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A CLASS OF S-MATRIX PROBLEMS WITH A WEAKENED CONDITION OF ANALYTICITY

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1. Introduction

In this paper we shall discuss the question of the existence and uniqueness of the solutions of a certain problem from S -matrix theory. The problem is to find N functions $h^a(z)$, $a = 1,2 \dots N$ (the partial scattering amplitudes) of the complex variable $z = x + iy$ which satisfy the following conditions: a) real analyticity in some sub-region of the plane b) unitarity; c) crossing symmetry; d) $h^{a*}(z) = h^a(z^*)$; e) a condition of behaviour at infinity. This problem a), b), c), d), e) which will be formulated more precisely below, is called Low's problem in this paper. It is a generalization of the problem solved by means of the integral equations of Low [1] Chew and Mandelstam [2], Shirkov [3,4] and the like.

We shall make use of the fact that the problem a), b), c), d), e) can be re-formulated as the algebraic system (4). Although this system is nonlinear and infinite it is in some respect sufficiently simple and can be investigated by means of the fixed-point theorems [5,6]. Following this method we shall prove, with the help of Schauder's theorem, the existence of solutions of (4).

Recently several authors have shown interest in similar questions. For instance, Warnock [7], and MacDaniel and Warnock [8,9] have studied the conditions under which there exist solutions of Low's integral equation, while in Refs. 10 and 11 Atkinson has made a detailed mathematical analysis of the integral equation of Chew and Mandelstam, and of Shirkov et al., respectively.

These authors examine the question of the existence and uniqueness of the solutions $h^a(z)$ of the integral eqiations with the assumtpion that $h^a(z)$ have at most one pole in the cut plane z.

Some of the results they obtained are less general than those obtained in the present work, because here it is supposed that $h^a(z)$ may have not only poles but also more complicated singularities, e.g., cuts.

The approach in this work differs from the usual approach, by the way, in which the analytical functions are represented. For instance, in the integral equation of Low, the functions $h^{a}(z)$ are represented through the Cauchy integral, while here Laurent's series are being used.

The algebraic approach has some peculiarities which manifest themselves both in the theoretical studies and in the numerical calculations (Refs. 12, 13, 21).

Because of the specific features of the algebraic system (4) it is appropriate to use the conventional methods of nonlinear functional analysis, such as Newton's method and the principle of contraction mapping [5,6] as, for example, applied in the Low amplitude method [13]. On the other hand, the integral equations of dispersive type are solved numerically exclusively by means of the N/D method or the inverse Low amplitude method [8], which techniques are specific for that class of problems. The theorems proved in this work justify the applicability of the numerical methods of Refs. 12 and 13.

In Section II, the precise formulation of Low's problem is given. Besides that, it is shown that under certain conditions it is equivalent to the algebraic system (4). In Section III, by means of Schauder's theorem, the existence of solutions of the system (4) is proved. In conclusion possible applications are briefly discussed. It is pointed out that the Low problem, considered as a model problem can be useful when we are interested in the investigation of general 2 particle-2 particle reactions [22,23] or in the calculations of partial wave amplitudes in the framework of the Mandelstam theory [4].

II. Formulation of Low's Problem

Here we shall give the basic results of Ref. 13. By Low's problem we mean the problem in which N functions $h^{a}(z)$, $a = 1,2,\dots N$ of the complex variable $z = x + iy$ are sought to obey the following conditions:

a) Analyticity: $h^{a}(z)$ are analytic in $p - s_I^a$ where the region p is the plane z from which the points belonging to the cuts $-\infty \le x \le -1$ and $1 \le x \le +\infty$ have been taken away, and the closed region s_I^a is a subregion of the region p.

b) Unitarity: $Im\, h^{a}(x) = f(x)\,|h^{a}(x)|^2$, $1 \le x \le \infty$ where $f(x)$ is a real function, the properties of which are specified below.

c) Crossing symmetry: $h^{a}(-z) = \sum_{\beta=1}^{N} C^{a\beta} h^{\beta}(z)$, where the

crossing matrix $C^{a\beta}$ is equal to the square root of the unit N-dimensional matrix, but otherwise is arbitrary.

d) $h^{a*}(z) = h^{a}(z^*)$.

e) Behaviour at infinity: The integrals in (1) converge. The contribution of the contour integrals $\int \frac{h^{a}(z)\,dz}{z}$ taken on a semicircle with an infinite radius in the upper half-plane is zero.

The problem a), b), c), d), e) is a generalization of the problem which is solved by means of Low's integral equation [1,8,9]

$$h^{a}(z) = \frac{\lambda_a}{z} + \frac{1}{\pi}\int_1^{\infty} dz'\, f(z')\left\{\frac{|h^{a}(z')|^2}{z'-z} + \frac{\sum_{\beta=1}^{N} C^{a\beta}\,|h^{\beta}(z')|^2}{z'+z}\right\}, \quad (1)$$

where $C^{a\beta}$, $a = 1,2,\dots N$ is the crossing matrix and $\lambda_a = -\sum_{\beta=1}^{N} C^{a\beta}\lambda_\beta$ are numbers proportional to the coupling constant f^2.

With an appropriate choice of λ^{a} and $C^{a\beta}$ one could describe by means of (1) the partial scattering amplitudes of various processes, e.g., of the $\pi - N$ scattering [1,16,17].

By the conformal mapping

$$z = \frac{2Z}{1+Z^2} \quad (2)$$

where $Z = X + iY = Re^{i\phi}$, the cut plane p goes over into the interior P of the unit circle C_0 of the Z plane, the functions $h^{a}(z)$ are transformed into the functions $H^{a}(Z)$, the regions s_i^a into the regions S_I^a.

The regions S_I^a contain all singularities of $H^{a}(Z)$ which lie inside C^0. By analogy we shall denote by S_{II}^a closed regions which contain all singularities of $H^{a}(Z)$ lying outside C^0. (All singularities in S_{II}^a were situated on the second sheet of $h^{a}(z)$ before the conformal mapping. Some of them correspond to the resonances, if any).

Let the curves dS_I^a and dS_{II}^a denote the boundaries of S_I^a and S_{II}^a, respectively. The functions $H^{a}(Z)$ are analytic in the annular regions D^{a} which are bounded from the inside by the curves dS_I^a and from the outside by the curves dS_{II}^a.

For several purposes instead of regions D^{a} their subregions D_c^a are preferred. The D_c^a are defined as the circular rings $R_i^a < |Z| < R_e^a$, $R_i^a \le 1$, $R_e^a \ge 1$ the R_i^a being the radii of the circles $|Z|=R_i^a$ which are tangent to the curves dS_i^a and R_e^a, the radii of the circles $|Z|=R_e^a$ which are tangent to the curves dS_e^a.

After the conformal transformation the problem a), b), c), d), e) turns into a problem for the transformed functions $H^a(Z)$. This problem, after some generalization, will be formulated in the following way.

Find the functions $H^a(Z)$, $a=1,2,\dots N$ which satisfy the conditions:

A) Analyticity: $H^a(Z)$, $Z \in D$ are analytic.

B) Unitarity: $Im H^a(Z) = F(\phi)\,|H^a(\phi)|^2$ where $F(\phi)=f(\frac{1}{\cos\phi})$, $-\frac{\pi}{2} \le \phi \le \frac{\pi}{2}$ $\quad H^a(\phi)=H^a(e^{i\phi})$.

C) Crossing symmetry: $H^a(\phi+\pi)=\sum_{\beta=1}^{N} C^{a\beta} H^{\beta}(\phi)$, $-\pi \le \phi \le \pi$.

D) $H^{a+}(Z)=H^a(Z^*)$.

Further on we shall suppose that $H^a(\phi)$ are Holder-continuous: under this notion we shall mean functions $H^a(\phi)$ which satisfy the conditions:

$$|H^a(\phi_2)-H^a(\phi_1)| \le K|\phi_2-\phi_1|^{\epsilon}, \qquad (3)$$

where $K>0$ is a suitable constant and $0<\epsilon\le 1$ and $-\pi-\eta \le \phi_1$, $\phi_2 \le \pi+\eta$.

Under the hypothesis the functions $Re H^a(\phi)$ and $Im H^a(\phi)$ are Holder-continuous and coincide with their Fourier series. This is sufficient to assert that $H^a(Z)$, $|Z|=1$ can be expanded in Laurent series

$$H^a(Z)=\sum_{n=-\infty}^{\infty} H_n^a Z^n, \quad |Z|=1. \qquad (4)$$

Taking into account the conditions B), C) and D), one can derive the following algebraic system which is to be satisfied by the unknown coefficients H_n^a

$$H_\nu^a = H_\nu^a + \sum_{m,k=\infty}^{\infty} F(\nu,k)\,E_\nu^a(H_m; H_{m+k}),\ a=1,2,\dots N;\ \nu=1,2,\dots\infty, \qquad (5)$$

where

$$F(\nu,k)=\frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\nu\,\sin\nu\phi\,\cos k\phi\,F(\phi) \qquad (6)$$

and

$$E_\nu^a(H_m;H_{m+k})=H_m^a H_{m+k}^a+(-1)^\nu\sum_{\beta=1}^{N} C^{a\beta} H_m^\beta H_{m+k}^\beta.$$

The system (5) has been derived in Ref. 13.

The following theorem which is based on the corresponding theorem in Ref. 13 has a more precise formulation that the latter. It clarifies the equivalence between the analytical formulation of Low's problem through the conditions A), B), C), D) and its algebraic formulation given by the system (5).

Theorem 1. Let the functions $H^a(\phi)$, $a=1,2,\dots N$ satisfy the conditions (3), B), C), and D) and let $F(\phi)$ satisfy the condition

$$|F(\phi_2)-F(\phi_1)| \le K_1\,|\phi_2-\phi_1|^{\epsilon_1},$$

where $K_1>0$ is a suitable constant and $0<\epsilon_1\le 1$

$$F(\pm\frac{\pi}{2})=0. \qquad (7)$$

Then the coefficients of the series (4) H_n^a, $a=1,2,\dots N$, $n=0,\pm1,\pm2,\dots\pm\infty$ will satisfy the algebraic system (5). With certain modifications of the theorem the opposite assertion is also true:

Let the system (5) have real roots satisfying the following conditions:

The series $\sum_{n=1}^{\infty} H_n^a \sin n\phi$ and $\sum_{n=1}^{\infty} H_{-n}^a \sin n\phi$ converge on the whole interval $-\pi \le \phi \le \pi$ to certain functions $V_+^a(\phi)$, and $V_-^a(\phi)$, respectively, which are known to satisfy the Holder-condition with the exponent ϵ, $0<\epsilon<1$ on the interval $[-\pi-\eta, \pi+\eta]$, where η is some positive number (8)

$$H_n^a=(-1)^n\sum_{\beta=1}^{N} C^{a\beta} H_n^\beta,\quad a=1,2,\dots N;\ n=0,-1,-2,\dots-\infty \qquad (9)$$

and let $F(\phi)$ satisfy the condition (7).

Then the series (4) converge to the functions $H^a(Z)$, $a=1,2,\dots N$ which satisfy the conditions (3), B), C), D).

If besides that, the roots of (5) satisfy the conditions

$$|H_{-n}^a| \le H(R_i^a)^n,\quad a=1,2,\dots N,\ n=0,1,2,\dots\infty,$$

$$|H_n^a| \le H(R_e^a)^{-n},\quad a=1,2,\dots N,\ n=1,2,\dots\infty, \qquad (10)$$

where H is a positive constant and R_i^a and R_e^a are the inner and the outer radii of the annual region, then the functions $H^a(Z)$, $Z \in D_c^a$ are analytic and satisfy the conditions B), C), D).

R e m a r k 1. Using the result of Ref. 19, Chapter II, §3 we conclude that if $|H_n^a| \leq const \frac{1}{|n|(1+\epsilon)}$ the condition (8) is automatically fulfilled.

R e m a r k 2. Condition (10) induces the analyticity of $H^a(Z)$ in the region D_c^a which is a subregion of D^a. This condition is introduced because it is convenient for the proof of the existence of theorems in Sec. 3.

If however, we are interested only in numerical investigation of Low's problem then condition (10) is superfluous. In this case the determination of the roots of (5) permits one to obtain numerically the Holder-continuous functions $H^a(Z)$ on the unit circle C_0. After that the $H^a(Z)$ can be calculated in principle in all points of D^a through analytic continuation. The analytic continuation can actually be performed by a slight modification of the procedure used successfully in Ref. 14, where numerical values of $H^a(Z)$ were obtained in a region which is larger than D_c^a.

In what follows it is advisable instead of system (5) to investigate its equivalent system /13/

$$t = A(t), \tag{11}$$

where $t \to t_\nu^a, a = 1,2,\dots N; \nu = 1,2,\dots\infty$ is an element of the metric space, and the operator A is defined by the right-hand side of the system:

$$\begin{aligned} t_\nu^a = & \sum_{\lambda,\mu} F(\nu,\lambda-\mu) E_\nu^a(t_\lambda; t_\mu) + 2\sum_{\lambda,\mu} F(\nu;\lambda-\mu) E_\nu^a(r_\mu; t_\lambda) + \\ & + 2\sum_{\xi,\lambda} F(\nu;\xi+\lambda) E_\nu^a(R_{-\xi}; t_\lambda) + 2\sum_{\xi,\lambda} F(\nu;\xi+\lambda) E_\nu^a(R_{-\xi}; r_\lambda) + \\ & + \sum_{\lambda,\mu} F(\nu;\lambda-\mu) E_\nu^a(r_\lambda; r_\mu) + \sum_{\xi,\eta} F(\nu;\xi-\eta) E_\nu^a(R_{-\xi}; R_{-\eta}) + \\ & + R_{-\nu}^a - r_\nu^a \end{aligned} \tag{12}$$

In (12), as well as below $\xi, \eta, \lambda, \mu, \nu$ and a are indices. Furthermore, ξ and η take the values 0, 1, 2,... ∞; λ, μ, ν take the values 1, 2, ... ∞ and $a = 1, 2, \dots N$, unless states otherwise. In (12) the values $R_{-\xi}$ and r_λ are known, and the values t_ν^a are sought. Moreover, $R_{-\xi}^a$ denotes $H_{-\xi}^a$ and $r_\nu^a + t_\nu^a$ is equal to H_ν^a. The values $H_{-\xi}^a = R_{-\xi}^a$ are considered to be known. For example, $R_{-1}^a = \lambda_a / 2$ where λ_a is the baryon pole residue, which is written explicitly in (1).

It is supposed that approximate values are known for H_ν^a which are denoted by r_ν^a. Therefore, in (12) the small corrections t_ν^a to the approximate values r_ν^a are sought.

III. Application of Schauder's Theorem for Proving the Existence of Solutions of Low's Problem

System (12) is very convenient for numerical determination of the solutions of Low's problem /13/ In the present paragraph we shall use it in order to prove the existence of such solutions. For this purpose, we shall make use of one of the fixed-point theorems - Shauder's theorem.

Schauder's theorem is formulated in the following way /5/ :

Let the operator A from (11) have the properties:

1) A maps the bounded, closed convex set M belonging to the Banach space B into itself, i.e., if $t \in M$ then $A(t) \in M$.
2) A is a completely continuous operator.

Then at least one element of the set M exists, which is a solution of (11).

The application of Schauder's theorem to Low's problem is facilitated by making use of the function

$$\chi_k(n) = \chi(j_k; n) = |n|^{-j_k}, \quad n = \pm 1, \pm 2, \dots \pm\infty$$

$$\chi_k(0) = \chi(j_k; 0) = 1.$$

In our case, j_k, $k = 1,2,3,4$, are numbers larger than 1.

The sets M_t, M_r and M_R which we use below are defined respectively by the inequalities

$$|t_\lambda^a| \leq t^* \chi(j_1, \lambda), \quad a = 1,2,\dots N; \quad \lambda = 1,2,\dots\infty, \tag{13}$$

$$|r_\mu^a| \leq r^* \chi(j_2, \mu), \quad a = 1,2,\dots N; \quad \mu = 1,2,\dots\infty. \tag{14}$$

$$|R_{-\xi}^a| \leq R^* \chi(j_3, \xi), \quad a = 1,2,\dots N; \quad \xi = 0,1,2,\dots\infty. \tag{15}$$

In (13), (14) and (15), t^*, r^* and R^* are positive numbers. The function χ is also convenient for the estimation of $F(\nu;\xi^*)$. This expression is defined by the integral (6), which in this case is conveniently put down in the form

$$F(\nu;\xi^*)=\frac{1}{2\pi}\int_{-\pi/2}^{+\pi/2}d\phi\,\{\sin(\nu+\xi^*)\phi+\sin(\nu-\xi^*)\phi\}F(\phi).$$

Further on we shall suppose that

$$|F'(\phi_2)-F'(\phi_1)|\le const|\phi_2-\phi_1|^{\epsilon},\ -\frac{\pi}{2}\le\phi_1,\phi_2\le\frac{\pi}{2},\ 0<\epsilon\le 1$$

$$F(\pm\frac{\pi}{2})=F'(\pm\frac{\pi}{2})=0\qquad F'=\frac{dF}{d\phi}. \tag{16}$$

Let us consider the auxiliary function

$$\tilde F(\phi)=F(\phi)\qquad -\frac{\pi}{2}\le\phi\le\frac{\pi}{2}$$

$$\tilde F(\phi)=0\qquad \frac{\pi}{2}\le\phi\le 3\frac{\pi}{2}$$

It is obvious that $\tilde F'(\phi)$, $-\pi\le\phi\le\pi$ is Holder-continuous with an exponent ϵ, $0<\epsilon\le 1$. This means that the Fourier coefficients of the function $\tilde F(\phi)$ obey condition

$$\tilde F_n=O\left[\frac{1}{|n|^{(1+\epsilon)}}\right]$$

(for proof, see Sec. III in Ref. 20).

Hence $F(\nu,\xi^*)$ can be majorized by the inequality

$$F(\nu;\xi^*)\le const[(\nu+\xi^*)^{-j_4}+(\nu-\xi^*)^{-j_4}],\ j_4>1\ if\ \nu\pm\xi^*\ne 0$$

$$F(\nu;\xi^*)=const,\quad if\ \nu\pm\xi^*=0$$

Using the function $\chi(j_4,n)$ introduced above, at $n=\nu+\xi^*$ and $n=\nu-\xi^*$, and choosing an appropriate positive constant F we obtain the inequality

$$F(\nu;\xi^*)\le F[\chi_4(\nu+\xi^*)+\chi_4(\nu-\xi^*)]. \tag{17}$$

By means of (13), (14), (15) and (17), the equation (12) is majorized by inequalities containing χ. In order to simplify these inequalities it is convenient to use the formula

$$\sum_{n=-\infty}^{\infty}\chi_1(n)\chi_2(n+m)<K_{12}\chi_1(m)+K_{21}\chi_2(m), \tag{18}$$

where

$$K_{12}=(2^{j_1}+1)\zeta(j_2)+1;\quad K_{21}=(2^{j_2}+1)\zeta(j_1)+1$$

and $\zeta(j_1)$ and $\zeta(j_2)$ are the Riemann ζ-functions from the theory of numbers.

When proving (18) it is convenient to proceed from the expression

$$\sum_{n=-\infty}^{\infty}\chi_1(n)\chi_2(n-m)$$

which is numerically equal to the expression

$$\sum_{n=-\infty}^{\infty}\chi_1(n)\chi_2(n+m).$$

The inequality (18) is proved by majorizing for $m\ge 2$ the right-hand side of the equality

$$\sum_{n=-\infty}^{\infty}\chi_1(n)\chi_2(n-m)=S_1+\chi_1(0)\chi_2(-m)+S_2+S_3+\chi_1(m)\chi_2(0)+S_4,$$

where

$$S_1=\sum_{n=-\infty}^{-1}|n|^{-j_1}|n-m|^{-j_2}<m^{-j_2}\sum_{k=-\infty}^{-1}|k|^{-j_1}=\chi_2(m)\zeta(j_2)$$

$$S_2=\sum_{n=1}^{n'}n^{-j_1}|n-m|^{-j_2}<m^{-j_2}\sum_{k=-\infty}^{-1}|k|^{-j_1}=\chi_2(m)\zeta(j_2)$$

$$S_3=\sum_{n=n'}^{m-1}n^{-j_1}|n-m|^{-j_2}<2^{j_1}m^{-j_1}\sum_{n=n'}^{m-1}|n-m|^{-j_2}<$$

$$<2^{j_1}m^{-j_1}\sum_{k=1}^{\infty}|k|^{-j_2}=2^{j_1}\chi_1(m)\zeta(j_2)$$

$$n'=0{,}5\,m$$

if m is even and $n'=0{,}5(m+1)$ if m is uneven

$$S_4=\sum_{n=m+1}^{\infty}n^{-j_1}(n-m)^{-j_2}<m^{-j_1}\sum_{k=1}^{\infty}|k|^{-j_2}=\chi_1(m)\zeta(j_2).$$

For $m=0$ and $m=1$ the fulfilment of (18) is proved directly. For $m<0$ the proofs are analogous.

In order to satisfy the first condition for the operator A we substitute in (12) t^a_λ, r^a_μ, R^a_μ and $F(\nu;\xi^*)$ with the expressions from (13), (14), (15) and (17).

Having the inequality (18) we can easily apply the Schauder's theorem to Low's problem.

For this purpose we choose the Banach space to be a subspace of the space of the bounded sequences of numbers [5]. More precisely, we use a space Y, the elements of which are the sequences of numbers Y_ξ, $|Y_\xi| < A\chi(j,\xi)$, $\xi=-\infty,\dots -1,0,1,\dots\infty; j>1$ the norm being defined by equality $\|Y\| = \sup_\xi |Y_\xi|$.

As (18) holds only for $j>1$ further we shall suppose in (13), (14), (15) and (17), $j_1>1$, $j_2>1$, $j_3>1$ and $j_4>1$, respectively. To satisfy the condition $A(t)\in t$, it is enough to put down

$$(1+NC)Ft^*\sum_{\lambda,\mu}[\chi_4(\nu+\lambda-\mu)+\chi_4(\nu-\lambda+\mu)]\chi_1(\lambda)\chi_1(\mu)+$$

$$+2(1+NC)Ft^*\tau^*\sum_{\lambda,\mu}[\chi_4(\nu+\lambda-\mu)+\chi_4(\nu-\lambda+\mu)]\chi_2(\mu)\chi_1(\lambda)+$$

$$+2(1+NC)Ft^*R^*\sum_{\lambda,\xi}[\chi_4(\nu+\lambda+\xi)+\chi_4(\nu-\lambda-\xi)]\chi_3(\xi)\chi_1(\lambda)+$$

$$+2(1+NC)F\tau^*R^*\sum_{\lambda,\xi}[\chi_4(\nu+\xi+\lambda)+\chi_4(\nu-\xi-\lambda)]\chi_3(\xi)\chi_2(\lambda)+$$

$$+(1+NC)F\tau^{*2}\sum_{\lambda,\mu}[\chi_4(\nu+\lambda-\mu)+\chi_4(\nu-\lambda+\mu)]\chi_2(\mu)\chi_2(\lambda)+$$

$$+(1+NC)FR^{*2}\sum_{\xi,\eta}[\chi_4(\nu+\xi-\eta)+\chi_4(\nu-\xi+\eta)]\chi_3(\xi)\chi_3(\eta)+$$

$$+R^*\chi_3(\nu)+\tau^*\chi_2(\nu)<t^*\chi_1(\nu),$$

where $C=\max C$; $\alpha, \beta=1,2,\dots N$. When deducing the latter inequality it is advisable to suppose at first that $N=1$, and $C^{\alpha\beta}=0$. In this case in the inequality we would have 1 instead of the factors $1+NC$. In the last expression NC accounts for the contribution of the term

$$(-1)^\nu\sum_{\beta=1}^{N}C^{\alpha\beta}H_m^\beta H_{m+k}^\beta$$

in the formula which defines E_ν^α.

Summing over all indices from $-\infty$ to ∞ and using (18), we obtain

$$2(1+NC)Ft^{*2}[K_{41}^2\chi_4(\nu)+(K_{41}K_{14}+2K_{14}K_{11})\chi_1(\nu)]+$$

$$+2(1+NC)F\tau^{*2}[K_{42}^2\chi_4(\nu)+(K_{42}K_{24}+2K_{24}K_{22})\chi_2(\nu)]+$$

$$+2(1+NC)FR^{*2}[K_{43}^2\chi_4(\nu)+(K_{43}K_{34}+2K_{34}K_{33})\chi_3(\nu)]+$$

$$+4(1+NC)Ft^*\tau^*[K_{42}K_{41}\chi_4(\nu)+(K_{43}K_{24}+K_{34}K_{23})\chi_2(\nu)+$$

$$+K_{34}K_{32}\chi_3(\nu)]+4(1+NC)F\tau^*R^*[K_{43}K_{42}\chi_4(\nu)+(K_{43}K_{24}+$$

$$+K_{34}K_{23})\chi_2(\nu)+K_{34}K_{32}\chi_3(\nu)]+4(1+NC)FR^*t^*[K_{43}K_{41}\chi_4(\nu)+$$

$$+(K_{43}K_{14}+K_{34}K_{13})\chi_3(\nu)+K_{34}K_{31}\chi_1(\nu)]+\tau^*\chi_2(\nu)+$$

$$+R^*\chi_3(\nu)<t^*\chi_1(\nu),\quad \nu=1,2,\dots\infty.$$

We suppose that $j_2\ge j_1$; $j_3\ge j_1$ and $j_4\ge j_1$. Under this assumption $\chi_2(\nu)\le\chi_1(\nu)$; $\chi_3(\nu)\le\chi_1(\nu)$ and $\chi_4(\nu)\le\chi_1(\nu)$. If we put $\tau^*=pt^*$ and $R^*=qt^*$ and suppose that $p+q<1$, the above inequality is transformed into the inequality

$$t^*=t_1^*<\frac{1-p-q}{2(1+NC)F(U_1+pU_2+qU_3+p^2U_4+q^2U_5+pqU_6)},\qquad(19)$$

where

$$U_1=K_{41}^2+K_{41}K_{14}+2K_{14}K_{41},$$
$$U_2=2(K_{42}K_{41}+K_{42}K_{14}+K_{24}K_{21}+K_{24}K_{12}),$$
$$U_3=2(K_{43}K_{41}+K_{43}K_{14}+K_{34}K_{31}+K_{34}K_{13}),$$
$$U_4=K_{42}^2+K_{42}K_{24}+2K_{24}K_{22},$$
$$U_5=K_{43}^2+K_{43}K_{34}+2K_{34}K_{33},$$
$$U_6=2(K_{43}^2+K_{43}K_{24}+K_{34}K_{24}+K_{34}K_{32}).$$

Let us suppose that t^* is so chosen that inequality (19) is satisfied. In respect to (13) that means that the absolute values of the left-hand sides of the system (12) are less than the absolute values of those on the right-hand side. In other words, if inequality (19) is satisfied, the set $M_1=M_t$ is such that $A(M_1)\in M_1$. And because by (13) M_1 is a bounded and convex set, it follows that condition 1 of Schauder's theorem has been satisfied. The second condition of Schauder's theorem demands that A should be a completely continuous operator.

Let us recall the definition of a completely continuous operator /5/. The operator A is completely continuous on the set M_1 if it is continuous on M_1 and compact on M_1, i.e. when A maps every bounded subset of M_1 into a compact one. The operator A is continuous on M_1. This is easily proved with regard to formula (15) from /21/. The compactness of M_1 is proved when taking into consideration that according to its definition M_1 is compact /18/.

Therefore, if condition (19) is satisfied, which with an appropriate choice of the parameters t, p and q can always be achieved, then all the requirements for the applicability of Schauder's theorem are also satisfied. This result is expressed in the following theorem:

Theorem 2. Let (16) be satisfied. Let sequences of numbers $R^a_{-\xi}$, $a = 1,2,\dots N$; $\xi = 0,1,\dots\infty$ be known such that

$$|R^a_{-\xi}| \le const\,\chi(j_3;\xi). \tag{20}$$

Then the algebraic system (12) has at least one solution t^a_ν, $a = 1,2,\dots N$; $\nu = 1,2,\dots\infty$ such that $t^a_\nu = O[\frac{1}{|\nu|^{j_1}}]$, $j_1 > 1$.

Let in addition the condition (9) be satisfied. Then the series (4) converge to the functions $H^a(Z)$, $a = 1,2,\dots N$, which satisfy the conditions (3), B), C), D). In the particular case when the sequences H^a_n, $n = 0, -1, -2, \dots$ are finite, the functions $H^a(Z)$, $Z \in D^a_c$ $(1 = R^a_e > |Z| \ge R^a_i = 0)$ are analytical.

Remark 1. Condition (16) can be replaced by the stronger condition.

The function $F'(\phi)$, $-\frac{\pi}{2} \le \phi \le \frac{\pi}{2}$ is bounded and

$$\lim_{\phi\to\pi/2} [F(\phi)(\frac{\pi}{2}-\phi)^{(1+\epsilon)}] < \infty. \tag{16}$$

Proof. In addition to the above motivation in proving the theorem we remark that $j_3 > 1$ and $j_4 > 1$, firstly because of (20), and secondly because $\tilde{F}_n = O[\frac{1}{|n|^{(1+\epsilon)}}]$, $0 < \epsilon \le 1$.

Then choosing $j_2 \ge j_1$ and $j_1 > 1$ we can write $j_4 \ge j_1$ and $j_2 \ge j_1$, which was supposed in deriving (19). So we complete the proof of the first part of Theorem 2.

To prove the second part of the theorem, it is sufficient to demonstrate that the conditions of the inverse part of Theorem 1 are satisfied:

Condition (8) is indeed fulfilled. This is a consequence of the fact that according to the first part of Theorem 2 $|H^a_n| \le const\,\chi(j_1;n)$, $n = 1,2,3,\dots$. Condition (20) means that $|H^a_n| \le const\,\chi(j_3, n)$. From here, in connection with a theorem of Ref. 19, Chapt. II 3, it follows that $V^a_+(\phi)$ and $V^a_-(\phi)$ exist and satisfy the Holder condition with exponent ϵ, $0 < \epsilon \le 1$ on the interval $[-\pi-\eta, \pi+\eta]$, where η is some positive number.

Condition (19) of Theorem 1 is also fulfilled because it figures in Theorem 2 as well.

Condition (7) of Theorem 1 is satisfied because it is a consequence of condition (16) of Theorem 2.

With this the proof of Theorem 2 is completed.

This theorem is an improved version of Theorem 2 in Ref. 21.

Let us consider the application of this theorem in two special cases.

a) Suppose that $R^a_{-\xi} = 0$, $\xi = 2,3,\dots\infty$ and $R^a_{-1} = \frac{\lambda_a}{2}$.

With these assumptions and the appropriate choice of $C^{a\beta}$ Low's problem corresponds to the problem resolved by means of the integral equation of Chew and Low. The existence of solution to this problem depends mainly on the properties of the cut-off function.

So in the case of the G.Salzman and F.Salzman's choice of cut-off function $f(x) = \frac{(x-1)^{3/2}}{12\pi} \exp \frac{-(x^2-1)}{4m_\pi}$, where m_π is the meson mass, passing from $f(x)$ to $F(\phi)$, we conclude that the condition of Remark 1 to Theorem 2 is satisfied. Hence the problem has at least one solution $H^a(Z)$, which is analytic at least in region $0 < |Z| < 1$. The existence of at least one soluition to (1) was proved by Warnock /7/ through its direct investigation.

b) Suppose that $R^a_{-\xi} = 0$, $\xi = 1,2,3,\dots$, $a = 1,2,\dots N$, i.e. the partial scattering amplitudes have no pole at the origin. For $N=3$ and with the appropriate choice of $C^{a\beta}$ this problem is equivalent to the integral equation of Shirkov et al. /3,4/ for the $\pi-\pi$ scattering in the low-energy region. If instead of the function $f(x)$ from Ref. 10 we use the function $f(x)e^{-kx}$, $k\to 0$ the results of Theorem 2 could be transferred directly to that case. If we conjecture that, we can put in the solution $k=0$, we may conclude that the Shirkov equation has at least one solution.

In particular cases as, for instance, in the case of the applications a) and b) of Theorem 2, the condition of analyticity A) is satisfied. But in general Theorem 2 does not guarantee the fulfilment of this condition.

As is seen from Remark 2 to Theorem 1 the conventional approach of simultaneous fulfilment of the conditions of analyticity, unitarity, crossing symmetry and of the condition D) can be abandoned. This is just the case with Theorem 2 in its general formulation, where the last three conditions are considered to define one problem while the fulfilment of condition A) must be regarded as a second problem. (The second problem can be attacked differently, for example, by the Padde approximation or by other methods for analytical continuation [14]).

Such an approach may be fruitful, especially when numerical investigation is intended, because it provides regions D^a, which are larger than the region of analyticity resulting from the direct consideration of the integral equation (1).

As far as the proof of the existence of functions $H^a(Z)$ which fulfil all conditions of the problem A), B), C), D) is concerned, it is given in particular cases in Theorem 2. More general conditions assuring the fulfilment of the four conditions are defined in the next theorem.

Theorem 3. Let (16) be satisfied. Let sequence of numbers $R^a_{-\xi}$, $a = , 1,2,\dots N$, $\xi = 0,1,2,\dots \infty$ be known such that

$$|R^a_{-\xi}| \leq const\,(R^a_i)^{\xi}, \quad \text{where } 1 > R^a_i > 0 \text{ are constants} \tag{21}$$

Then the algebraic system (12) has at least one solution t^a_ν, $a = 1,2,\dots N$; $\nu = 1,2,\dots \infty$ such that $t^a_\nu = O[\frac{1}{\nu^{j_1}}]$, $j_1 > 1$. Let in addition condition (9) be satisfied. Then the series (4) converge for $1 \geq |Z| > R^a_i$ to the functions $H^a(Z)$, $a = 1,2,\dots N$, which are analytic for $Z \in D^a_c\,(1 - R^a_e > |Z| > R^a_i)$ and satisfy the conditions B), C), D).

Proof. Having in view condition (21) we introduce instead of $\chi(j_3, n)$ the function $(R^a_i)^{-|n|}$. We remark that the relations (13) and (14) hold also for Theorem 3 if relation (15) is substituted by (15′)

$$|R^a_{-\xi}| \leq R'*(R^a_i)^{\xi}, \quad a = 1,2,\dots N; \quad \xi = 0,1,2,\dots \infty. \tag{15}$$

The proof of Theorem 3 can be carried out merely as a literal repetition of the proof of the Theorem 2. For this

purpose the relation (18) must be substituted with an analogous relation for the expression $\sum_{n=-\infty}^{\infty} \chi_1(n)(R^a_i)^{-|n+m|}$. This is easily achieved, observing that for K large enough $(R^a_i)^{-|n|} \leq K\chi(j_3\ n)$ so that we get the relation

$$\sum_{n=-\infty}^{\infty} \chi_1(n)(R^a_i)^{-|n+m|} \leq K'_{13}\chi_1(m) + K'_{31}\chi_3(m),$$

where

$$K'_{13} = KK_{13}, \quad K'_{31} = KK_{31}.$$

With (15′) and (18′) instead of (15) and (18) we repeat the reasoning leading to the proof of Theorem 2 and get the proof of Theorem 3. In Theorem 3 an extra moment is the proof of the analyticity of $H^a(Z)$, which is trivial.

IV. Conclusion

In the conclusion we would like to stress the following points.

1) Theorem 2 concerning the existence of at least one solution of Ref. 21 was splittereu here into Theorem 2 and Theorem 3, differing in the properties of the solution. While in Theorem 2 the solution $H^a(Z)$ is supposed to be Holder-continuous on C^0, in the next theorem $H^a(Z)$ is proved to be analytic in an annular region. In the equal way Theorem 3 of Ref. 21, guaranteeing the existence and uniqueness of the solution, could be splittered into theorems 4 and 5, the first being connected with the appurtenance of $H^a(Z)$ to the class of Holder-continuous functions and the second with the analyticity of $H^a(Z)$.

2) In the Bros-Epstein-Glaser theory it is proved that eventually there can exist a finite non-analytical domain [22,23]. The present investigation allows us to give an semiempiric answer to the question whether a domain where $H^a(Z)$ is non-analytical really exists or not. For this purpose we must consider $ReH^a(\phi)$ and $ImH^a(\phi)$ as known by the measurements. In this condition the coefficients S^a_ν and C^a_ν in the Fourier series

$$ImH^a(\phi) = \sum_{\nu=1}^{\infty} S^a_\nu \sin\nu\phi$$

$$ReH^a(\phi) = \sum_{\nu=0}^{\infty} C^a_\nu \cos\nu\phi$$

are known. Therefore the coefficients H^{a}_{-n} , $n=0,1,2,\dots\infty$ related to S^{a}_{ν} and C^{a}_{ν} by the equalities $H^{a}_{\nu}-H^{a}_{-\nu}=S^{a}_{\nu}$ and $H^{a}_{\nu}+H^{a}_{-\nu}=C^{a}_{\nu}$ can be determined.

But $H^{a\nu}_{-n}=R^{a}_{-n}$ must obey (9). Because among the eigenvalues of $C^{\alpha\beta}$ there exist the eigenvalue $+1$ with the eigenvector E^{a}_{+} and the eigenvalue -1 with the eigenvector E^{a}_{-} , $a=1,2,\dots N$, (9) could be rewritten in the following way:

$$R^{a}_{-\nu}=f^{2}_{\nu}E^{a}_{+}\,,\qquad \nu \text{ odd}$$

$$R^{a}_{-\nu}=g^{2}_{\nu}E^{a}_{-}\,,\qquad \nu \text{ even}$$

Clearly, if the suggestion of $H^{a}(Z)$, being nonanalytical in the vicinity of $Z=0$, is true then some f^{2}_{ν} and g^{2}_{ν} will be not zero. Moreover the components $R^{a}_{-\nu}$, $a=1,2,\dots N$ should be proportional to the components of E^{a}_{+} or E^{a}_{-} if the condition C) would hold for the whole interval $-\pi\leq\phi\leq\pi$.

Another example for partial amplitudes with restricted domain of analyticity are the partial scattering amplitudes derived from the Mandelstam representation /4/. This question also may be of interest for the application of some of the results obtained here.

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