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ON THE BOUND STATES OF THE RELATIVISTIC TWO-BODY PROBLEM IN INFINITE MOMENTUM FRAME

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1. Introduction

The relativistic two-body problem is one of the main problems in quantum field theory. The Bethe-Salpeter [1] equation, describing the same problem in four-dimensional formalism, displays a number of undesirable features. First of all, it involves a relative time which has no clear physical meaning. Several years ago, Logunov and Tavkhelidze [2], Kadyshevsky [3,4] and Todorov [5] proposed a quasipotential approach to the relativistic two-body problem. The important features of the quasipotential equations are that the wave function depends on one-time argument and allows a probability interpretation. The finding of the solutions of the corresponding quasipotential equations is not a simple mathematical problem,

Sisskind [6], Bardakci and Halpern [7], and De and Kim [8] showed that the two-body relativistic problem can be described in the infinite momentum frame. Having in mind an isomorphism between a subgroup of the Poincare group and the two-dimensional galilean group they suggest a two-dimensional description of the relativistic system. The corresponding equations are two-dimensional non-relativistic equations which contain Lorentz covariant quantities. This method allows one to make use of some results of the non-relativistic potential theory. In a series of publications [9,10,11] Namylowski discussed some problems of infinite momentum dynamics and covariant quasipotential approach.

In the present paper we suggest a three-dimensional equation for describing the two-body relativistic problem in infinite momentum frame. The longitudinal fractions which are analogous to the masses in non-relativistic theory are expressed by the center-of-mass energies of constituent particles.

In Section 2 a simple equation for relativistic two-body problem with arbitrary interaction in infinite momentum frame is considered. The quasipotential approach in infinite momentum frame is discussed in Section 3. The potential is defined as an infinite power series in the coupling constant which fits the perturbative expansion of the on-energy shell scattering amplitude.

2. Equation for Relativistic Two-Body Problem in Infinite Momentum Frame

Consider the system of two spinless particles of masses m_1 and m_2 and four-momenta p_1 and p_2. The total momentum $P = m_1 + m_2$ of the system is related to the total energy by

$$P^2 = s = w^2 . \tag{1}$$

The center-of-mass energies E_1 and E_2 of particles 1 and 2 can be defined by the covariant relations

$$E_1 = \frac{1}{w} P \cdot p_1 , \quad E_2 = \frac{1}{w} P \cdot p_2 \tag{2}$$

or

$$E_1 = \frac{w^2 + m_1^2 - m_2^2}{2w} , \quad E_2 = \frac{w^2 - m_1^2 + m_2^2}{2w} . \tag{3}$$

For the relative four-momentum q we use the Wightman-Gŏrding vector

$$q = \frac{E_2}{w} p_1 - \frac{E_1}{w} p_2 . \tag{4}$$

Consequently, if both the particles are on their mass-shells then

$$q . P = 0 \tag{5}$$

and in the center of mass system $q_0 = 0$. If we consider a minimal way out of the mass-shell assuming that the four-momenta squared in the initial, final, or intermediate states satisfy the relation /12/

$$p_1^2 - p_2^2 = m_1^2 - m_2^2 \tag{6}$$

then the condition (5) will be valid always.

We shall consider the description of the system of two particles from two reference frames. The system of the particles L' is supposed to be moving with respect to the center-of-mass-frame L, at a high velocity (near the velocity of light) in the $-\vec{P}$ direction. Hence in the frame L the total three-momentum is $P = (0,0,|\vec{P}|)$, $|\vec{P}| \to \infty$. The finite quantities η_1 and η_2 defined by relations

$$\eta_1 = \frac{1}{2} e^{-\omega} (p_1)_3 , \quad \eta_2 = \frac{1}{2} e^{-\omega} (p_2)_3 \tag{7}$$

will be called the longitudinal fractions, where $(p_1)_3$ and $(p_2)_3$ are the third components of the momenta of particles 1 and 2, and the hyperbolic angle $th\,\omega = v$ becomes very large. Let us assume that in the frame L' for the zero component of the Wightman-Gŏrding vector the conditions

$$q_0 = \frac{E_2}{w} (p_1)_0 - \frac{E_1}{w} (p_2)_0 = 0 \tag{8}$$

should holed. Then, having in mind (5), we obtain that the total three-momentum $\vec{P}$ will be orthogonal to the relative three-momentum

$$\vec{P} . \vec{q} = 0 . \tag{9}$$

From (8) we get the approximate relations

$$\frac{E_1}{E_2} = \frac{(p_1)_0}{(p_2)_0} \simeq \frac{(p_1)_3}{(p_2)_3} = \frac{\eta_1}{\eta_2} \tag{10}$$

or

$$\frac{E_1}{E_2} = \frac{\eta_1}{\eta_2} . \tag{11}$$

According to (9) for the energy of the system

$$P_0 = \sqrt{(\frac{\eta_1}{M} \vec{P} + \vec{q})^2 + m_1^2} + \sqrt{(\frac{\eta_2}{M} \vec{P} - \vec{q})^2 + m_2^2} =$$
$$= |\vec{P}| + \frac{M}{2\eta_1 |\vec{P}|} (\vec{q}^{\,2} + m_1^2) + \frac{M}{2\eta_2 |\vec{P}|} (\vec{q}^{\,2} + m_2^2) , \tag{12}$$

where

$$M = \eta_1 + \eta_2 . \tag{13}$$

It is an easy exercise to verify that for the quantities $P_0 + P_3$ and $P_0 - P_3$ under a Lorentz transformation in the Z direction

$$P_0 + P_3 \to e^{\omega}(P_0 + P_3), \tag{14}$$

$$P_0 - P_3 \to e^{-\omega}(P_0 - P_3). \tag{15}$$

If we choose the Lorentz transformation such that

$$2|P| = M e^{\omega} \tag{16}$$

and using (12), (14) and (15) we have in the infinite momentum frame

$$H_0 = P_0 - P_3 = \frac{\vec{q}^{\,2}}{\mu^*} + \frac{m_1^2}{\eta_1} + \frac{m_2^2}{\eta_2}, \tag{17}$$

$$P_0 + P_3 = M, \quad \mu^* = \frac{\eta_1 \eta_2}{\eta_1 + \eta_2}, \tag{18}$$

where H is the Hamiltonian of the system of the non-interacting particles,

$$q = \frac{\eta_2 p_1 - \eta_1 p_2}{M} = \frac{E_2}{w} p_1 - \frac{E_1}{w} p_2 \tag{19}$$

is the three dimensional relative momentum. Multiplying (17) by (18) we have

$$E = \frac{\vec{q}^{\,2}}{\mu^*}, \tag{20}$$

where

$$E = \frac{w^2}{M} - \frac{m_1^2}{\eta_1} - \frac{m_2^2}{\eta_2}. \tag{21}$$

On the basis of (11) and (18) it is easy to verify that

$$b^2 = \mu^* E = \frac{w^4 - 2(m_1^2 + m_2^2)w^2 + (m_1^2 - m_2^2)^2}{4w^2} \tag{22}$$

is the on-shell value of the center of mass momentum squared of each of the two particles.

The next question is how to introduce an interaction into the system. Keeping the non-relativistic analogy for an arbitrary potential V we get

$$E = \frac{\vec{q}^{\,2}}{\mu^*} + \frac{1}{\mu^*} V. \tag{23}$$

When we substitute

$$\vec{q} \to -i\nabla, \quad E \to i\frac{\partial}{\partial t} \tag{24}$$

in (23) we have

$$i\frac{\partial \phi}{\partial t} = -\frac{\Delta\phi}{\mu^*} + \frac{1}{\mu^*} V\phi. \tag{25}$$

For stationary states there is an equation describing the system of two-spinless particles interacting by local potential in the configuration space in infinite momentum frame

$$\frac{w^2}{M}\psi + \frac{\Delta\psi}{\mu^*} = \left(\frac{m_1^2}{\eta_1} + \frac{m_2^2}{\eta_2}\right)\psi + \frac{1}{\mu^*} V\psi. \tag{26}$$

Having in mind that the fraction η_1/η_2 in (26) is expressed by E_1/E_2 we have

$$wE_2\psi = -\Delta\psi\frac{w}{E_1} + \left(m_1^2\frac{E_2}{E_1} + m_2^2\right)\psi + \frac{w}{E_2} V\psi. \tag{27}$$

The equation (26) is of the type of the non-relativistic Schrö-dinger equation

$$\left(-\frac{\Delta}{\mu^*} + \frac{1}{\mu^*} V - E\right)\psi = 0. \tag{28}$$

The factor $\frac{1}{\mu^*}$ in the term of the potential is such that in the non-relativistic limit when

$$w \to E' + m_1 + m_2$$
$$|E'| \ll m_1, m_2 \qquad (29)$$

we obtain the non-relativistic Schrödinger equation

$$(\Delta + k^2)\psi(r) = V(r)\psi(r). \qquad (30)$$

In (30) k^2 is the value of the non-relativistic center of mass momentum squared of each of the two particles.

3. Quasipotential Approach in Infinite Momentum Frame

In quasipotential approach the two-body propagator can be written in the following form /5/

$$G_0 = c_w \frac{1}{\frac{\vec q^{\,2}}{\mu^*} + i0 - E} \qquad (31)$$

and the corresponding equation for the scattering amplitude is of the Lippman-Schwinger type:

$$T = V + c_w V \frac{1}{\vec q^{\,2}/\mu^* + i0 - E} T. \qquad (32)$$

We now choose the constant c_w such that for a hermitian potentials eq. (32) should imply the on-shell elastic unitarity condition

$$T(\vec q, \vec q\,') - T^*(\vec q, \vec q\,') = \frac{\pi i}{w\mu^*} \int T(\vec q, \vec k) T^*(\vec k, \vec q\,') \delta\left(\frac{\vec k^2}{\mu^*} - E\right) d^3\vec k \qquad (33)$$

Using (32) after some transformation, we get

$$T - T^* = c_w T\left(\frac{1}{\vec q^{\,2}/\mu^* - E + i0} - \frac{1}{\vec q^{\,2}/\mu^* - E - i0}\right) T^*. \qquad (34)$$

If we identify (33) with (34) we have

$$c_w = \frac{1}{2w\mu^*}. \qquad (35)$$

The wave function defined by the relation

$$\psi_E(\vec q) = \delta(\vec p - \vec q) + \frac{1}{2\mu^* w} \frac{1}{\vec q^{\,2}/\mu^* - E + i0} T(\vec q, \vec p) \qquad (36)$$

satisfies the homogeneous equation

$$2\mu^* w\left(\frac{\vec q^{\,2}}{\mu^*} - E\right)\psi_E(\vec q) = \int V(\vec q, \vec k)\psi_E(\vec k) d^3\vec k. \qquad (37)$$

The corresponding equation in configuration space in infinite momentum frame is

$$\frac{w^2}{M}\psi = -\frac{\Delta\psi}{\mu^*} + \left(\frac{m_1^2}{\eta_1} + \frac{m_2^2}{\eta_2}\right)\psi + \frac{1}{2\mu^* w} V\psi. \qquad (38)$$

This equation coincides with the quasipotential equation obtained and discussed by Todorov /5/.

The potential is defined as an infinite power series in the coupling constant which fits the perturbative expression of the one-energy-shell scattering amplitude found from (32). Consequently, if $T = \Sigma T_n$ is the perturbative expansion of T when $V = \Sigma V_n$ with

$$V_1 = T_1, \quad V_2 = -T_2 - \frac{1}{2\mu^* w} T_1 \frac{1}{\vec q^{\,2}/\mu^* - E + i0} T_1 \ldots \qquad (39)$$

In the simple model of two complex scalar fields interacting via a neutral scalar field of mass μ the potential obtained by (39) in ladder approximation is of the Yukawa type.

$$V(r) = m_1 m_2 \alpha \frac{e^{-\mu r}}{r}. \qquad (40)$$

A detailed investigation of the relativistic Coulomb problem is given in the Todorov's paper [5] and we repeat some results for completeness.

For $\mu = 0$ eq. (38) is exactly soluble and leads to the following relativistic Balmer formula:

$$w_n^2 = m_1^2 + m_2^2 + 2m_1 m_2 \left(1 - \frac{\alpha^2}{n^2}\right)^{1/2}. \qquad (41)$$

This formula is a known relation [13,14,15] which permits to find the spectrum of masses of two spinless particles interacting by Coulomb potential. It should be noted that this result cannot be obtained directly from the ladder approximation of the Bethe-Salpeter equation.

In the case, when the mass of the exchange particle is not large the potential (40) may be expanded in powers of μ. Using the perturbative method of the solution of the Schrödinger equation suggested by Muller [16,17] we get

$$w_{n\ell} = \left\{ m_1^2 + m_2^2 + 2m_1 m_2 \left(1 - \frac{\alpha^2}{(n+\ell+1)^2}\right)^{1/2} \right\}^{1/2} + \frac{\alpha\mu}{\left(1 - \frac{\alpha^2}{(n+\ell+1)^2}\right)^{1/2}}. \qquad (42)$$

In this relation the second term is a correction due to the non-zero mass of exchange particle.

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