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ЛАБОРАТОРИЯ ТЕОРЕТИЧЕСКОЙ ФИЗИКИ

V.A.Matveev, D.Robaschik, A.N.Tavkhelidze,
E.Wieczorek

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Summary

On the basis of the Dyson-Jost-Lehmann representation the connection between the automodel behaviour in deep-inelastic scattering region and on-shell high-energy asymptotics is investigated.

Weight functions are given which lead to automodel behaviour for deep inelastic e-p scattering and to a constant total cross section for real Compton scattering. In general the corresponding weight functions $\psi(\vec{u},\lambda^2)$ have singularities at $\vec{u}=0$ (in the sense of generalized functions) which determine the high energy behaviour on the mass shell, whereas the automodel behaviour depends on the λ^2 asymptotics of $\psi(\vec{u},\lambda^2)$.

Examples of causal structure functions exhibit the importance of the time-like region $q^2>0$ for the determination of the leading light cone singularity of the current commutator.

In the e^2-approximation the cross section of deep inelastic e-p scattering is given by the imaginary part $W_{\mu\nu}(q,p)$

$$W_{\mu\nu}(q,p) = \frac{1}{8\pi} \sum_{\sigma} \int \langle p,\sigma | [j_\mu(x), j_\nu(0)] | p,\sigma \rangle \, e^{iqx} \, dx \qquad (1.1)$$

of the forward scattering amplitude for the virtual Compton process. Here using the notations of /2/ $j_\mu(x)$ are the electromagnetic current components, q is the four momentum of a virtual photon; $q^2 < 0$, and the matrix elements are taken between identical one nucleon states $|p,\sigma\rangle$ with the four momentum p of mass 1 $(p^2 = 1)$ and spin σ $(\sigma = \pm\frac{1}{2})$. We use here the usual relativistic normalization of the nucleon state

$$\langle p,\sigma | p',\sigma' \rangle = 2p^0 (2\pi)^3 \, \delta(\vec{p} - \vec{p}') \, \delta_{\sigma\sigma'} \qquad (1.2)$$

We represent the tensor $W_{\mu\nu}(q,p)$ in the conventional forms

$$W_{\mu\nu}(q,p) = \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right) W_1 + \left(p_\mu - \frac{qp}{q^2} q_\mu\right)\left(p_\nu - \frac{qp}{q^2} q_\nu\right) W_2 \qquad (1.3)$$

or

$$W_{\mu\nu}(q,p) = (-g_{\mu\nu} q^2 + q_\mu q_\nu) V_1 + \qquad (1.4)$$
$$+ [p_\mu p_\nu q^2 - (p_\mu q_\nu + q_\mu p_\nu) qp + g_{\mu\nu} (qp)^2] V_2 ,$$

where the structure functions are connected by

$$V_1 = \frac{1}{q^2}\left[W_1 + \frac{(qp)^2}{q^2} W_2\right], \quad V_2 = \frac{W_2}{q^2} . \qquad (1.5)$$

On the other hand V_i and W_i can be expressed by the components of the tensor $W_{\mu\nu}$

$$W_1 = -\frac{F_1}{2(1+\frac{4s}{v})} + \frac{1}{2} F_2 ,$$

$$W_2 = \frac{1}{2(1+\frac{v}{4s})^2}\left[(2-\frac{v}{4s}) F_1 + (1+\frac{v}{4s}) F_2\right] ,$$

$$V_1 = \frac{2}{v^2(1+\frac{4s}{v})}\left[\frac{3}{1+\frac{4s}{v}} F_1 + F_2\right] , \qquad (1.6)$$

$$V_2 = V_1 - \frac{4 F_1}{v^2(1+\frac{4s}{v})} ,$$

where

$$F_1 = \Sigma g_{\mu\mu} g_{\nu\nu} p_\mu p_\nu W_{\mu\nu} = p^\mu p^\nu W_{\mu\nu} ,$$
$$F_2 = F_1 - \Sigma g_{\mu\mu} W_{\mu\mu} , \qquad (1.7)$$
$$v = 2qp , \quad s = -\frac{q^2}{v} , \quad g^{00} = +1 .$$

From experiments on deep inelastic e-p scattering in the asymptotic region (Bjorken region)

$$\nu > 0 \quad , \quad \nu \to \infty$$
$$\xi > 0 \quad , \quad \xi \quad \text{fixed} \tag{1.8}$$

the following automodel behaviour for the structure functions W_i is derived

$$W_1(\nu,\xi) \approx f_1(\xi) \quad , \quad W_2(\nu,\xi) \approx \frac{4}{\nu} f_2(\xi) , \tag{1.9}$$

where f_1 and f_2 are different from zero. For the other structure functions the resulting behaviour is (compare equs. (1.5) and (1.7))

$$V_1(\nu,\xi) \approx \frac{1}{\nu\xi}\left[\frac{f_2}{\xi} - f_1\right] , \quad V_2(\nu,\xi) \approx -\frac{1}{\nu^2}\frac{4f_2}{\xi} ,$$
$$F_1(\nu,\xi) \approx \frac{\nu}{4\xi}\left[\frac{f_2}{\xi} - f_1\right] , \quad F_2(\nu,\xi) \approx \frac{\nu}{4\xi}\left(\frac{f_2}{\xi} - f_1\right) \tag{1.10}$$

and especially

$$F_2(\nu,\xi) - F_1(\nu,\xi) = 2 f_1(\xi) \tag{1.11}$$

If one furthermore assumes

$$f_2(\xi) = \xi f_1(\xi) \tag{1.12}$$

then instead of (1.1o) we have.

$$F_1(\nu,\xi) \approx g_1(\xi) \,, \qquad F_2(\nu,\xi) \approx g_2(\xi) \tag{1.13}$$

and

$$V_1(\nu,\xi) \sim \frac{1}{\nu^2} \,, \qquad V_2(\nu,\xi) \sim \frac{1}{\nu^2} \,. \tag{1.14}$$

The relations (1.12 - 1.14) are characteristic for the free field case /1,3/.

Let us now express the total cross section of real Compton scattering (i.e. $q^2=0$) with the help of already introduced structure functions

$$\sigma_{tot}(\nu) = \frac{4C}{\nu} W_1 \quad \text{or} \quad \sigma_{tot}(\nu) = C\nu \lim_{q^2\to 0} \frac{W_2}{-q^2}$$

$$\sigma_{tot}(\nu) = -C\nu V_2 \,, \qquad \sigma_{tot}(\nu) = \frac{2C}{\nu}(F_2 - F_1) \,. \tag{1.15}$$

Consequantly a constant total cross section requires the following on shell limits (Regge limits) for the structure function

$$W_1(\nu,\xi)\Big|_{\substack{q^2=0\\ \xi=0}} \sim \nu \,, \tag{1.16}$$

$$V_2(\nu,\xi)\Big|_{q^2=0,\,\xi=0} \sim \frac{1}{\nu} \,, \tag{1.17}$$

$$\Big[F_2(\nu,\xi) - F_1(\nu,\xi)\Big]_{\substack{q^2=0\\ \xi=0}} \sim \nu \,. \tag{1.18}$$

It is interesting to understand in the framework of Local Quantum Field Theory the mechanism which leads

a) to the automodel asymptotic in the deep inelastic region and

b) to the "Regge asymptotics" on the mass shell.

For this purpose we follow the methods of former investigations of N.N.Bogolubov et.al. /1,2/. In /2/ the causality of the structure functions W_i and V_i has been deduced from general principles of Local Quantum Field Theory and the automodel asymptotics is studied on the basis of the Dyson-Jost-Lehmann representation+) :

$$F(q) = \int \varepsilon(q_0)\, \delta\left[q_0^2 - (\vec{q}-\vec{u})^2 - \lambda^2\right] \psi(\vec{u},\lambda^2)\, d\vec{u}\, d\lambda^2$$

$$\left[(\vec{u},\lambda^2) : |\vec{u}| \le 1,\ \lambda^2 \ge \left(1-\sqrt{1-\vec{u}^2}\right)^2\right]. \tag{1.19}$$

Here $F(q)$ denotes any of the functions W_i and V_i in the rest system of the proton ($\vec{p} = 0$).

In the following we investigate additional conditions for the weight functions $\psi(\vec{u},\lambda^2)$ leading to the

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+) A heuristic derivation of the results found in /1,2/ is given in the lectures by A.A.Logunov at the Grado Summer School, May 1971.

"Regge asymptotics" taking into account the restrictions /2/ which are sufficient for the automodel behaviour in deep inelastic region. We give examples for weight functions with suitable λ^2 asymptotics and singularities at $\vec{u}=0$ (in the sense of generalized functions) such that the resulting structure functions show automodel behaviour (1.1o) in deep inelastic region (1.8) and the "Regge limit" (1.17, 1.18) for real Compton scattering. Whereas the large λ^2 -behaviour of $\psi(\vec{u},\lambda^2)$ determines the automodel asymptotics the on mass shell high energy limit depends on the character of the singularities at $\vec{u}=0$. This is in accordance with the fact the leading light cone singularity must not determine the on shell limit (see W.Rühl /4/).

Using the generalization to non forward scattering /5/ we see that by this mechanism a t-dependent Regge asymptotics can be realised.

Finally on the basis of the DJL representation we study the question whether the experimentally known structure functions in the space-like region determine the leading light cone singularity uniquely. For this reason we give some examples where the space-like region of the scaling function does not determine the leading light-cone singularity. The structure functions calculated with the help of suitably choosen weight functions show different scaling behaviour for space-like and time-like regions respectively. In one example the space-like behaviour is the canonical one whereas the time-like behaviour may be a quite different one determining the leading light cone singularity. Remark that causality and spectral conditions are fulfilled in these examples. So we conclude that causality and spectral conditions alone do not exclude such an unexpected behaviour.

2.

In the following we consider the absorptive part of a causal invariant amplitude in the case of forward scattering. From the representation (1.19) the following result for the automodel limit has been obtained /2/:

$$\text{If} \quad \lim_{\lambda^2\to\infty} \frac{\Psi(\rho,\lambda^2)}{\lambda^{2\kappa}} = \Psi_0(\rho) \quad , \quad \int d\rho\, \rho^2\, \Psi_0(\rho)\, \varphi(\rho) < \infty \tag{2.1}$$

$$\text{then} \quad F(\nu,\xi) \approx \frac{2\pi}{\kappa+1}\, \nu^{\kappa} \int_{\xi}^{1} d\rho\, \rho\, \Psi_0(|\rho|)\, (\rho-\xi)^{\kappa+1} \tag{2.2}$$

$(\kappa > -1\ ,\ \varphi(\rho)$ testfunction, $|\vec{u}| = \rho\ ,\ \Psi(\vec{u},\lambda^2) = \Psi(|\vec{u}|,\lambda^2))$

Let us assume that one and the same term in $\Psi(u,\lambda^2)$ gives both the automodel limit $(\xi > 0)$ and the Regge limit $(\xi = 0)$. If the integral in eq.(2.2) is convergent at $\xi = 0$ the Regge limit is simply

$$F(\nu, m^2) \approx \frac{2\pi}{\kappa+1} \nu^\kappa \int_0^1 d\rho\, \rho\, \psi_0(\rho)\, \rho^{\kappa+1}. \tag{2.3}$$

The question arises how to get the difference between the automodel behaviour (1.11) for $\rho > 0$ and the Regge limit (1.18) for $\rho = 0$. For this purpose we consider weight functions fulfilling condition (2.1) with a singularity at $\rho = 0$ such that expression (2.3) does not exist. Typical examples are the generalized functions

$$\psi_0 = \rho_+^{-\sigma}, \quad \psi_0 = \rho_+^{-\sigma} \log^m \rho, \quad \psi_0 = \Delta^\kappa \delta(\vec{u}) \tag{2.4}$$

(σ non-integer. Remark that the generalized function $\rho_+^{-\sigma}$ in 3 dimensions has simple poles at $\sigma = 3+2k$, $(k=0,1,2,\ldots)$ with residua proportional to $\Delta^\kappa \delta(\vec{u})$, which may interpolate if $\rho_+^{-\sigma}$ is suitably normed.)

In this case we turn back to the DJL representation (1.19) and obtain the following expression for the on shell amplitude

$$F(\nu, m^2) = \frac{2\pi}{2q} \int_0^1 d\rho\, \rho \int_{-2q\rho+m^2-\rho^2}^{2q\rho+m^2-\rho^2} d\lambda^2\, \psi(\rho, \lambda^2) \tag{2.5}$$

$$q = (q_0, \vec{q}), \quad |\vec{q}| = q, \quad q^2 = m^2, \quad \nu = 2q_0 = 2q(1 + O(\tfrac{1}{q^2})).$$

Let us evaluate the contribution from the special weight function

$$\psi(\rho,\lambda^2) = \rho_+^{-\sigma}\,\lambda^{2\kappa}\,\theta\left(\lambda^2-(1-\sqrt{1-\rho^2})^2\right) \tag{2.6}$$

to the Regge limit.

$$F(\nu,m^2) \approx 2\pi\int_0^1 d\rho\,\rho^2\,\rho_+^{-\sigma}\,\frac{1}{\nu\rho}\int_{-\nu\rho+m^2-\rho^2}^{\nu\rho+m^2-\rho^2} d\lambda^2\,\theta\left(\lambda^2-(1-\sqrt{1-\rho^2})^2\right)\lambda^{2\kappa} \tag{2.7}$$

Taking into account the support restrictions (see Fig.1)

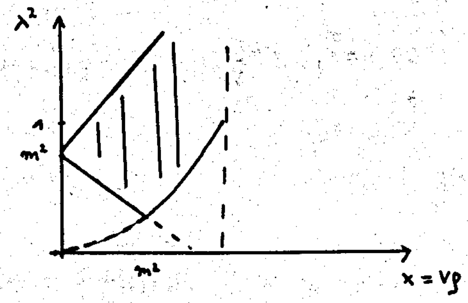


Fig.1.

the function

$$\phi(x,\nu) = \frac{1}{x}\int_{-x+m^2-\frac{x^2}{\nu^2}}^{x+m^2-\frac{x^2}{\nu^2}} d\lambda^2\,\theta\left(\lambda^2-(1-\sqrt{1-\tfrac{x^2}{\nu^2}})^2\right)\lambda^{2\kappa} \tag{2.8}$$

turns out to be analytic in the interval $0\le x< m^2$ and at $x=m^2$ it goes over continuously into a function which behaves asymptotically as x^{κ}. These properties guarantee that the usual subtructions defining the generalised function $\varrho_+^{-\sigma}$ /6/ may be performed. Therefore we have no convergence difficulties. Using homogeneity property of $\varrho_+^{-\sigma}$ (compare /6/) we write

$$F(\nu, m^2) \approx 2\pi\, \nu^{\sigma-3} \int_0^{\nu} dx\, x_+^{2-\sigma}\, \phi(x,\nu) \tag{2.9}$$

Depending on the convergence properties of this integral for $\nu \to \infty$ we have

$$F(\nu, m^2) \sim \begin{cases} \nu^{\sigma-3} \int_0^{\infty} dx\, x_+^{2-\sigma} \phi(x,\infty) & K+3-\sigma<0 \\ \nu^K \log \nu & K+3-\sigma=0 \\ \nu^K & K+3-\sigma>0 \end{cases} \tag{2.10}$$

As it could be expected the Regge behaviour dominates in general the automodel behaviour $F \sim \nu^K$.

Special considerations needs the case $q^2 = m^2 = 0$. From Fig.1. it is obvious that in the region $\varrho \approx 0$ also the λ^2 -behaviour at $\lambda^2 = 0$ becomes important. Therefore we must take into account special properties of $\psi(\varrho, \lambda^2)$ at $\lambda^2 = 0$ too.
For simplicity we choose

$$\psi(\varrho, \lambda^2) = \varrho_+^{-\sigma} \frac{\lambda^{2(K+K')}}{1+\lambda^{2K'}} , \quad K' > 0 \tag{2.11}$$

For the automodel behaviour the analysis of /2/ is again applicable with the result

$$F(\nu,\xi) \approx \frac{2\pi\nu^{\kappa}}{\kappa+1}\int_{\xi}^{1} d\varrho\,\varrho\, \psi_0(|\varrho|)\,(\varrho-\xi)^{\kappa+1}\ , \quad \xi\neq 0 \tag{2.12}$$

For the determination of the Regge limit we do the same calculations as in the case $m^2 \neq 0$. However instead of formula (2.8) we have to use

$$\phi(x,\nu) = \frac{1}{x}\int_{\left(1-\sqrt{1-\frac{x^2}{\nu^2}}\right)^2}^{x-\frac{x^2}{\nu^2}} d\lambda^2\, \frac{\lambda^{2(\kappa+\kappa')}}{1+\lambda^{2\kappa'}} \tag{2.13}$$

which behaves near $x=0$ as $x^{\kappa+\kappa'}$. Let us choose a suitable κ' fulfilling the condition $2-\sigma+\kappa+\kappa' > -1$ then there are no difficulties with the convergence of the integral (2.9) at $x=0$. So we obtain the result (2.10) also in the mass-less case.

Now we consider the case of weight functions $\psi(\varrho,\lambda^2)$ such that their s-primitive $\psi_{(s)}(\varrho,\lambda^2)$ $(s>-1)$ with respect to λ^2 is integrable over λ^2. In this case the automodel behaviour is /2/

$$F(\nu,\xi) \approx \frac{2\pi(-1)^{s+1}}{\nu^{s+1}}\left[\xi\,\psi_0(|\xi|)\right]^{(s-1)} \tag{2.14}$$

with

$$\int_{0^-}^{\infty} \psi_{(s)}(\rho,\lambda^2)\, d\lambda^2 = \psi_{0(s)} \quad . \tag{2.15}$$

In most cases this formula is not applicable at $\rho = 0$. Therefore we again start from formula (2.5) and use the special weight function

$$\psi = \rho_+^{-\sigma} f(\lambda^2)\, \theta(\lambda^2 - (1-\sqrt{1-\rho^2})^2) \tag{2.16}$$

$$\int_0^{\infty} d\lambda^2 f_{(s)}(\lambda^2) < \infty \quad \text{and} \quad |f(\lambda^2)| \underset{\lambda^2\to\infty}{\leq} (\lambda^2)^{-s-1-\varepsilon}, \quad \varepsilon > 0 .$$

If furthermore continuity of $f(\lambda^2)$ at $\lambda^2 = 0$ is imposed then the function

$$\phi(x,v) = \frac{1}{x} \int_{-x+m^2-\frac{x^2}{v^2}}^{x+m^2-\frac{x^2}{v^2}} d\lambda^2\, \theta(\lambda^2 - (1-\sqrt{1-\rho^2})^2)\, f(\lambda^2) \tag{2.17}$$

is continuous at $x = 0$ (for both cases $m^2 \neq 0$ and $m^2 = 0$) and vanishes for $x \to \infty$ as $x^{-s-1-\varepsilon}$. If

$$-1 < 2-\sigma \leq s \tag{2.18}$$

then the integral (2.9) converges and we obtain

$$F(v,m^2) \sim v^{\sigma-3} \quad . \tag{2.19}$$

The condition (2.18) means that the Regge behaviour equals or dominates the automodel behaviour (see eq. (2.14)). In order to get a growing Regge asymptotics one must impose further conditions on $f(\lambda^2)$.

Let us finally apply these considerations to the structure functions $F_2 - F_1$ and V_2 . The properties of $F_2 - F_1$ namely

$$F_2 - F_1 \approx \begin{cases} 2 f_1(\xi) & \xi > 0 \\ \sim \nu & \xi = 0,\ q^2 = 0 \end{cases} \qquad (2.20)$$

can be realized with a weight function $\psi(\rho, \lambda^2)$

$$\psi(\rho,\lambda^2) \underset{\lambda^2 \to \infty}{\sim} \psi_0(\rho) \quad , \quad \psi_0(\rho) \underset{\rho \to 0}{\sim} \rho_+^{-4} \qquad (2.21)$$

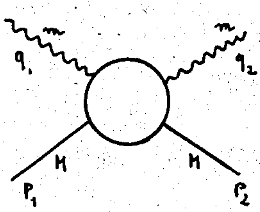
(This corresponds $k = 0$, $\sigma = 4$ in eq. (2.11)). To the structure function V_2 with

$$V_2(\nu, \xi) \asymp \begin{cases} -\frac{1}{\nu^2} \frac{4 f_2(\xi)}{\xi} & \xi > 0 \\ \sim \frac{1}{\nu} & \xi = 0,\ q^2 = 0 \end{cases} \qquad (2.22)$$

corresponds a weight function $\psi(\rho, \lambda^2)$ with an integrable 1-primitive with respect to λ^2 and a singularity $\psi_0(\rho) \sim \rho^{-2}$ ($s = 1$; $\sigma = 2$ in eq. (2.16)).

3.

For elastic non-forward scattering some non essential complications occur. Following /5/ we specialize the kinematical variables in the Breit-system (see Fig.2.).



$$P = \frac{P_1+P_2}{2} = (E_P, 0) = (\sqrt{\vec{p}^2+M^2}, 0)$$

$$Q = \frac{q_1+q_2}{2}$$

$$\Delta = q_2 - q_1 = (0, 2\vec{p})$$

$$\nu = 2PQ = 2E_p Q_0$$

$$t = \Delta^2 = -4\vec{p}^2$$

$$\delta = 2\Delta Q = -4\vec{Q}\vec{p}$$

Fig.2.

The mass shell constraints $q_1^2 = q_2^2 = m^2$ lead to $\vec{Q}\vec{p} = 0$, $Q^2 = m^2 - \frac{t}{4} = m_t^2$. The DJL representation for this case reads

$$F(\nu, m^2, t) = \int d\vec{u} \int d\lambda^2\, \epsilon(Q_0)\, \delta(Q_0^2 - (\vec{Q}-\vec{u})^2 - \lambda^2)\, \psi(\vec{u}, \vec{p}, \lambda^2)$$

$$[(\vec{u}, \lambda^2):\ |\vec{u}| \leq E_p\,,\ \lambda^2 \geq (E_p - \sqrt{E_p^2 - \vec{u}^2})^2 = \bar{\lambda}^2\ ] \qquad (3.1)$$

We choose $\vec{Q} = (0,0,Q)$, $\vec{P} = (P,0,0)$, $\vec{u}\vec{Q} = |u| Q z$ and write therefore $\vec{u}\vec{P} = \rho P \sqrt{1-z^2} \cos\phi$. Introducing $\vec{u} = E_P \vec{\rho}$ and taking into account the relation $Q = (2E_P)^{-1} \nu \,(1 + O(\frac{1}{\nu^2}))$ we have finally

$$F(\nu, m^2) \approx E_P^3 \int_0^1 d\rho\, \rho^2 \int d\lambda^2 \int_{-1}^{+1} dz\, \delta(m_t^2 - E_P^2\rho^2 - \lambda^2 + \rho\nu z) \int_0^{2\pi} d\phi\, \psi(\rho, z, \phi, t, \lambda^2) \qquad (3.2)$$

Performing the z-integration we get

$$F(\nu, m^2) \approx \frac{E_P^3}{\nu} \int_0^1 d\rho\, \rho \int_{-\rho\nu + m_t^2 - E_P^2\rho^2}^{\rho\nu + m_t^2 - E_P^2\rho^2} d\lambda^2 \int_0^{2\pi} d\phi\, \psi(\rho, \bar{z}, \phi, t, \lambda^2) \qquad (3.3)$$

$$\bar{z} = \frac{1}{\rho\nu}(\lambda^2 + E_P^2\rho^2 - m_t^2)$$

The contribution to the on shell limit from the weight function

$$\psi = \psi_0(\rho, z, \phi, t)\, \lambda^{2K}\, \theta(\lambda^2 - \bar{\lambda}^2) \qquad (3.4)$$

with ψ_0 regular at $\rho = 0$ is

$$F(\nu, m^2, t) \approx \frac{E_P^3}{K+1} \nu^K \int_0^1 d\rho\, \rho^{K+2} \int_0^1 dz \int_0^{2\pi} d\phi\, z^K \psi_0(\rho, z, \phi, t) \qquad (3.5)$$

which coincides with the automodel behaviour found in /5/. We now consider generalized functions ψ_0 singular at $\rho = 0$ allowing a t-dependent power of the singularity. As an example we take

$$\mathcal{H} = \rho_+^{-\sigma(t)} g(z,\phi,t)\, \lambda^{2\kappa}\, \theta(\lambda^2 - \bar{\lambda}^2). \tag{3.6}$$

Similar arguments as in the case of forward scattering lead to

$$F(\nu, m^2, t) \approx \nu^{\sigma(t)-3} \int_0^{\check{}} dx\, x_+^{2-\sigma(t)}\, \phi(x,\nu,t) \tag{3.7}$$

where for regular g :

$$\phi(x,\nu,t) = \frac{1}{x} \int_{-x+m_t^2-\frac{E_p^2x^2}{\nu^2}}^{x+m_t^2-\frac{E_p^2x^2}{\nu^2}} d\lambda^2 \; \lambda^{2\kappa}\, \theta\left(\lambda^2 - E_p^2\left(1-\sqrt{1-\tfrac{x^2}{\nu^2}}\right)^2\right) \cdot \int_0^{2\pi} d\phi\, g\left(\frac{\lambda^2+E_p^2x^2\nu^{-2}-m_t^2}{x}, \phi, t\right) \tag{3.8}$$

has the same properties as expression (2.8). Consequently analogous results (see eq.(2.1o)) are obtained. We conclude that such a property of the weight function seems to be a suitable mechanism to get t dependent on shell asymptotics without a variable light cone singularity. By this way one could construct a Regge behaved scattering amplitude which is causal and has furthermore an acceptable off-shell extrapolation.

4.

For the determination of the leading light cone singularity the full support of the structure functions has to be taken into account./7/ For causal theories under certain conditions the automodel behaviour for $\xi > 0$

(q^2 space-like) determines the current commutator on the light cone. Here we will give examples were the light cone singularity in an essential way depends on the behaviour of the structure functions in the time-like region.

The common properties of these and other possible examples are that the conditions given in /2/ (eqs. (2.12), (2.28)) are not fulfilled and therefore their analysis is not applicable. From the DJL representation (2) with $\psi(\vec{u},\lambda^2) = \psi(|\vec{u}|,\lambda^2)$ we obtain

$$F(\nu,\xi) = \frac{2\pi}{\nu}\int_0^1 d\rho\,\rho \int_{-\nu(\rho-\xi)-\chi_2}^{\nu(\rho-\xi)-\chi_1} d\lambda^2\, \psi(\rho,\lambda^2)\,, \tag{4.1}$$

$$\chi_{1,2} = \rho^2 \mp \rho\nu\left(\sqrt{1-\frac{4\xi}{\nu}}-1\right).$$

For large ν taking into account that χ remains bounded we may write

$$F(\nu,\xi) \approx \frac{2\pi}{\nu}\int_0^1 d\rho\,\rho \left\{\theta(\rho-\xi)\,\theta(\rho+\xi)\int_0^{\nu(\rho-\xi)} d\lambda^2\,\psi(\rho,\lambda^2) + \right.$$
$$\left. + \,\theta(-\rho-\xi)\int_{-\nu(\rho+\xi)}^{\nu(\rho-\xi)} d\lambda^2\,\psi(\rho,\lambda^2)\right\} \tag{4.2}$$

which is our starting point for the treatment of the following examples. If condition (2.1) is fulfilled the results of /2/ may be reproduced from eq.(4.2).

Example 1

We choose as weight function

$$\psi = \frac{\lambda^2}{\rho^\mu \lambda^2 + 1}\, \theta(1-\rho)\, \theta(\lambda^2 - 1)\,. \tag{4.3}$$

For $\mu \geqslant 3$ the function $\psi_0(\rho) = \lim_{\lambda^2 \to \infty} \psi = \rho^{-\mu}\theta(1-\rho)$ is not integrable, i.e. condition (2.1) is not fulfilled. Let us determine the limit $\nu \to \infty, \xi$ fix in the space-like and time-like regions and the leading light cone singularity. In the space-like region, i.e. $\xi > 0$ we find

$$\begin{aligned} F(\nu,\xi) &\approx \frac{2\pi}{\nu} \int_\xi^1 d\rho\, \rho \int_0^{\nu(\rho-\xi)} d\lambda^2\, \theta(\lambda^2-1)\, \frac{\lambda^2}{\rho^\mu \lambda^2 + 1} \\ &\approx 2\pi \int_{\xi+\frac{1}{\nu}}^1 d\rho\, \rho^{1-\mu} \left[\rho - \xi - \frac{1}{\nu} - \frac{\rho^{-\mu}}{\nu} \log \frac{1+\nu\rho^\mu(\rho-\xi)}{1+\rho^\mu}\right] \\ &\approx 2\pi \int_\xi^1 d\rho\, \rho^{1-\mu}(\rho-\xi) + O\left(\frac{\log \nu}{\nu}\right) \end{aligned} \tag{4.4}$$

which is just eq.(2.25) of /2/. In the time-like region, i.e. for $\xi < 0$ the first term from eq.(4.2) may be evaluated like expression (4.4). However the seoond term turns out to be the leading one.

$$\begin{aligned} F^{II}(\nu,\xi) &\approx \frac{2\pi}{\nu} \int_0^{-\xi} d\rho\, \rho \int_{-\nu(\rho+\xi)}^{\nu(\rho-\xi)} d\lambda^2\, \theta(\lambda^2-1)\, \frac{\lambda^2}{\rho^\mu \lambda^2 + 1} \\ &\approx \nu^{\frac{\mu-3}{\mu}} \int_0^\infty dy\, y^{\frac{3-2\mu}{\mu}} \frac{\xi y}{1 - 2^{\frac{1+\mu}{\mu}} \xi y}\,, \quad \mu > 3 \end{aligned} \tag{4.5}$$

(for details see Appendix).

The light cone singularity can be determined from the Fourier transform of $F(q)$

$$\tilde{F}(x) = \frac{1}{(2\pi)^4}\int dq\, e^{-iqx} F(q)$$
$$= -\frac{i}{2\pi}\int_0^\infty d\lambda^2\, \mathcal{D}(x,\lambda^2)\, \Delta(\vec{x},\lambda^2), \qquad (4.6)$$

where $\mathcal{D}(x,\lambda^2)$ is the wellknown free field commutation function and

$$\Delta(r,\lambda^2) = 4\pi \int_0^1 d\rho\,\rho\, \frac{\sin r\rho}{r}\, \psi(\rho,\lambda^2) \qquad (4.7)$$

We now use one of the results of /2/:

If $\Delta(r,\lambda^2) \underset{\lambda^2\to\infty}{\sim} \lambda^{2K} G(r)$ then $\tilde{F}(x) \underset{x^2\approx 0}{\sim} \frac{2i}{\pi} G(x)(-\Box)^K \left[\frac{\mathcal{D}(x)}{x^2}\right]$. (4.8)

To evaluate $\Delta(r,\lambda^2)$ for large λ^2 we write

$$\Delta(r,\lambda^2) = \frac{4\pi}{r}\int_0^1 d\rho\,\rho\left[\sin\rho r - \left(\rho r - \frac{(\rho r)^3}{3!} + \ldots\right)\right]\frac{\lambda^2}{\rho^\mu\lambda^2+1} \qquad (4.9)$$
$$+ \frac{4\pi}{r}\int_0^1 d\rho\,\rho\left[\rho r - \frac{(\rho r)^3}{3!} + \ldots\right]\frac{\lambda^2}{\rho^\mu\lambda^2+1}$$

such that the first integral converges in the limit $\lambda^2 \to \infty$. The remaining integrals have the behaviour

$$\frac{1}{r}\int_0^1 d\rho\, \rho\,(\rho r)^n \frac{\lambda^2}{\rho^\mu \lambda^2 + 1} = \frac{r^{n-1}}{n+1}\lambda^2\, {}_2F_1\left(1, \frac{n+2}{\mu}, 1+\frac{n+2}{\mu}, -\lambda^2\right)$$

$$\sim r^{n-1}\left\{ \begin{matrix} \lambda^{2\left(1-\frac{n+2}{\mu}\right)} \\ \lambda^0 \end{matrix} \right\}. \tag{4.10}$$

So finally we find (from the term $n=1$)

$$\Delta(r,\lambda^2) \sim \lambda^{2\frac{\mu-3}{\mu}} \tag{4.11}$$

which yields the leading light cone singularity

$$\tilde{F}(x) \sim (-\Box)^{\frac{\mu-3}{\mu}}\left[\frac{\vartheta(x,0)}{x^2}\right]. \tag{4.12}$$

We see that the leading light cone singularity (4.12) is determined by the behaviour of $F(q)$ in the time-like region.

Example 2

Here we consider an example where a canonical behaviour expected from the space-like region is essentially modified by the behaviour in the time-like region. We choose

$$\Psi(\rho,\lambda^2) = \frac{\theta(1-\rho)\,\theta(\lambda^2-1)}{(\rho^\mu \lambda^2+1)^2} \tag{4.13}$$

Because of

$$\int_0^\infty d\lambda^2\, \Psi(\rho,\lambda^2) = \Psi_0(\rho) = \frac{\rho^{-\mu}}{1+\rho^\mu} \tag{4.14}$$

condition (2.28) in /2/ is not fullfilled for $\mu \gtrsim 3$.
From eq.(4.2) we have for $\zeta > 0$

$$\begin{aligned} F(v,\zeta) &\approx \frac{2\pi}{v} \int_{\zeta+\frac{1}{v}}^{1} d\rho\, \rho \int_{1}^{v(\rho-\zeta)} d\lambda^2 \frac{1}{(\rho^\mu \lambda^2+1)^2} \\ &\approx \frac{2\pi}{v} \int_{\zeta}^{1} d\rho\, \rho\, \Psi_0(\rho) \end{aligned} \tag{4.15}$$

For $\zeta < 0$ we get from the second term

$$F(v,\zeta) \approx \frac{2\pi}{v} \int_0^{-\zeta} d\rho\, \rho \int_{-v(\rho+\zeta)}^{v(\rho-\zeta)} d\lambda^2 \frac{\theta(\lambda^2-1)}{(\rho^\mu \lambda^2+1)^2} \sim v^{-\frac{3}{\mu}} \tag{4.16}$$

(see Appendix).

An analogous calculation of the light cone singularity gives

$$\tilde{F}(x) \sim (-\Box)^{-3/4} \left[\frac{\mathcal{D}(x,0)}{x^2} \right] , \tag{4.17}$$

It is interesting to note that the contributions from the leading light cone singularities to the space-like scaling region may compensate each other as these examples show.

In this sense we have no one to one correspondence between space-like scaling behaviour and the leading light cone singularity (see Ref. /7/), despite the fact that causality and spectral conditions are fulfilled.

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Appendix

For the first example we have to evaluate $(\xi < 0,\ |\xi| = z)$

$$F^{II}_{(v,\xi)} \approx \frac{2\pi}{v} \int_0^z d\rho\,\rho \int_{v(z-\rho)}^{v(z+\rho)} d\lambda^2\ \theta(\lambda^2-1) \frac{\lambda^2}{\rho^\mu \lambda^2 + 1}$$

$$\approx \frac{2\pi}{v} \int_0^z d\rho\,\rho^{1-\mu} \left\{ \int_{v(z-\rho)}^{v(z+\rho)} d\lambda^2 \frac{\lambda^2}{\rho^\mu \lambda^2 + 1} - \theta(1 - v(z-\rho)) \int_{v(z-\rho)}^{1} d\lambda^2 \frac{\lambda^2}{\rho^\mu \lambda^2 + 1} \right\}.$$

The second term is negligible

$$-\frac{2\pi}{v} \int_{z-\frac{1}{v}}^{z} d\rho\,\rho^{1-\mu} \int_{v(z-\rho)}^{1} d\lambda^2 \frac{\lambda^2}{\rho^\mu \lambda^2 + 1} \sim \frac{1}{v} .$$

Therefore

$$F^{II}_{(v,\xi)} \approx \frac{2\pi}{v} \int_0^z d\rho\,\rho^{1-2\mu} \left[2v\rho^{1+\mu} - \log\left(1 + \frac{2v\rho^{1+\mu}}{1 + v\rho^\mu (z-\rho)}\right) \right]$$

with the substitution $y = 2^{\frac{\mu}{1+\mu}} v \rho^\mu$ we get

$$F^{II} \approx \frac{2\pi}{v} 2^{\frac{2\mu-1}{\mu+1}} v^{\frac{\mu-3}{\mu}} \int_0^{2^{\frac{2\mu}{\mu+1}} v z^\mu} dy\, y^{\frac{2-3\mu}{\mu}} \left\{ y^{1+\frac{1}{\mu}} - \right.$$

$$\left. - v^{\frac{1}{\mu}} \log\left[1 + v^{-\frac{1}{\mu}} y \left(1 + 2^{-\frac{\mu}{1+\mu}} \left[z - 2^{-\frac{1}{1+\mu}} \left(\frac{y}{v}\right)^{\frac{1}{\mu}}\right] y\right)^{-1}\right] \right\} .$$

Dividing the integration interval $(0, cv) = (0, v^{\frac{2}{1+\mu}}) + (v^{\frac{1}{1+\mu}}, cv)$ such that in the first integral we may perform the limit $v \to \infty$ under the integral sign, therefore

$$\int_0^{v^{\frac{1}{1+\mu}}} dy \ldots \to \int_0^\infty dy\, y^{\frac{2-3\mu}{\mu}} \left\{ y^{1+\frac{1}{\mu}} - \frac{y^{1+\frac{1}{\mu}}}{1 + 2^{-\frac{\mu}{1+\mu}} z y} \right\} .$$

This leads to Eq. (4.5). The second integral vanishes if we take into account the convergence properties due to the factor $y^{\frac{2-3\mu}{\mu}}$.

For the second example we have to evaluate expression (4.16)

$$F(v,z) \approx \frac{2\pi}{v} \int_0^z d\rho\,\rho \int_{v(z-\rho)}^{v(z+\rho)} d\lambda^2 \; \frac{\theta(\lambda^2-v)}{(\rho^\mu \lambda^2+1)^2} \, .$$

Again we suppress the θ function neglecting terms of order $\log v \cdot v^{-2}$. Therefore we have to consider only

$$F(v,z) \approx 4\pi \int_0^z d\rho\,\rho^2 \; \frac{1}{(1+v\rho^\mu(z+\rho))(1+v\rho^\mu(z-\rho))} \, .$$

The contribution from $0<\varepsilon\leq\rho<z$ behaves asymptotically as $\log v \; v^{-2}$. In the remaining integral we substitute $x = zv\rho^\mu$ so that

$$F(v,z) \sim (zv)^{-3/\mu} \int_0^{\varepsilon v} dx \; \frac{x^{3/\mu-1}}{(1+x(1+z^{-1-1/\mu}(\frac{x}{v})^{1/\mu}))(1+x(1-z^{-1-1/\mu}(\frac{x}{v})^{1/\mu}))} \, .$$

The integral converges for $v \to \infty$.

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