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REGULARIZATION OF THE SCATTERING AMPLITUDE FOR EXPONENTIAL INTERACTIONS (Third Order)

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I. Introduction

In the present paper we continue to discuss the problem of the construction of the higher orders in the major coupling constant G in quantum field theories with non-polynomial Lagrangians. We have begun this discussion in previous papers /1–3/. At present there is a number of papers in which similar problems are investigated /1–8/. However, a lot of questions remain still unsolved in these theories.

The following problems are the chief ones in our opinion: the introduction of intermediate regularization with the help of which we get the finite results in our theory and the test of unitary of the S-matrix. The solution for the latter problem is given in our paper /3/ and partly in /2/. There we have proved the unitarity of the S-matrix in nonpolynomial theories in the third order in G. Here we would like to focus our attention on the former problem, introduction and subsequent removing of intermediate regularization.

We investigate the scattering amplitude of two massless scalar particles in the third order in G for the exponential interaction. The attractive feature of this process is the following. In this case we can perform explicitly all the integrations over the intermediate momenta. We are able thereby to demonstrate most clearly our method of introducing and subsequent removing of intermediate regularization.

As far as we are dealing with the case of massless particles, in addition to ultraviolet divergences we meet here infrared divergences. The suggested method of analytical regularization is found to permit removing not only ultraviolet divergences, but also infrared ones *.

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* Similar problems in renormalizable theories are solved by the Speer's method (see /9/).

In the case of the exponential interaction the higher perturbation orders are expressed through the integrals of the product of superpropagators /1,2/ . We show that if we take these superpropagators in the form obeying the principle of minimal singularity introduced by H.Lehmann and K.Pohlmeyer /10/ , then the scattering amplitude constructed from these superpropagators in the third order in G agrees automatically with this principle.

This result is in agreement with similar results, obtained in the third order in G by K.Pohlmeyer /7/ and M.Daniel and P.K.Mitter /8/ . Their approaches differ essentially from ours /1-3/ . They have suggested to use the principle of minimal singularity for constructing independently every perturbation order. We propose to use the same principle of minimal singularity, but only for selecting a superpropagator in the "minimal" form. Then we construct all the orders of perturbation theory with the help of these "minimal" superpropagators. It seems to us that this approach is more convenient for constructing the higher perturbation orders in G .

In Section 5 we show how the result obtained for the scattering amplitude of the two particles is generalized to more complicated cases.

2. The Superpropagator of Scalar Massless Particles

Let us consider the exponential Lagrangian of the scalar massless field $\phi(x)$

$$\mathcal{L}_{int}(x) = G : \{ \exp[g\phi(x)] - 1 - g\phi(x) - \frac{g^2}{2}\phi^2(x) \} : \tag{1}$$

When we construct the scattering amplitude (the Green function) in the higher orders in G , the main difficulty is connected with the definition of the integrals of the product of the generalized functions - superpropagators. Note here, that the scattering amplitude in the higher orders in G is expressed in terms of the product of the superpropagators only in the case of the exponential interaction of the fields. In the cases of more general interactions, the higher orders in G are expressed via more complicated functions, "mixed" superpropagators. (see /2/).

The superpropagator $F(x)$ in the exponential case (1) takes the form

$$F(x) = i \sum_{1}^{\infty} \frac{g^{2n}}{\Gamma(n+1)} [-i \Lambda^c(x)]^n , \tag{2}$$

where $\Lambda^c(x)$ is the propagator of the free scalar massless field, and $\Gamma(n+1)$ is the Gamma function. For this superpropagator we can write the following integral representation in the momentum space /1-3,11/

$$\tilde{F}(p) = \lim_{\gamma \to 1} \tilde{F}_{\gamma}(p), \tag{3}$$

$$\tilde{F}_{\gamma}(p) = i \frac{8\pi^2 \kappa}{p^2 + i\epsilon} \int_{-0-i\infty}^{-0+i\infty} dz \, ctg \pi z [\kappa (p^2 + i\epsilon) e^{-i\pi}]^z \frac{\Gamma(-\gamma z)}{\Gamma(z)\Gamma(z+2)} . \tag{4}$$

Here $\kappa = \left(\frac{g}{4\pi}\right)^2$ and the parameter γ is real and larger than four in formula (4). When we want to go over to the limit $\gamma = 1$, we should use some other representation for the $\tilde{F}_{\gamma}(p)$. For instance, we can use the representation like (4), but with a contour bending to the positive real axis. The representation (4) is very useful for the calculations of the integrals over the intermediate momenta in the higher orders in G.

In general, some ambiguity can appear in the formula for the superpropagator in the momentum space (see /11/). However, utilizing the principle of minimal singularity we can remove this ambiguity and obtain formula (4). Further, we shall use this principle in obtaining the expression for the superpropagators in the second order in G and shall construct the higher orders, using these expressions. The formula, obtained for the scattering amplitude in the third order in G with the help of this receipt, automatically agrees with the principle of minimal singularity.

3. The Scattering Amplitude of Scalar Particles in the Third Order in G

Let us consider the diagram, given on Fig.I.

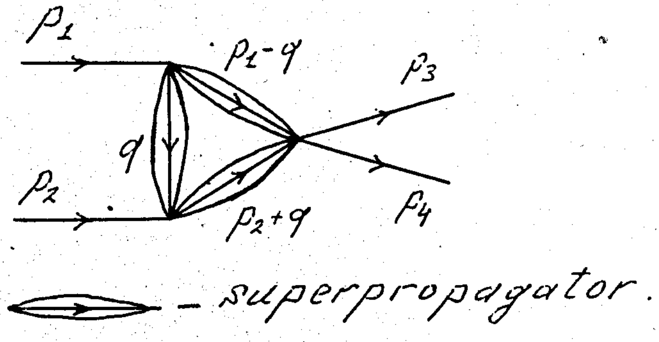

Fig.I

This diagram describes the scattering of two particles. The scattering amplitude corresponding to this diagram, in the x-space can be written in the form

$$F_{2;1}(x_1,x_2,x_3) = g^4 \sum_0^\infty \sum_0^\infty \sum_0^\infty g^{2(n_1+n_2+n_3)} \times \tag{5}$$
$$\times \frac{[-i\Delta^c(x_1-x_2)]^{n_3}[-i\Delta^c(x_1-x_3)]^{n_2}[-i\Delta^c(x_1-x_3)]^{n_1}}{n_1!\, n_2!\, n_3!}.$$

With the help of the representation (4) we can write the following expression for the scattering amplitude (5) in the momentum space depending on the three arbitrary parameters γ_i $(i = 1,2,3)$

$$F_\gamma(p_1,p_2) = F_\gamma^0(p_1,p_2) + F_\gamma(s),$$

$$F_\gamma(s) = c \iiint_{-0-i\infty}^{-0+i\infty} dz_3\, dz_2\, dz_1 \prod_1^3 \left\{ \mathrm{ctg}\, \pi z_i [\kappa e^{-i\pi}]^{z_i} \times \tag{6}\right.$$
$$\left. \times \frac{\Gamma(-\gamma_i z_i)}{\Gamma(z_i)\Gamma(z_i+2)} \right\} f(s \mid z_1, z_2, z_3).$$

Here

$$f(s \mid z_1, z_2, z_3) = \int d^4q\,(q^2+i\epsilon)^{z_3-1}\,[(p_1-q)^2+i\epsilon]^{z_2-1}\,[(p_2+q)^2+i\epsilon]^{z_1-1}, \tag{7}$$

$$s = (p_1+p_2)^2, \qquad \gamma = \{\gamma_1, \gamma_2, \gamma_3\},$$

$c = g^{10}/8$ and $F^0_\gamma(p_1, p_2)$ is the part of the scattering amplitude $F_\gamma(p_1, p_2)$, which corresponds to these diagrams of Fig.1, where one or more superpropagators are replaced by δ-functions, $\delta^{(4)}(q)$. The function $F^0_\gamma(p_1, p_2)$ is investigated by us in paper /2/, and we shall not consider it here.

The function $f(s \mid z_1, z_2, z_3)$ is expressed through the integral in which ultraviolet and infrared divergences can be contained. However, we can choose such z_i-values for which all these divergences will be absent in the integral (7) *.

Let us believe that we have found these z_i. Now we can calculate the integral (7). Here it is convenient to use the representation

$$(q^2 + i\epsilon)^{z-1} = \frac{\Gamma(z)}{i\pi} e^{i\frac{\pi}{2}z} \sin\pi z \int_0^\infty d a\, a^{-z} e^{iaq^2 - \epsilon a} . \tag{8}$$

With the help of (8) we can get

$$f(s \mid z_1, z_2, z_3) = \frac{1}{\pi} \prod_1^3 [e^{i\frac{\pi}{2}z_i} \sin\pi z_i \Gamma(z_i) \int_0^\infty da_i\, a_i^{-z_i}] \times$$
$$\times (a_1 + a_2 + a_3)^{-2} \exp\{ i \frac{a_1 a_2}{a_1 + a_2 + a_3} S - \epsilon (a_1 + a_2 + a_3) \}. \tag{9}$$

After replacing the variables $a_i = \lambda t_i$, $\lambda = a_1 + a_2 + a_3$ we are led to the integral

$$f(s \mid z_1, z_2, z_3) = (-is)^{z-1} \Gamma(1-z) \iiint_0^1 dt_1\, dt_2\, dt_3\, t_1^{z_2+z_3-1} t_2^{z_1+z_3-1} \times$$
$$\times t_3^{-z_3} \delta(1 - \sum_1^3 t_i) . \qquad (z = \sum_1^3 z_i) \tag{10}$$

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* The problems connected with the existence of the integrals of the type (7) and with removing the ultraviolet and infrared divergences in these expressions are investigated in paper /9/.

Now we can easily see, that the integral (10) exists in the domain, where $Re\, z_1 < 0$,

$Re\, z_2 < 0$, $1 > Re\, z_3 > max(|Re\, z_1|, |Re\, z_2|) > 0$.

Calculating it in this domain we come to the following expression for the function $f(s|z_1, z_2, z_3)$

$$f(s|z_1, z_2, z_3) =$$

$$= i\pi s^{z-1} \frac{\sin\pi z_1 \sin\pi z_2 \Gamma(z_1)\Gamma(z_2)\Gamma(z_1+z_3)\Gamma(z_2+z_3)}{\sin\pi z\, \Gamma(z)\Gamma(z+1)} . \qquad (11)$$

Now we pay our attention to the definition of the integral (6). This is the iterated integral over the three complex variables z_1 , z_2 and z_3 . The integration domain over the two first variables z_1 , and z_2 in (6) coincides with the domain, where the integral (10) exists. But the integration domain over variables z_3 in (6) does not coincide with the latter and is somewhat shifted to the left. This is a consequence of the infrared divergences contained in our scattering amplitude.

Taking into account this consideration we suggest the following receipt for defining the iterated integral (6). At first, we calculate the integrals over z_1 and z_2 in the domain, where $Re\, z_3 \geq \beta > 0$. Further we consider the obtained integrand as a function of one complex variable z_3 and continue it analytically through the domain $-1 < Re\, z_3 \leq 0$, where the integral (6) is specified. Then we calculate the last integral over z_3 .

We can go over successively to the limits $\gamma_i = 1$ after performing the integrations over the corresponding variable z_i . We can also keep all the auxiliary parameters γ_i until the integrals over all variables z_i are taken. Then it is seen, that the $F_\gamma(s)$ exists as a finite function of real γ_i in the following region

$$\gamma_3 \geq \gamma_2 \geq \gamma_1 \geq 1. \qquad (12)$$

Further we shall go successively to the limits $\gamma_i = 1$.

It is important to note, that the procedure suggested by us for the defiinition of the integral (6) completely agrees with the unitarity condition of the S-matrix (see /2,3/). If one uses other ways of defining the integral (6), in the expression for the $F(s)$ terms can appear which are in contradiction with unitarity.

4. Removal of Intermediate Regularization

Let us now define the limit $\gamma_i = 1$ of the function $F_\gamma(s)$ following the procedure which was described in Section 3.

Using formulae (6) and (11), we can write the $F_\gamma(s)$ in the form

$$F_\gamma(s) = i\frac{c}{s}\int_{-0-i\infty}^{-0+i\infty} dz_3 (\kappa s)^{z_3} \frac{\Gamma(-\gamma_3 z_3)\Gamma(1-z_3)}{\Gamma(2+z_3)} \times \Big[\int\int_{-0-i\infty}^{-0+i\infty} dz_2 dz_1 \times$$

$$\times \prod_1^2 \left((\kappa s)^{z_i} \frac{\Gamma(-\gamma_i z_i)}{\Gamma(2+z_i)}\right) \frac{\Gamma(z_1+z_3)\Gamma(z_2+z_3)\prod_1^3(\cos\pi z_i)}{\sin\pi z\,\Gamma(z)\Gamma(1+z)} e^{-i\pi z}\Big], \tag{13}$$

where the square brackets mean, that the integrals in between are calculated in the domain $Re\, z_3 \geq \beta > 0$. Then the obtained function of one variable z_3 is continued analytically in the domain $-1 < Re\, z_3 \leq 0$. After that we calculate the integral over z_3.

First we obtain the limit $\gamma_i = 1$ for the imaginary part of the $F_\gamma(s)$

$$Im F_\gamma(s) = -i\frac{c}{s}\int_{-0-i\infty}^{-0+i\infty} dz_3 (\kappa s)^{z_3} \frac{\Gamma(-\gamma_3 z_3)\Gamma(1-z_3)}{\Gamma(2+z_3)} \times \Big[\int\int_{-0-i\infty}^{-0+i\infty} dz_2 dz_1 \times$$

$$\times \prod_1^2 \left((\kappa s)^{z_i} \frac{\Gamma(-\gamma_i z_i)}{\Gamma(2+z_i)}\right) \frac{\Gamma(z_1+z_3)\Gamma(z_2+z_3)}{\Gamma(z)\Gamma(1+z)} \prod_1^3(\cos\pi z_i)\Big]. \tag{14}$$

We see, that we can bend all the contours in the integrals (14) to the positive real axises. Successively performing the integrations and going to the limits $\gamma_i = 1$, we obtain the following expression for the $Im F(s)$*

$$Im F(s) = -(2\pi)^3 \frac{c}{s} \sum\sum_0^\infty\sum \frac{(-1)^{n_3} (\kappa s)^{n_1+n_2}}{\Gamma(n_1+1)\Gamma(n_1+2)\Gamma(n_2+1)\Gamma(n_2+2)} \times$$

$$\times \frac{\partial}{\partial n_3} \left[\frac{(\kappa s)^{n_3} \Gamma(n_1+n_3)\Gamma(n_2+n_3)}{\Gamma(n_3)\Gamma(n_3+1)\Gamma(n_3+2)\Gamma(n)\Gamma(n+1)} \right], \quad (n = \sum_1^3 n_i) \tag{15}$$

Here all the series are absolutely convergent. The asymptotic behaviour of the imaginary part of the $F(s)$ when $s \to \infty$ is

$$\sim \phi(s) \cos\left[\frac{3\sqrt{3}}{2} (\kappa s)^{1/3}\right] exp\left[\frac{11}{2} (\kappa s)^{1/3}\right], \tag{16}$$

where $\phi(s)$ increases slower than an exponential.

Now let us find the limit $\gamma_i = 1$ for the real part of the $F_\gamma(s)$. We show that the limit of the $Re F_\gamma(s)$ is equal to

$$Re F(s) = \Delta(s) + i \frac{c}{s} \int_{-0-i\infty}^{-0+i\infty} dz_3 (\kappa s)^{z_3} \frac{\Gamma(-z_3)\Gamma(1-z_3)}{\Gamma(2+z_3)} \times$$

$$\times \left[\iint_{-0-i\infty}^{-0+i\infty} dz_2 \, dz_1 \prod_1^2 \left((\kappa s)^{z_i} \frac{\Gamma(-z_i)}{\Gamma(2+z_i)}\right) \frac{\Gamma(z_1+z_3)\Gamma(z_2+z_3)}{\sin\pi z \, \Gamma(z)\Gamma(1+z)} \right], \tag{17}$$

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* Note, that the series which appear after integration over z_1 and z_2 are uniformly convergent (see Appendix). Therefore we can integrate them term by term.

where

$$\Delta(s) = -4\pi^4 \frac{c}{s} \sum_0^\infty \frac{(\kappa s)^n \quad (2+n)}{\Gamma^2(1+n)\Gamma(2+n)}. \qquad (18)$$

The integral in formula (17) is a decreasing function of s when $s \to \infty$. The triple and double infinite series, which are increasing functions of the type (15), completely cancel one another and disappear in the limit $\gamma_i = 1$. The remaining increasing in s term, $\Delta(s)$, is obviously a trace of the infrared divergences contained in our amplitude. The obtained expression for the $Re\,F(s)$ differs from a similar expression, obtained in /7,8/ only by that term *.

Let us show now that the expression (17) is the limit of the $Re\,F_\gamma(s)$ when $\gamma_i \to 1$. One of the possible ways of obtaining formula (17) is the following. The cosines standing in the numerator of the integrand of the $Re\,F_\gamma(s)$ can be rewritten in the form

$$\cos\pi z \prod_1^3 (\cos\pi z_i) = 1 + \sin\pi z_1 \sin\pi z_2 \cos\pi(z_1 + z_2) -$$
$$- \sin^2\pi(z_1 + z_2) - \cos\pi z_1 \cos\pi z_2 \sin\pi z_3 \sin\pi z . \qquad (19)$$

Correspondingly the real part of the $F_\gamma(s)$ is also decomposed into four parts. In the first one we can at once go to the limit $\gamma_i = 1$ and obtain the integral standing in the formula (17). The third and the fourth parts of the $Re\,F_\gamma(s)$ almost completely cancel each other. From them only the term $-8\pi^4 \frac{c}{s}$ survives. At last the limit of the second part is equal to $\Delta(s)$ without the term $-8\pi^4 c/s$.

Let us obtain these limits. We shall begin with the second part of the $Re\,F_\gamma(s)$,

$$Re\,F_\gamma^{(2)}(s) = i\pi^2 \frac{c}{s} \int_{-0-i\infty}^{-0+i\infty} dz_3 \,(\kappa s)^{z_3} \frac{\Gamma(-\gamma_3 z_3)\Gamma(1-z_3)}{\Gamma(2+z_3)} \times \Big[\iint_{-0-i\infty}^{-0+i\infty} dz_2\, dz_1 \prod_1^2 \times$$

$$\times \Big(\frac{(\kappa s)^{z_i} \sin\pi z_i}{\sin\pi\gamma_i z_i \;\Gamma(1+\gamma_i z_i)\Gamma(2+z_i)}\Big) \frac{\cos\pi(z_1+z_2)\Gamma(z_1+z_3)\Gamma(z_2+z_3)}{\sin\pi z\,\Gamma(z)\Gamma(z+1)}\Big]. \qquad (20)$$

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* Note, that in the case $p_i^2 = 0$ $(i = 1,2)$ investigated by Pohlmeyer the infrared divergences are absent. Therefore the term, like $\Delta(s)$, must not appear in his results.

We bend the z_1 contour to the positive real axis. Then we integrate over z_1 and go the limit $\gamma_1 = 1$

$$Re F^{(2)}_{\tilde{\gamma}}(s) = 2\pi^2 \frac{c}{s} \int_{-0-i\infty}^{-0+i\infty} dz_3 \frac{\Gamma(-\gamma_3 z_3)\Gamma(1-z_3)}{\Gamma(2+z_3)} \left[\int_{-0-i\infty}^{-0+i\infty} dz_2 \frac{\sin\pi z_2 \cos\pi z_3 \Gamma(z_2+z_3)}{\sin\pi\gamma_2 z_2 \Gamma(1+\gamma_2 z_2)\Gamma(2+z_2)} \times \right.$$

$$\left. \times \sum_1^\infty \frac{(\kappa s)^{n_1} \Gamma(n_1 - z_2)}{\Gamma(n_1)\Gamma(n_1+1)\Gamma(1+n_1-z_2-z_3)\Gamma(2+n_1-z_2-z_3)} \right] \qquad (21)$$

$(\tilde{\gamma} = \{\gamma_2, \gamma_3\})$.

The infinite series which appears in (21) is uniformly convergent (see Appendix). Therefore we can integrate it term by term. We again bend the z_2 -contour to the positive real axis, integrate over z_2 and go to the limit $\gamma_2 = 1$. Then we get the double series

$$\sin^2\pi z_3 \sum_0^\infty (-1)^{n_2} \frac{\Gamma(n_2+z_3)\Gamma(n_2-1+z_3)}{\Gamma(n_2+1)} \times$$

$$\times \sum_1^\infty \frac{(\kappa s)^{n_1} \Gamma(n_1-n_2+z_3)}{\Gamma(n_1)\Gamma(n_1+1)\Gamma(n_1+n_2+1)\Gamma(n_1+n_2+2)} . \qquad (22)$$

This series is also uniformly convergent (see Appendix). Therefore we can integrate it also term by term. We continue analytically each term of this series throughout the domain $Re z_3 < 0$ and perform the integration over z_3. Going to the limit $y_3 = 1$, we obtain

$$Re F^{(2)}(s) = -4\pi^4 \frac{c}{s} \sum_1^{\infty} \frac{(\kappa s)^n (2+n)}{\Gamma^2(n+1)\Gamma(n+2)}. \qquad (23)$$

A similar calculation made for the $Re F_{\gamma}^{(3)}(s)$ gives the following result

$$Re F^{(3)}(s) = -8\pi^4 \frac{c}{s} \Big[1 - \sum\sum\sum_0^{\infty} \frac{(-1)^{n_3} (\kappa s)^n \Gamma(n_1+n_3)\Gamma(n_2+n_3)}{\Gamma(n_1+1)\Gamma(n_1+2)\Gamma(n_2+1)\Gamma(n_2+2)} \times$$

$$\times \left(\Gamma(n_3)\Gamma(n_3+1)\Gamma(n_3+2)\Gamma(n)\Gamma(n+1)\right)^{-1} \Big]. \qquad (n = \sum_1^3 n_i) \qquad (24)$$

Finally we can find very simply the limit of the fourth part of $Re F_{\gamma}(s)$ in the same way as the limit of the $Im F_{\gamma}(s)$. It is equal to

$$Re F^{(4)}(s) = -8\pi^4 \frac{c}{s} \sum\sum\sum_0^{\infty} \frac{(-1)^{n_3} (\kappa s)^n \Gamma(n_1+n_3)\Gamma(n_2+n_3)}{\Gamma(n_1+1)\Gamma(n_1+2)\Gamma(n_2+1)\Gamma(n_2+2)} \times \qquad (25)$$

$$\times \left(\Gamma(n_3)\Gamma(n_3+1)\Gamma(n_3+2)\Gamma(n)\Gamma(n+1)\right)^{-1}.$$

Summarizing all these results we obtain the formula (17) we looked for.

Now we shall show how the results obtained here are generalized for the case of the triangular diagram with external momenta $p_i^2 > 0$ $(i = 1,2,3)$.

5. The Triangular Diagram with External Momenta $p_i^2 > 0$ $(i = 1, 2, 3)$

In the general case the parametrical Green function or the scattering amplitude in the third order in G is represented by the following integrals /7/.

$$F_\gamma^\sigma(p_1, p_2, p_3) = i\delta^{(4)}\left(\sum_1^3 p_i\right) \iiint_{-0-i\infty}^{-0-i\infty} dz_3\, dz_2\, dz_1 \prod_1^3 \left[\frac{\cos\pi z_i\,(\kappa e^{-i\pi})^{z_i}}{\sin\pi\gamma_i z_i\,\Gamma(1+\gamma_i z_i)\Gamma(2+z_i)}\right] \times$$

$$\times\, 1/\pi\,\Gamma(1-z)\, f(z_3, z_2, z \mid \{p\}_\sigma), \tag{26}$$

$$\left(z = \sum_1^3 z_i\right).$$

where

$$f(z_3, z_2, z \mid \{p\}_\sigma) =$$

$$= \int_0^1 dt_2\, t_2^{z_2} \int_0^1 dt_3\, t_3^{-z_3}\, (1+t_2+t_2 t_3)^{-2} \left[\frac{p_i^2 + t_3 p_j^2 + t_3 t_2 p_k^2}{1+t_2+t_2 t_3} + i\epsilon\right]^{z-1}. \tag{27}$$

Here $\sigma = \{i, j, k\}$ i, j, k are not equal to one another and take the values 1,2,3. The behaviour of the function $f(z_3, z_2, z \mid \{p\}_\sigma)$ has been carefully investigated in the paper /7/ .

We consider here the most simple case when all the $p_i^2 > 0$. On the basis of this instance we show that our result coincides with Pohlmeyer's results obtained in the paper /7/ . Like Pohlmayer's formula the obtained by us expression for the scattering amplitude in the third order in G agrees with the principle of minimal singularity.

In the case when all $p_i^2 > 0$ always $(p_i^2 + t_3 p_j^2 + t_3 t_2 p_k^2) \neq 0$ in (27). Therefore we can easily estimate the behaviour of the function $f(z_3, z_2, z \mid \{p\}_\sigma)$ in the domain Ω .

$$\Omega = \{Re z_3 > 0, \quad \infty > Re z_2 > -\infty, \quad Re z > 0\}. \tag{28}$$

The function $f(z_3, z_2, z \| \{p\}_\sigma)$ is meromorphic in Ω. We can express it through a holomorphic function $\bar{f}(z_3, z_2, z \| \{p\}_\sigma)$ in the following way

$$f(z_3, z_2, z \| \{p\}_\sigma) = \frac{\Gamma(1+z_2)}{\sin\pi z_3 \Gamma(z_3)} \bar{f}(z_3, z_2, z \| \{p\}_\sigma), \tag{29}$$

where

$$\bar{f}(z_3, z_2, z \| \{p_\sigma\}) =$$

$$= -\frac{\Gamma(z_3)\Gamma(-z_2)}{4} e^{i\pi(z_3 - z_2)} \int_C dt_2 t_2^{z_2} \int_C dt_3 t_3^{-z_3} \frac{(p_i^2 + t_3 p_j^2 + t_3 t_2 p_k^2)^{z-1}}{(1 + t_2 + t_2 t_3)^{z+1}}. \tag{30}$$

Here the contour C encircles the real interval $[0\,1]$. It is so close to it that the expressions in the brackets in (30) do not vanish in the domain bounded by the contours C.

There exist positive constants M_1, M_2 and M_3 such that in Ω

$$\bar{f}(z_3, z_2, z \| \{p\}_\sigma) < M_1^{Re z} M_2^{|Re z_2|} M_3^{Re z_3} \times$$

$$\times |z_2|^{-Re z_2} |z_3|^{Re z_3} \exp\frac{\pi}{2}(|Im z_2| + |Im z_3|). \tag{31}$$

Using this estimate for the behaviour of the function $\bar{f}$ in Ω we can carry out the procedure of going to the limit $\gamma_i = 1$ in the function $F_\gamma^\sigma(p_1, p_2, p_3)$, in the same way as we did it in the Section 4.

We shall again consider the real and the imaginary parts of the F_γ^σ independently. Using the formula (19) let us decompose the $Re F_\gamma^\sigma$ on four parts. In the first part we can go immediately to the limit $\gamma_i = 1$. After integrating over z_1 and going to the limit $\gamma_1 = 1$ we obtain for the $Re F_\gamma^{\sigma(2)}$

$$Re F_{\tilde{\gamma}}^{\sigma(2)} = 2\delta^{(4)}\left(\sum_1^3 p_i\right) \iint_{-0-i\infty}^{-0+i\infty} dz_3 dz_2 \frac{ctg\pi z_3 \sin\pi z_2 \Gamma(1+z_2)}{\sin\pi\gamma_3 z_3 \sin\pi\gamma_2 z_2 \Gamma(z_3)\Gamma(2+z_3)\Gamma(1+\gamma_3 z_3)\Gamma(1+\gamma_2 z_2)} \times \tag{32}$$

$$\times \frac{1}{\Gamma(2+z_2)} \times \sum_1^{\infty} \frac{\kappa^{n_1}}{\Gamma(n_1)} \quad \frac{\bar{f}(z_3, z_2, n \| \{p\}_\sigma)}{\Gamma(1+n_1-z_2-z_3)\Gamma(2+n\ -z_2-z_3)}. \tag{32}$$

It can be easily seen that the next step, the integration over z_2 and going to the limit $\gamma_2 = 1$, gives the result

$$Re\, F_{\gamma_1}^{\sigma(2)} = 0. \tag{33}$$

Performing successive transitions to the limits $\gamma_i = 1$ for the $Re\ F_\gamma^{\sigma(3)}$ we obtain

$$Re\, F^{\sigma(3)} = -8\,\delta^{(4)}(\sum_1^3 p_i)\ \sum_1^{\infty} \frac{(-\kappa)^{n_3}}{\Gamma(n_3)\Gamma(n_3+1)\Gamma(n_3+2)} \sum_0^{\infty} \frac{\kappa^{n_2}}{\Gamma(n_2+2)} \times$$

$$\times \sum_0^{\infty} \frac{\kappa^{n_1}\bar{f}(n_3, n_2, \sum_1^3 n_i \| \{p\}_\sigma)}{\Gamma(n_1+1)\Gamma(n_1+2)\,\Gamma(\sum_1^3 n_i)}. \tag{34}$$

For the $ReF^{\sigma(4)}$ we obtain the same expression but with other sign. Summarizing all these results we get for the $Re\, F^\sigma(p_1, p_2, p_3)$

$$ReF^\sigma(p_1, p_2, p_3) = \frac{\delta^{(4)}(\sum_1^3 p_i)}{i\pi^4} \iiint_{-0-i\infty}^{-0+i\infty} dz_3\, dz_2\, dz_1 \prod_{1}^{3} [\kappa^{z_i} \frac{\Gamma(-z_i)}{\Gamma(2+z_i)}] \times \tag{35}$$

$$\times \Gamma(1-z)\, f(z_3, z_3, z \| \{p\}_\sigma).$$

This function decreases when any $p_i^2 \to \infty$.

Let us give also the expression for the $Im F^{\sigma}(p_1, p_2, p_3)$

$$Im F^{\sigma}(p_1, p_2, p_3) = \frac{8}{\pi}\delta^{(4)}\left(\sum_1^3 p_i\right) \sum\sum\sum_0^{\infty} \frac{(-1)^{n_3}\ \kappa^{n_2+n_1}}{\Gamma(n_1+1)\Gamma(n_1+2)\Gamma(n_2+2)} \times$$

$$\times \frac{\partial}{\partial n_3}\left[\frac{\kappa^{n_3}\bar{f}(n_3, n_2, \sum_1^3 n_i \mid \{p\}_{\sigma})}{\Gamma(n_3)\Gamma(n_3+1)\Gamma(n_3+2)\Gamma(\sum_1^3 n_i)} \right]. \tag{36}$$

At $p_i^2 \to \infty$ this function increases as an exponential.

These results are in agreement with the principle of minimal singularity /7,10/ .

In conclusion of this Section we would like to note that in a similar way we can investigate also the most general case, when a part of $p_i^2 > 0$ and a part of $p_i^2 < 0$ (i = 1,2,3). In this case it is convenient to divide the integral over t_3 in the function $f(z_3, z_2, z \mid \{p\}_{\sigma})$ into two parts. In the first part $(p_i^2 + t_3 p_j^2 + t_3 t_2 p_k^2) \geq 0$ and in the second $(p_i^2 + t_3 p_j^2 + t_3 t_2 p_k^2) \leq 0$.*

6. Conclusion

Thus, we have shown that the scattering amplitude constructed with the help of the "minimal" superpropagators does not contradict the principle of minimal singularity in the third order in G . The removal of the intermediate regularization can conveniently be made by passing successively to the limits $\gamma_1 = 1$, $\gamma_2 = 1$ and finally $\gamma_3 = 1$.

The S -matrix is unitary in this theory. We can verify it using the spectral representation for the superpropagator (see /2,3/ **)

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*A similar way was used by Pohlmayer (see /7/).

** Note, that the function (37) is identically equal to the function (4).

$$\tilde{F}_\gamma(p) = -\frac{(4\pi)^2\kappa}{p^2 + i\epsilon} -$$

$$-i8\pi\kappa \int_{+0-i\infty}^{+0+i\infty} dz \cos\pi z\, \kappa^z \frac{\Gamma(-\gamma z)}{\Gamma(z)\Gamma(z+2)} \int_0^\infty d\mu^2 \frac{\mu^{2(z-1)}}{\mu^2 - p^2 - i\epsilon}. \tag{37}$$

With the help of this representation any amplitude in the n-order in G consisting of superpropagators is expressed through a similar amplitude consisting of ordinary propagators of the massive particles. Owing to this the problem of unitary of the S-matrix in the nonpolynomial theory reduces to an analogous one in the ordinary polynomial theory (see /2,3/).

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Appendix

Let us show that the series of Sections 4 and 5 are uniformly convergent. Here we consider the two most typical series and prove that they uniformly convergent

$$A_1(z_2, z_3) = \sum_{n_1=1}^{\infty} \frac{(\kappa s)^{n_1}\, \Gamma(n_1 - z_2)}{\Gamma(n_1)\Gamma(n_1+1)\Gamma(1+n_1-z_2-z_3)\Gamma(2+n_1-z_2-z_3)}, \tag{A.1}$$

$$A_2(z_3) = \sum_{n_2=0}^{\infty} \frac{(-1)^{n_2}\, \Gamma(n_2+n_1+z_3)\, \Gamma(n_2+z_3)\Gamma(n_2-1+z_3)}{\Gamma(1+n_2)\Gamma(1+n_2+n_1)\Gamma(2+n_2+n_1)}. \tag{A.2}$$

We can believe that the variables z_i are imaginary and change in the infinite limits. We use the following formulae /13/

$$|\Gamma(x+iy)]| = \Gamma(x)\left[\prod_0^\infty \left(1+\frac{y^2}{(x+k)^2}\right)\right]^{-1/2} \quad (x>0) \tag{A.3}$$

$$sh\,\pi y = \pi y \prod_1^\infty \left(1+\frac{y^2}{k^2}\right). \tag{A.4}$$

With the help of them we can obtain the following estimate for the series A_1

$$A_1 \le \sum_1^\infty \frac{(\kappa s)^{n_1}}{\Gamma^2(n_1+1)\Gamma(n_1+2)}\left[\prod_0^\infty \frac{\left(1+\frac{(y_2+y_3)^2}{(1+n_1+k)^2}\right)\left(1+\frac{(y_2+y_3)^2}{(2+n_1+k)^2}\right)}{\left(1+\frac{y_2^2}{(n_1+k)^2}\right)}\right]^{1/2} < \tag{A.5}$$

$$< \prod_1^\infty\left(1+\frac{(y_2+y_3)^2}{k^2}\right)\sum_1^\infty \frac{(\kappa s)^{n_1}}{\Gamma^2(n_1+1)\Gamma(n_1+2)} = \frac{sh\,\pi(y_2+y_3)}{\pi(y_2+y_3)}\sum_1^\infty \frac{(\kappa s)^{n_1}}{\Gamma^2(n_1+1)\Gamma(n_1+2)}.$$

The last series is absolutely convergent. The y-dependence is uniformly separated from all the terms. Therefore the series A_1 are uniformly convergent.

Let us proceed now to the series A_2. Using (A.3) we can show that

$$A_2 < \sum_{n_2=2}^\infty \frac{\Gamma(n_2-1)}{n_2(n_2+n_1)\Gamma(n_2+n_1+2)}. \tag{A.6}$$

The condition of the absolute convergence of this series $n_1 > -4$ is fulfilled. Therefore the series A_2 is uniformly convergent too.

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