$ and $U^{(p_1,\dots,p_n)}$:

$$
\begin{aligned}
p_1 &= q_1(+)U^{(p_1\cdots p_n)}\\
p_2 &= q_2(+)U^{(p_1\cdots p_n)}\\
&\dots\dots\dots\dots\\
p_n &= q_n(+)U^{(p_1\cdots p_n)},
\end{aligned}
\qquad (I.19)
$$

where

$$
(q_1+q_2+\dots+q_n)_L = \left((q_1+\dots+q_n)_\lambda,\ q_{14}+\dots+q_{n4}\right) =
= \left(0_\lambda,\ \sqrt{(p_1+\dots+p_n)^2_M}\right), \qquad (I.20a)
$$

$$
(p_1+\dots+p_n)^2_M = n + \sum_{\substack{i,j=1\\ i\neq j}}^{n} \sqrt{1-(p_i(-)p_j)^2}, \qquad (I.20b)
$$

$$
\left(\frac{(p_1+\dots+p_n)_\lambda}{\sqrt{(p_1+\dots+p_n)^2_M}},\ \frac{p_{14}+\dots+p_{n4}}{\sqrt{(p_1+\dots+p_n)^2_M}}\right) \equiv (U_\lambda, U_4) = U_L^{(p_1\cdots p_n)}, \qquad (I.21a)
$$

$$
\left(U_L^{(p_1\cdots p_n)}\right)^2 = U^2 + U_4^2 = 1 \qquad (I.21b)
$$

and

$$
d\Omega_{p_1}\dots d\Omega_{p_n} =
= \delta\left(U^{(q_1\cdots q_n)},0\right) d\Omega_{q_1}\dots d\Omega_{q_n}\, d\Omega_{U^{(p_1\cdots p_n)}} \qquad (I.22)
$$

If:

$$
\begin{aligned}
p_1(+)b &= p_1'\\
&\dots\dots\\
p_n(+)b &= p_n',
\end{aligned}
\qquad (I.23)
$$

then

$$
U^{(p_1'\cdots p_n')} = U^{(p_1\cdots p_n)}(+)b. \qquad (I.24)
$$

In the case of $n=2$:

$$
q_1 = p_1(-)U^{(p_1p_2)} = \frac{\mu_2 p_1 - \mu_1 p_2}{\mu_1+\mu_2} \qquad (I.25a)
$$

$$
q_2 = p_2(-)U^{(p_1p_2)} = \frac{\mu_1 p_2 - \mu_2 p_1}{\mu_1+\mu_2},
$$

where

$$
\begin{aligned}
\mu_1 &= \frac{1}{2}\left(p_{14} + \frac{1}{2}\sqrt{(p_1+p_2)^2_M}\right)\\
\mu_2 &= \frac{1}{2}\left(p_{24} + \frac{1}{2}\sqrt{(p_1+p_2)^2_M}\right).
\end{aligned}
\qquad (I.25b)
$$

The eq.(I.22) when $n=2$:

$$
d\Omega_{p_1}d\Omega_{p_2} = \delta\left(U^{(q_1q_2)},0\right) d\Omega_{q_1}d\Omega_{q_2}\, d\Omega_{U^{(p_1p_2)}} =
= d\Omega_q \frac{d^4P}{\sqrt{1-q^2-P^2/4}}\ \theta\left(1-q^2-\frac{P^2}{4}\right), \qquad (I.25c)
$$

where $P = p_1+p_2$, $q=q_1$ (see (I.25a)). The appearance of $\theta(1-q^2-\frac{P^2}{4})$ is connected with the condition $\left(U_4^{(p_1p_2)}\right)^2 = 1-\left(U^{(p_1p_2)}\right)^2 \geq 0$.

6) Generators of the group $SO(2,3)$:

$$
M^{KL} = -M^{LK} = \left(M^{\varkappa\lambda}, M^{\lambda 4}\right)
$$

$$
(K,L = 0,1,2,3,4\ ;\ \varkappa,\lambda = 0,1,2,3). \qquad (I.26)
$$

$$M^{\varkappa\lambda} = i\left(p^{\varkappa}\frac{\partial}{\partial p_\lambda} - p^{\lambda}\frac{\partial}{\partial p_\varkappa}\right)$$

$$M^{\lambda 4} = i p_4 \frac{\partial}{\partial p_\lambda} \equiv -\xi^{\lambda}$$ (Snyder's 4 -coordinate)

We have also:

$$[\xi^{\varkappa}, \xi^{\lambda}] = i M^{\varkappa\lambda}. \tag{I.27}$$

The time operator ξ^0 in "semispherical" coordinates (I.13):

$$\xi^0 = -i\frac{\partial}{\partial\omega} \tag{I.28}$$

The eigenfunctions of the operator ξ^0, periodical on the segment $-\pi \leq \omega \leq \pi$:

$$\langle n|\omega\rangle = e^{in\omega}, \quad n = 0, \pm 1, \pm 2, \dots \tag{I.29a}$$

This corresponds to the spectrum:

$$\xi^0 = n \tag{I.29b}$$

7) Casimir's operator:

$$\frac{1}{2}M^{KL}M_{KL} = \frac{1}{2}M^{\varkappa\lambda}M_{\varkappa\lambda} + \xi^2 = \tag{I.30}$$

$$= -\frac{1}{\sqrt{|g|}}\frac{\partial}{\partial p^{\mu}}\left(g^{-1}_{\mu\nu}\sqrt{|g|}\,\frac{\partial}{\partial p_\nu}\right),$$

where $\|g_{\mu\nu}\|$ is the metric tensor, calculated with

the help of (I.4) ; $g = \det\|g_{\mu\nu}\|$

The eigenvalue problem:

$$-\frac{1}{\sqrt{|g|}}\frac{\partial}{\partial p_\mu}\left(g^{-1}_{\mu\nu}\sqrt{|g|}\,\frac{\partial}{\partial p_\nu}\right)\langle\lambda,\dots|p\rangle = \lambda\langle\lambda,\dots|p\rangle, \tag{I.31}$$

where the dots denote the variables which together with λ form the correspondent complete set of observables.

For the unitary representations of the group $SO(2,3)$ from the maximal degenerate series, the spectrum of λ is of the form:

$$\lambda = \begin{cases} L(L+3) & L = -1, 0, 1, \dots \quad \text{(discrete series)}^{+)} \quad \text{(I.32a)} \\ -\left(\frac{3}{2}\right)^2 - \Lambda^2 & 0 \leq \Lambda < \infty \quad \text{(continuous series)}^{+)} \quad \text{(I.32b)} \end{cases}$$

8) The eigenfunctions, corresponding to (I.32)[++)].

<u>Spherical basis (see I.12)</u>:

+) For brevity the discrete series we shall call L -series and the continuous correspondingly Λ -series.

++) We shall investigate the properties of these functions in a separate paper. Here we would like just to mention that the eigenfunctions in the L -spectrum are square integrable (I.5).

L -series:

$$\langle L,n,\ell,m|p\rangle = \langle L,n,\ell,m|\omega,\chi,\theta,\varphi\rangle =$$

$$= e^{in\omega} Y_{\ell m}(\theta,\varphi)(\mathrm{th}\chi)^{\ell}(\mathrm{ch}\chi)^{-L-3}\,{}_2F_1\left(\frac{L+3+\ell+n}{2},\frac{L+3+\ell-n}{2};\ell+\frac{3}{2};\mathrm{th}^2\chi\right) \qquad \text{(I.33a)}$$

with

$$|n| \geqslant L+3. \qquad \text{(I.33b)}$$

Λ -series:

$$\langle \Lambda,n,\ell,m|p\rangle = e^{in\omega}\, Y_{\ell m}(\theta,\varphi)(\mathrm{th}\chi)^{\ell}(\mathrm{ch}\chi)^{i\Lambda-3/2}\cdot {}_2F_1\left(\frac{-i\Lambda+\frac{3}{2}+\ell+n}{2},\frac{-i\Lambda+\frac{3}{2}+\ell-n}{2};\ell+\frac{3}{2};\mathrm{th}^2\chi\right). \qquad \text{(I.33c)}$$

<u>"Plane waves"</u>

L -series:

$$\langle L,N|p\rangle = (p_4 - i\,p\cdot N)^{-L-3}$$
$$N=(N_0,\vec{N});\; N^2 = N_0^2-\vec{N}^2 = 1 \qquad \text{(I.34a)}$$

Λ -series:

$$\langle \Lambda,N|p\rangle = (p_4 + p\cdot N)^{-\frac{3}{2}+i\Lambda}$$
$$N=(N_0,\vec{N});\; N^2 = N_0^2-\vec{N}^2 = -1 \qquad \text{(I.34b)}$$

Functions (I.34) satisfy the following differential-difference equations in the variables (L,N) and (Λ,N) :

L -series:

$$2(K_L - p_4)\langle L,N|p\rangle = 0,$$

$$K_L = 2\,\mathrm{ch}\frac{\partial}{\partial L} + \frac{3}{L+\frac{3}{2}}\,\mathrm{sh}\frac{\partial}{\partial L} - \frac{e^{-\frac{\partial}{\partial L}}}{(L+\frac{3}{2})(L+2)}\,\Delta_{(N)}, \qquad \text{(I.35)}$$

where $\Delta_{(N)}$ is the Laplace operator on the hyperboloid and $N_0^2-\vec{N}^2 = 1$.

Λ -series:

$$2(K_\Lambda - p_4)\langle \Lambda,N|p\rangle = 0, \qquad \text{(I.36)}$$

$$K_\Lambda = 2\,\mathrm{ch}\,i\frac{\partial}{\partial \Lambda} - \frac{3}{i\Lambda}\,\mathrm{sh}\,i\frac{\partial}{\partial \Lambda} - \frac{e^{-i\frac{\partial}{\partial\Lambda}}}{i\Lambda(i\Lambda-\frac{1}{2})}\,\Delta_{(N)},$$

where $\Delta_{(N)}$ is the Laplace operator on the hyperboloid and $N_0^2-\vec{N}^2=-1$.

In the "classical limit"

$$\langle L,N|p\rangle \longrightarrow e^{i\sqrt{\xi^2}\,N\cdot p} = e^{i\xi\cdot p} \quad (\xi_\mu = \sqrt{\xi^2}\,N_\mu) \qquad \text{(I.37)}$$

$$\langle \Lambda,N|p\rangle \longrightarrow e^{i\sqrt{-\xi^2}\,N\cdot p} = e^{i\xi\cdot p} \quad (\xi_\mu = \sqrt{-\xi^2}\,N_\mu). \qquad \text{(I.38)}$$

Appendix II

Absolute Value Estimate of the $\mathcal{D}^{(-)}$- Function

Consider the function (see (5.14)):

$$\mathcal{D}^{(-)}(\xi,-\xi)=\frac{i}{(2\pi)^3}\int(\langle\xi|k\rangle)^2\theta(-k^0)\delta(2k_4-2m_4)d\Omega_k \qquad \text{(II.1)}$$

and let us estimate its modulus in the timelike region $\xi=\xi_L$. As:

$$|\langle\xi_L|k\rangle|^2=\left(k_4^2+(k.N)^2\right)^{-L-3}$$

$$\left(N^2=N_0^2-\vec{N}^2=1\right) \qquad \text{(II.2)}$$

(see (I.34a)), then

$$|\mathcal{D}^{(-)}(\xi_L,-\xi_L)|\leq$$

$$\leq\frac{1}{(2\pi)^3}\int\left(k_4^2+(k.N)^2\right)^{-L-3}\theta(-k_0)\delta(2k_4-2m_4)\,d\Omega_k. \qquad \text{(II.3)}$$

Taking into account the relativistic invariance of this inequality we can write the right-hand side in the form [17]:

$$\frac{1}{(2\pi)^3}\int\left(k_4^2+k_0^2\right)^{-L-3}\theta(-k_0)\delta(2k_4-2m_4)d\Omega_k= \qquad \text{(II.4)}$$

$$=\frac{1}{(2\pi)^3}\int\frac{d\vec{k}}{2\sqrt{m_4^2+\vec{k}^2}\,(1+\vec{k}^2)^{L+3}}=\frac{1}{(2\pi)^2}\int_0^\infty\frac{\vec{k}^2d|\vec{k}|}{\sqrt{m_4^2+\vec{k}^2}\,(1+\vec{k}^2)^{L+3}}=$$

$$=\frac{1}{(2\pi)^2}\frac{m^2}{2}B\left(\tfrac{3}{2},L+2\right){}_2F_1\left(L+3,\tfrac{3}{2};L+\tfrac{7}{2};m_4^2\right).$$

Therefore:

$$|\mathcal{D}^{(-)}(\xi_L,-\xi_L)|\leq$$

$$\leq\frac{m^2}{16\pi^{3/2}}\frac{\Gamma(L+2)}{\Gamma(L+\frac{7}{2})}\,{}_2F_1\left(L+3,\tfrac{3}{2};L+\tfrac{7}{2};m_4^2\right). \qquad \text{(II.5)}$$

It is essential that for all possible values of $L=-1,0,1,2,\ldots$ the right-hand side of (II.5) has no singularities.

When $L\gg1$ the inequality (II.5) is simplified:

$$|\mathcal{D}^{(-)}(\xi_L,-\xi_L)|\leq \mathrm{const.}\,L^{-3/2}. \qquad \text{(II.6)}$$

With the help of (II.5) we get the following estimate for the matrix element (5.13):

$$|\langle0|[j(\xi_L),j(-\xi_L)]|0\rangle|\leq n!\,2\,|\mathcal{D}^{(-)}(\xi_L,-\xi_L)|^n\leq$$

$$\leq\frac{n!\,m^{2n}}{2^{4n-1}\pi^{\frac{3n}{2}}}\left(\frac{\Gamma(L+2)}{\Gamma(L+\frac{7}{2})}\right)^n\left[{}_2F_1\left(L+3,\frac{3}{2};L+\frac{7}{2};m_4^2\right)\right]^n. \qquad \text{(II.7)}$$

When $L\gg1$ we get:

$$|\langle0|[j(\xi_L),j(-\xi_L)]|0\rangle|\leq\mathrm{const.}\,L^{\frac{3n}{2}} \qquad \text{(II.8)}$$

+) Let us notice that the estimate (II.6) is valid, as it should be from the correspondence principle, also for the "classical" $\mathcal{D}^{(-)}$ -function [1] .

References

[1] Bogolubov, N.N., Shirkov, D.V., Introduction in Quantum Field Theory, Moscow, Gostechizdat, 1957.

[2] Bogolubov, N.N., Medvedev, B.V., Polivanov, M.K., Problems in the Theory of Dispersion Relations, Moscow, Fizmatgiz, 1958.

[3] Bogolubov, N.N., Logunov., A.A. Todorov, I.T., Axiomatic Approach to Quantum Field Theory, Mosoow, Nauka, 1969.

[4] Medvedev, B.V., Pavlov, V.P., Polivanov, M.K., Sukhanov, A.D., Theoretical and Mathematioal Physics, 13, 3, 1972.

[5] Kadyshevsky, V.G., JINR, P2-5717, Dubna, 1971.

[6] Kadyshevsky, V.G., Quantum Field Theory and Momentum Space of Constant Curvature, in Problems of Theoretical Physics, dedioated to the memory of I.E. Tamm, Moscow, Nauka, 1972.

[7] Blokhintsev, D.I., Spaoe and Time in Microworld, Moscow, Nauka, 1970.

[8] Snyder, H., Phys.Rev. 71, 38, 1947; 72, 68, 1947.

[9] Yang , C.N., Phys.Rev. 72, 874, 1947.

[10] Gol'fand, Yu.A., JETP, 37, 504, 1959; 43, 256, 1962; 44, 1248, 1963.

[11] Kadyshevsky, V.G., JETP 41, 1885, 1961; Sov,Ac.Sc. Doklady 147, 588, 1336, 1962.

[12] Mir-Kasimov, R.M., JETP 49, 905,1161, 1965; 52 533, 1967.

[13] Kirzhnits, D.A., Chechin, V.A., Soviet Nucl.Phys. 7, 431, 1968.

[14] Kadyshevsky, V.G., Mir-Kasimov, R.M., Skachkov N.B., Nuovo Cimento, 55A, 233, 1968.

[15] Cicogna, G., Nuovo Cimento, 42A, 656, 1966.

[16] Limic, N., Niederle, J., Raczka, R.; J.Math.Phys., 8, 1079, 1967.

[17] Gradshtein, I.S., Ryzhik, I.M., Tables of Integrals, Summs, Series and Products, Moscow, Fizmatgiz, 1962.

[18] Logunov, A.A., Tavkhelidze, A.N., Nuovo Cimento 29, 380, 1963.

[19] Kadyshevsky, V.G., Nuclear Physics, B6, 125, (1968).

[20] Kadyshevsky V.G.and Tavkhelidze A.N., in "Problems of Theoretical Physics", essays dedicated to the 60th birthday of N.N.Bogolubov, Moscow, Nauka, 1969.

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