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ON THE ESSENTIAL SELFADJOINTNESS OF DIFFERENT ALGEBRAS OF FIELD OPERATORS

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ON THE ESSENTIAL SELFADJOINTNESS OF DIFFERENT ALGEBRAS OF FIELD OPERATORS

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0. In the theory of * - algebras of unbounded operators the (essentially) selfadjoint algebras are of special interest, because, for example, their commutants have very good properties.

Vassiliev and Borissov [6] have shown, that the algebras generated by the generalized free field or the Wick square of the free field are selfadjoint.

In this paper we prove, that the Op^* -algebras generated by Wick powers (resp. Wick polynomials) of the free field and the number operator are essentially selfadjoint.

1. Let $\mathfrak{D}$ be a unitary space and $\mathcal{H}$ its completion. By $\mathfrak{L}^+(\mathfrak{D})$ [3] we denote the * -algebra of all operators A with $A\mathfrak{D} \subset \mathfrak{D}$ for which there is a $A^+ \in \mathfrak{L}^+(\mathfrak{D})$ with $<\Phi, A\Psi>=<A^+\Phi, \Psi>$ for all $\Phi, \Psi \in \mathfrak{D}$. A^+ is the restriction of A^* to $\mathfrak{D}$. A *- subalgebra $\mathfrak{A} = \mathfrak{A}(\mathfrak{D})$ of $\mathfrak{L}^+(\mathfrak{D})$ containing the identity will be called Op^* algebra

Definition 1 (/4.5/)

An Op^* -algebra $\mathfrak{A} = \mathfrak{A}(\mathfrak{D})$ will be called selfadjoint, if

$$\mathfrak{D} = \mathfrak{D}_* = \bigcap_{A \in \mathfrak{A}} \mathfrak{D}(A^*)$$

essentially selfadjoint, if

$$\mathfrak{D}_* = \tilde{\mathfrak{D}} = \bigcap_{A \in \mathfrak{A}} \mathfrak{D}(\bar{A})$$

Now we prove a simple lemma about the essential selfadjointness of an Op^*-algebra.

Lemma 1

Let $\mathfrak{A} = \mathfrak{A}(\mathfrak{D})$ be an Op^*-algebra. If $\mathfrak{A}$ contains a linear generating system Γ of essentially selfadjoint operators, then $\mathfrak{A}$ is essentially selfadjoint.

Proof:

For the proof we remark that

$$\tilde{\mathfrak{D}} = \bigcap_{A\in\Gamma} \mathfrak{D}(\bar{A}), \qquad \mathfrak{D}^* = \bigcap_{A\in\Gamma} \mathfrak{D}(A^*)$$

because Γ is a linear generating system. Therefore

$$\tilde{\mathfrak{D}} = \bigcap_{A\in\Gamma} \mathfrak{D}(\bar{A}) = \bigcap_{A\in\Gamma} \mathfrak{D}(A^*) = \mathfrak{D}^*$$

which means by definition the essential selfadjointness of $\mathfrak{A}$.

2. Let $\Psi(x) = :\phi^{\ell}(x):$ be a Wick power of the free field $\phi(x)$ with the mass μ For any test function $f\in\mathcal{S}$ ($\mathcal{S}$ - Schwartz' space) we have

$$\Psi(f) = \sum_{k+j=1} \binom{\ell}{k} \int a^+(p_1)\dots a^+(p_k)a(q_1)\dots a(q_j)\times$$

$$\times \tilde{f}(p_1+\dots+p_k-q_1-\dots-q_j)\,d\Omega_p\,d\Omega_q$$

where $d\Omega_p = dp_1\dots dp_k\cdot\omega(p_1)^{-1/2}\dots\cdot\omega(p_k)^{-1/2}$

$$d\Omega_q = dq_1\dots dq_j\,\omega(q_1)^{-1/2}\dots\omega(q_j)^{-1/2}$$

and $\omega(k) = (k^2+\mu^2)^{1/2}$. $\Psi(f)$ is a well-defined operator in the Fock space $\mathfrak{F} = \overset{\infty}{\underset{u=0}{+}} \mathfrak{F}_n$ where $\mathfrak{F}_n$ is the n-particle space

$$\mathfrak{F}_n = L_2(R^3)\underset{s}{\times}\dots\underset{s}{\times} L_2(R^3) \qquad (n \text{ factors}).$$

Let

$$\mathfrak{D} = \sum_{u=0}^{\infty} \mathfrak{F}_n \subset \mathfrak{F}$$

be the algebraic direct sum of the n -particle spaces, i.e., any state $\Phi \in \mathfrak{D}$ contains only a finite number of particles, $\Phi = \Phi_0 + \Phi_1(p_1) + + \Phi_n(p_1, \ldots, p_n)$. $\mathfrak{D}$ is invariant with respect to $\Psi(f)$ and therefore $\Psi(f) \in \mathfrak{L}^+(\mathfrak{D})$. Furthermore $\mathfrak{D}$ is invariant with respect to the number operator N , too. N is defined by

$$(N\Phi_n)(p_1, \ldots p_n) = n \cdot \Phi_n(p_1, \ldots, p_n)$$

$$N = \int a^+(p)\, a(p)\, dp \; .$$

Thus, $N \in \mathfrak{L}^+(\mathfrak{D})$.

By $\mathfrak{A}$ we denote the Op^*-algebra generated by the field operators $\Psi(f)$ and N. Any operator $A \in \mathfrak{A}$ is of the form

$$A = \sum_{i=1}^{k} a_i B_i$$

where a_i are complex numbers and

$$B_i = N^{\nu_1} \Psi(f_1^1) \ldots \Psi(f_{\tau_1}^1) N^{\nu_2} \Psi(f_1^2) \ldots N^{\nu_k} \Psi(f_1^k) \ldots \times \Psi(f_{\tau_k}^k) N^{\nu_{k+1}} \qquad (+)$$

ν_i , τ_i are nonnegative integers. If $\nu_i = 0$ or $\tau_i = 0$ the corresponding part in the product above is equal to 1. Any operator of the form $(+)$ is a partial bounded operator, i.e., such a B is bounded as an operator of $\mathfrak{F}_n$ into $\mathfrak{F}$

$$s_n(B) = \sup_{\Phi \in \mathfrak{F}_n} \frac{\|B\Phi\|}{\|\Phi\|} < \infty$$

For $n \to \infty$ we have the asymptotic behaviour

$$s_n(B) \sim n^{\beta/2},$$

where β is the number of factors $(n+p)^{1/2}$ contained in the product (+). From this we obtain the following lemma. For a detailed proof of the statement cf. /1/.

Lemma 2

For any operator of the form $(+)B, (N+1)^{-p/2} B(N+1)^{-q/2}$ is a bounded operator in $\mathfrak{F}$ if only $p + q \geq \beta$, where β is a natural number depending on B.

With the help of this lemma we prove

Lemma 3

For any operator $B \in \mathfrak{A}$ holds the following estimation
$\| B\Phi \| \leq C \, \| (N+1)^{\gamma} \Phi \|$,
where γ depends on B.

Proof:

In consequence of $B = \Sigma\, a_i B_i$ and by Lemma 2 it holds
$\| B_i (N+1)^{-q_i/2} \Psi \| < C_i \, \| \Psi \|$
which means $\| B_i \Phi \| \leq C_i \, \| (N+1)^{q_i/2} \Phi \|$ (++).
Therefore

$$\| B\Phi \| \leq \Sigma \, |a_i| \, \| B_i \Phi \| \leq \Sigma \, |a_i| \, C_i \, \| (N+1)^{q_i/2} \Phi \| \leq$$

$$\leq C \, \| (N+1)^{\gamma} \Phi \|,$$

where we can choose for example $C = \max_i (|a_i| \, C_i)$

$$\gamma = \max_i (q_i/2)$$

Now we need the following simple fact:

Lemma 4

The Op^*-algebra $\mathfrak{A}$ has a linear generating system Γ of symmetric operators of the form
$a(A)\, A \pm (N+1)^{\gamma}$ (+++)

where γ depends on A in coincidence with Lemma 3 and $a(A)$ is a real number.

Proof:

In an arbitrary $*$-algebra each element x has the decomposition $x = x_1 + i x_2$ where x_1, x_2 are symmetric, namely

$$x_1 = 1/2(x^* + x), \quad x_2 = 1/2i\,(x^* - x).$$

Therefore in our case we can choose $A = A^+$. Then $a(1)A \pm (N+1)^\gamma$ is symmetric, because, as is well known, $(N+1)^\gamma$ is essentially selfadjoint on $\mathfrak{D}$.

q.e.d.

For the proof of our main result we remember the following Theorem about the perturbation of an essentially selfadjoint operator (cf. [2], chap. V, Theorem 4.4).

Theorem

Let T be essentially selfadjoint. If A is symmetric and T-bounded, i.e.,

$\|A\Phi\| < a\|\Phi\| + b\|T\Phi\|$ for all $\Phi \in \mathfrak{D}(T)$

with T-bound smaller than 1 (T-bound, that is the greatest lower bound for the possible values of b), then $T + A$ is essentially selfadjoint and its closure $\overline{(T+A)}$ is equal to $\bar{T} + \bar{A}$.

Now we are able to prove

Theorem 1

The Op^*-algebra $\mathfrak{A}$ has a generating system of essentially selfadjoint operators and, consequently, $\mathfrak{A}$ is essentially selfadjoint.

Proof:

We choose $(+++)$ as a linear generating system of $\mathfrak{A}$ and prove, that each operator of $(+++)$ is essentially selfadjoint.
In fact, by Lemma 3 it holds

$$\|A\Phi\| \leq C(A)\,\|(N+1)^\gamma \Phi\|, \qquad \gamma \text{ depends on } A.$$

Therefore in consequence with the Theorem above it holds

$a(1)A \pm (N+1)^\gamma$

is essentially selfadjoint, where, for example, $a(A) = 1/2C(A)$. Now Lemma 1 completes the proof.

The considerations above show, that Theorem 1 holds also for any Wick polynomial of the free field and a big class of generalized free fields and their polynomials.

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