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HARMONIC ANALYSIS ON THE LORENTZ GROUP AND PARTICLES WITH SPIN IN QUASIPOTENTIAL APPROACH

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HARMONIC ANALYSIS ON THE LORENTZ GROUP AND PARTICLES WITH SPIN IN QUASIPOTENTIAL APPROACH

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Гармонический анализ на группе Лоренца и частицы со спином в квазипотенциальном подходе

Получены преобразования Шапиро в случае произвольного спина в удобной для квазипотенциального подхода параметризации, теорема сложения для ядер преобразований и дан пример локального квазипотенциала в релятивистском конфигурационным пространстве.

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Harmonic Analysis on the Lorentz Group and Particles with Spin in Quasipotential Approach

The Shapiro transformation for the case of an arbitrary spin is obtained in the parametrization suitable for quasipotential approach. Addition theorem was also obtained for transformation nuclei. An example of the local quasipotential in the relativistic configuration space is given.

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Introduction

In the paper [1] the unitary representations of the Lorentz were applied to formulate the equations of the quasipotential approach [2-4] for the relativistic two-body problem in the relativistic configurational representation. The role of the Fourier-transformation is there played by the Shapiro integral transformation [5], which, from the "group-theoretical point" of view, performs an expansion of the unitary representation of the Lorentz group in the irreducible unitary representations of the principal series. The generalization of this transformation to the case of the spin particles had been carried out by Chou Kuang-chao and L.G.Zastavenko [6]. Later this expansion was obtained by Popov [7] on the basis of the theory of the Lorentz group representations, developed by Gelfand and Naimark [8,9], in particular, on the basis of an analog of Plancherel's theorem for the Lorentz group, proved by them. But the formulas obtained by him, differ by their parametrization from those used in the quasipotential approach in order to introduce the relativistic radius-vector of the relative distance between two particles, because instead of integration over the angular-variables of the radius-vector they contain the integration over SU(2) group.

In [10] the expansion for the wave function of particles with spin, obtained in [6] was used to pass to the configurational representation in the quasipotential equation, which describes the interaction of the spin 1/2 and spinless particles. It is necessary to know the addition theorem for the kernels of "Shapiro transformation" for introducing a local quasi-potential in that equation, because these kernels play the role of plane waves.

The main aim of the present article is to derive such an addition theorem on the basis of Lorentz group representation theory, developed in [8,9] and to construct with its help a local quasipotential from the Feynman diagrams of quantum field theory.

In the first part of the article on the basis of the formulas, obtained in [9], we present a derivation of the expansion for the wave function of the particle with spin in the functions which transform under the Lorentz group irreducible unitary representations in the parametrization, which allows the transition to the relativistic configurational representation in the quasipotential approach. This expansion is obtained in helicity and canonical bases. In the second part, the plane waves, introduced in [10], describing free motion of particles with spin in the quasipotential approach are discussed. In the third part the "addition theorem" for such plane waves is derived. The fourth part is devoted to the application of the addition theorem for constructing the local quasi-potential in the configurational representation.

The wave function of the particle with spin S and projection σ on the Z axis transforms under the SL(2.c) group transformation in the following way:

$$T_a \psi_\sigma^{(s)}(\underset{\sim}{p}) = \sum_{\sigma'=-s}^{s} D_{\sigma\sigma'}^{(s)}\{V(a,p)\}\, \psi_{\sigma'}^{(s)}(\Lambda_a^{-1} p), \tag{1}$$

$p^2 = p^{o2} - \underline{p}^2 = m^2, \quad p^o > 0,$

where $D^{(s)}$ is the matrix of the irreducible representation of weight S of $SU(2)$ group, and unitary unimodular matrix $V(a,p) \in SU(2)$ which corresponds to the Wigner rotation, is defined by

$$V(a,p) = H_p^{-1} a\, H_{\Lambda_a^{-1} p}, \tag{2}$$

Hermitian matrix $H_p \in SL(2,C)$ is

$$H_p = \frac{m + \underset{\sim}{p}}{\sqrt{2m(p^o+m)}}, \quad \underset{\sim}{p} = p^o + \underline{p}.\underline{\sigma} \tag{3}$$

($\underline{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices), corresponds to the four-dimensional matrix Λ_p, of the pure Lorentz transformation, or "boost" i.e. that for if $P_R = (m, o)$ and $P = (p^o, \underline{p})$

$$\Lambda_p P_R = P \tag{4}$$

and

$$(\Lambda_k^{-1} p)^o = (p(-)k)^o = p^o k^o - \underline{p}.\underline{k} \equiv pk$$

$$\underline{(\Lambda_k^{-1} p)} = \underline{p(-)k} = \underline{p} - \frac{\underline{k}}{m}\, \frac{m p^o + pk}{m + k^o}$$

(5)

In the space of matrices from SL(2,C) the transformation (4) induces the next transformation

$$\underset{\sim}{P} = H_p \underset{\sim}{P}_R \overset{+}{H}_p \tag{6}$$

where in the spherical coordinates

$$p^0 = mc\,ch\chi \;, \quad \underline{p} = ms\,h\chi\, \underline{n}_p \;, \quad \underline{n}_p^2 = 1$$

$$H_p = u_p\, \varepsilon_p\, \overset{+}{u}_p \,, \qquad \varepsilon_p = \begin{pmatrix} e^{\chi/2} & 0 \\ 0 & e^{-\chi/2} \end{pmatrix} \tag{7}$$

$$u_p \in SU(2)$$

The group representation property of the transformation (1) follows from the group composition law for the Wigner rotations

$$V(a,p)\, V(a_1, \Lambda_a^{-1} p) = V(a a_1, p)\,. \tag{8}$$

The connection of the wave function $\psi_\sigma^{(s)}(p)$ in the canonical basis with the wave function in the helicity basis $\phi_\sigma^{(s)}(p)$ is given by the relation

$$\phi_\lambda^{(s)}(\underline{p}) = \sum_{\sigma=-s}^{s} D_{\lambda\sigma}^{(s)}(u_p)\, \psi_\sigma^{(s)}(\underline{p})\,, \tag{9}$$

where λ is a value of the projection of the spin on the momentum direction i.e. - helicity. From eq. (1) and eq. (9) the law of the transformations of $\phi_\lambda^{(s)}(\underline{p})$ follows:

$$T_a \phi^{(s)}_{\lambda}(p) = \sum_{\lambda'=-s}^{s} D^{(s)}_{\lambda\lambda'}\{W(a,p)\}\, \phi^{(s)}_{\lambda'}(\Lambda_a^{-1} p), \tag{10}$$

where

$$W(a,p) = u_p V(a,p)\, \overset{+}{u}_{\Lambda_a^{-1} p}\,.$$

Because of the Lorentz invariance of the scalar product

$$(\psi, \phi) = \sum_{\sigma=-s}^{s} \int \frac{d^3 p}{p^0}\, \overline{\psi^{(s)}_{\sigma}(p)}\, \phi^{(s)}_{\sigma}(p)$$

the representation T_a is the unitary one.

Thus the wave function can be expanded in the unitary irreducible representations of the Lorentz group.

In order to perform such an expansion let us pass from the functions on the hyperboloid (1) and (7) to the functions on the SL(2,C) group defining them as follows:

$$X_{\sigma}(a) = T_a\, \psi^{(s)}_{\sigma}(p_R)\,. \tag{11}$$

Under such a definition, as is clear from eq. (7) and eq. (8), the function on the group $X_{\sigma}(a)$ is independent of the choice of basis, i.e. the next formul is valid:

$$X_{\sigma}(a) = T_a\, \phi^{(s)}_{\sigma}(p_R)\,. \tag{12}$$

The function $X_{\sigma}(a)$ transforms according to the regular representation

$$T_{a_0}\, X_{\sigma}(a) = X_{\sigma}(a a_0) \tag{13}$$

and satisfies the following covariance constraint on the left cosets of the SU(2) subgroup:

$$X_\sigma(ua)=\sum_{\sigma'=-s}^{s} D^{(s)}_{\sigma\sigma'}(u)\, X_{\sigma'}(a),\quad u\in SU(2). \qquad (14)$$

From the existence of the analog of the Plancherel theorem for the SL(2,C) group follows the existance of the formulae, by which the right regular transformation (13) is expanded into the direct sum of irreducible unitary infinite-dimensional representations [9]:

$$X_\sigma(a)=\frac{1}{(2\pi)^4}\sum_{m=-\infty}^{\infty}\int_0^{\infty} d\rho\,(\rho^2+m^2)\int_{SU(2)} d\mu(u)\,\overline{d_{m\rho}(K_{a,u})}\,K_\sigma(u,u_{a,u}|m\rho) \qquad (15)$$

and

$$K_\sigma(u,v|m\rho)=\pi\int_K d\mu_\ell(K)\,d_{m\rho}(K)\,X_\sigma(u^{-1}K\,v), \qquad (16)$$

where for any $a\in$ SL(2,C)

$$d_{m\rho}(a)=|a_{22}|^{-m+i\rho-2}\,a_{22}^{m}, \qquad (17)$$

$d\mu(u)$ is the invariant measure on the SU(2) group,
$d\mu_\ell(K)$ is left-invariant measure on the subgroup of triangular matrices:

$$K=\begin{pmatrix}\lambda^{-1} & \mu\\ 0 & \lambda\end{pmatrix} \qquad \lambda,\mu \text{ — complex number} \qquad (18)$$

and $K_{a,u}$ and $\mathcal{U}_{a,u}$ are determined from matrix equation

$$u\,a = K_{a,u}\,\mathcal{U}_{a,u}. \tag{19}$$

From (16) it follows the following property of the function $K_\sigma(u, v/m\varsigma)$

$$K_\sigma(\gamma_1 u, \gamma_2 v/m\varsigma) = e^{im(\gamma_1-\gamma_2)} K_\sigma(u, v/m\varsigma), \tag{20}$$

where the diagonal matrices

$$\gamma = \begin{pmatrix} e^{i\gamma/2} & 0 \\ 0 & e^{-i\gamma/2} \end{pmatrix} \tag{21}$$

describe the rotation around the Z axis at γ angle.

Replacing in (15) the variables: $u \to \mathcal{U}_{a,u}$ and using the equality

$$d\mu(u) = |\alpha_{m\varsigma}(K_{a,u})|^{-2} d\mu(\mathcal{U}_{a,u}) \tag{22}$$

we arrive at

$$\chi_\sigma(a) = \frac{1}{(2\pi)^4} \sum_{m=-\infty}^{\infty} \int_0^\infty d\varsigma(\varsigma^2+m^2) \int_{SU(2)} d\mu(u)\, \alpha_{m\varsigma}(K_{a^{-1},u})\, K_\sigma(\mathcal{U}_{a^{-1},u}, u/m\varsigma). \tag{23}$$

The property (14) allows to represent eq. (22) and eq. (23) in the form

$$X_\sigma(a) = \frac{1}{(2\pi)^4} \sum_{m=-\infty}^{\infty} \int_0^\infty \rho\, d\rho\, (\rho^2 + m^2) \int d\mu(u)\, \alpha_{m\rho}(K_{a^{-1}u}) \sum_{\sigma'=-s}^{s} \overset{+}{D}{}^{(s)}_{\sigma\sigma'}(u_{a^{-1}u}) \times \tag{24}$$

$$\times\ K_{\sigma'}(u|m\rho)$$

$$K_\sigma(u|m\rho) = \pi \int_K d\mu_\ell(k)\, \alpha_{m\rho}(k)\, X_\sigma(ku). \tag{25}$$

From eq. (24) it is easy to obtain an expansion for the wave function, because by definition (11)

$$X_\sigma(H_p^{-1}) = \Psi_\sigma^{(s)}(p). \tag{26}$$

Now let us note that $\Psi_\sigma^{(s)}(p)$, as a function on the hyperboloid, depends on three parameters. In order to perform in eq. (24) the integration over an extra parameter let us consider the solution of the matrix equation:

$$u\, H_p = K_{H_p,u}\, u_{H_p,u}. \tag{27}$$

It is known that from this equation the matrices $u_{H_p,u}$ and $K_{H_p,u}$ are determined nonuniquely. However it is possible to write the solution of eq. (27) in such a manner that under the sign of integration the nonuniqueness is contained only in K_{12} (it can be seen from eq. (15) and eq. (17) that the element K_{12} plays no role). In this case, it is possible in the right-hand side of eq. (24) to pass to integration over the angular parameters of the unit vector, which is a spatial part of the four-vector $n=(1,\underline{n})$, belonging to the cone. Finally the expansion for the wave function of the particle with spin S in the canonical basis takes the form

$$\Psi^{(s)}_{\sigma}(p)=\sum_{\sigma'=-s}^{s}\int_0^{\infty} dr\,(r^2+\sigma'^2)\int d\omega_{\underline{n}}\,(pn)^{-1+ir}\,\overset{+}{D}{}^{(s)}_{\sigma\sigma'}(u_{p,r})\,\Psi^{(s)}_{\sigma'}(r)\,. \tag{28}$$

The matrix $u_{p,r}$, which stands in the argument of function, is determined from the equation:

$$u_r\,H_p = K_{p,r}\,u_{p,r}\,, \tag{29}$$

where the matrix u_r corresponds to the rotation of the vector $\underline{n}_0=(0,0,1)$ in the direction of the vector $\underline{n}$:

$\overset{+}{u}_r\,\underline{\sigma}\cdot\underline{n}_0\,u_r = \underline{\sigma}\cdot\underline{n}$

and

$$(u_{p,r})_{11}=\frac{(1+p^0-p^3)\cos\frac{\theta_r}{2}-p^+\sin\frac{\theta_r}{2}e^{-i\varphi_r}}{\sqrt{2(p^0+1)(p^0-\underline{p}\cdot\underline{n})}}\,,\quad (u_{p,r})_{12}=\frac{(1+p^0+p^3)\sin\frac{\theta_r}{2}e^{-i\varphi_r}-p^-\cos\frac{\theta_r}{2}}{2(p^0+1)(p^0-\underline{p}\cdot\underline{n})}\,. \tag{30}$$

In order to obtain the formula , inverse to (28), let us make use of the fact that any triangular matrix K with the element $K_{22}=|\lambda|e^{i\alpha}$ can be represented in the form $K=\gamma(\alpha)K_r$, where for the triangular matrix K_r the condition $\arg(K_r)_{22}=0$ is valid. In this case

$d\mu_e(K) = d\alpha \, d\mu_e(K_r)$. Let us put $u = u_r$ in eq. (25). Then, taking into consideration (14), we represent it in the form:

$$K_\sigma(u_r|m_\xi) = 2\pi^2 \delta_{\sigma, m/2} \int d\mu_e(K_r) \overline{\alpha_{m_\xi}(K_r)} \, X_\sigma\left(K_r^{-1} u_r\right) , \qquad (31)$$

where we have used the equality $\alpha_{m_\xi}(K) d\mu_e(K) = \overline{\alpha_{m_\xi}(K^{-1})} d\mu_e(K^{-1})$.
Next, let us introduce into eq. (31) new integration variables using the equality $K_r^{-1} u_r = u_{p,r} H_p^{-1}$.
Then, taking into account eq. (14), which leads to

$$X_\sigma(u_{p,r} H_p^{-1}) = \sum_{\sigma'=-s}^{s} D^{(s)}_{\sigma\sigma'}(u_{p,r}) \, X_{\sigma'}(H_p^{-1})$$

and after it is easy to arrive at:

$$\psi_\sigma^{(s)}(\underline{r}) = \frac{1}{(2\pi)^3} \sum_{\sigma'=-s}^{s} \frac{d^3 \underline{p}}{p^0} \left(pn\right)^{-1-ir} D^{(s)}_{\sigma\sigma'}(u_{p,r}) \, \psi^{(s)}_{\sigma'}(\underline{p}), \qquad (32)$$

We shall consider the formulas (28) and (32) as a transition from momentum representation, to relativistic configurational representation, introduced in [1] , in which the modulus of the radius vector $\underline{r} = r\underline{n}$ is defined as $r = \lambda/2$.
The matrix $u_{p,r}$ according to (29) can be represented in the form: $u_{p,r} = v_{p,r} \, u_p$,
where $v_{p,r}$ can be found from the matrix equation

$$u_r \, u_p^{+} \, \varepsilon_p = K_{p,r} \, v_{p,r} \qquad (33)$$

Thus taking into account (9), connecting the helicity and canonical basis, we get from eq. (29) and eq. (32) the expansion for the wave function in the helicity basis

$$\Psi_\lambda^{(s)}(\underline{r}) = \frac{1}{(2\pi)^3} \sum_{\gamma=-s}^{s} \int \frac{d^3 p}{p^0} (pn)^{-1-ir} D_{\lambda\gamma}^{(s)}(v_{p,r}) \phi_\gamma^{(s)}(p), \tag{34}$$

$$\phi_\lambda^{(s)}(p) = \sum_{\gamma=-s}^{s} \int_0^\infty dr\,(r^2+\nu^2) \int d\omega_{\underline{n}} (pn)^{-1+ir} \overset{+}{D}{}_{\lambda\gamma}^{(s)}(v_{p,r})\, \Psi_\gamma^{(s)}(\underline{r}) \,. \tag{35}$$

The unitary matrix $v_{p,r}$, as it follows from eq. (33), has the following values for its elements:

$$v_{22} = \frac{(u_r \overset{+}{u}_p)_{22}\, e^{-\chi/2}}{\sqrt{(pn)}} \quad , \quad v_{21} = \frac{(u_r \overset{+}{u}_p)_{21}\, e^{\chi/2}}{\sqrt{(pn)}} . \tag{36}$$

It is clear from this expression, that the rotation angle defined by the matrix $v_{p,r}$ depends only on scalar product of the vectors $\underline{p}$ and $\underline{n}$:

$$\cos\theta_v = |v_{22}|^2 - |v_{21}|^2 = \frac{p^0 \cos\theta_{pn} - p}{pn}, \tag{37}$$

where $\cos\theta_{pn} = \dfrac{\underline{p}\cdot\underline{n}}{|\underline{p}|}$.

It should be stressed, that the kernel of the transformation (32) automatically contains $\mathcal{D}^{(s)}$ function, which transforms the wave function from the canonical basis to the helicity one, then in fact the expansion for the wave function is performed only in the helicity basis. Thus, as the wave function $\psi_{\gamma}^{(s)}(\underline{r})$ index $\gamma = \frac{m}{2}$ in the configurational representation is the eigenvalue of the helicity operator on the cone [12], the indices λ and γ of the wave functions are eigenvalues of helicity.

II

The quasipotential equation for the wave function, describing the relative motion of particle with spin S and spinless particle, has in the c.m.s. the following form [11, 12]:

$$(2E_q - 2E_p)\psi^{(s)}_{q\sigma}(\underline{p}) = \frac{1}{(2\pi)^3}\sum_{\sigma'=-s}^{s}\int \frac{d^3k}{k^0} V_{\sigma\sigma'}(\underline{p},\underline{k};E_q)\,\psi^{(s)}_{q\sigma'}(\underline{k}) . \tag{38}$$

The wave function of the continuous spectrum $\psi_{\sigma}^{(s)}(p)$ is connected with the scattering amplitude in the c.m.s. $A_{\sigma\sigma'}(p,q)$ in the following way:

$$\psi^{(s)}_{q\sigma}(\underline{p}) = (2\pi)^3\delta(\underline{p(-)q})\chi^{(s)}_{\sigma} - 4\pi\sum_{\sigma'}\frac{A_{\sigma\sigma'}(\underline{p},\underline{q})\,\chi^{(s)}_{\sigma'}}{2E_q - 2E_p + i0} \tag{39}$$

here p and q are the momenta of the initial and final particles with equal masses in the c.m.s. and $\chi_\sigma^{(s)}$ is a normalized $(2s+1)$ component spinor in the canonical basis.

The expression $(2\pi)^3\delta(p(-)q)\chi_\sigma^{(s)}$ in eq. (39) describes free motion. On transforming eq. (32) to the configurational representation, this term gives the "plane wave" with spin

$$\xi_\gamma^{(s)}(p,r) = \sum_{\sigma=-s}^{s} \xi_{\gamma\sigma}(p,r)\chi_\sigma^{(s)} = \tag{40}$$
$$= (pr)^{-1-ir}\sum_{\sigma=-s}^{s} D_{\gamma\sigma}^{(s)}(u_{p,r})\chi_\sigma^{(s)}.$$

From this expression and the connection of the spinor $\mathcal{E}_\lambda^{(s)}(p)$ in helicity basis with the spinor $\chi_\sigma^{(s)}$ in canonical basis

$$\mathcal{E}_\lambda^{(s)}(p) = \sum_{\sigma=-s}^{s} D_{\lambda\sigma}^{(s)}(u_p)\,\chi_\sigma^{(s)}$$

We see that in this case as well, the kernel of transformation performs the "rotation" of the canonical basis to the helicity one, and then eq. (40) takes the form:

$$\xi_\gamma^{(s)}(p,r) = (pr)^{-1-ir}\sum_{\lambda=-s}^{s} D_{\gamma\lambda}^{(s)}(v_{p,r})\,\mathcal{E}_\lambda^{(s)}(p). \tag{41}$$

In the nonrelativistic limit $\xi_\gamma(p,r)$ turns into the plane wave: $e^{i\,p.r}\,\mathcal{E}_\lambda(p)$

which describes a free motion of nonrelativistic particle with spin S.

Let us emphasize, that just in the helicity basis the expansion for, the wave function is an analog of the quantum mechanical expansion in the plane waves $e^{i\underline{p}.\underline{r}}$ because in this basis the kernel

$$\xi_{\sigma\lambda}(\underline{p},\underline{r}) = (pn)^{-1-ir} D^{(s)}_{\sigma\lambda}(V_{pr})$$

contains the rotation by an angle, which (as it follows from eq. (37)) depends on three dimensional scalar product of the vectors $\underline{p}$ and $\underline{n}$ only. The kernel of the transformations (28), (32) is not the function of the scalar product of $\underline{p}$ and $\underline{n}$, because it contains, as is clear from eq. (29) also the rotation up from the $\underline{p}$, to the Z axis, along which the projection of spin is determined in the canonical basis. For this reason it cannot serve as an analog of the plane wave.

III

In order to construct a quasi-potential in the relativistic configurational representation, we need an addition theorem for the plane waves (40) [x], analogous to that for nonrelativistic plane waves $e^{i\underline{p}.\underline{r}}\left(e^{i\underline{k}.\underline{r}}\right)^{+} = e^{i(\underline{p}-\underline{k}).\underline{r}}$.

For this purpose let us notice, that the substitution into the right-hand side of the equality (1) of the expansion

[x] For the spinless case such an addition theorem has been obtained in [1].

allows one to represent the latter in the form

$$T_{H_K} \psi_\sigma^{(s)}(\underline{p}) = \sum_{\sigma'=-s'}^{s'} D_{\sigma\sigma'}^{(s')} \{ V(H_K, p) \} \times \tag{42}$$

$$\times \sum_{\sigma''=-s'}^{s'} \int_0^\infty dr (r^2 + \sigma''^2) \int d\omega_{\underline{n}} \, \xi^{+}(\underline{p(-)K}, r) D_{\sigma'\sigma''}^{(s')}(u_{p(-)K, r}) \psi_{\sigma''}^{(s')}(\underline{r}),$$

where, according [1], the notation

$$\xi(\underline{p}, \underline{r}) = (pn)^{-1-ir} = (p^0 - \underline{p} \cdot \underline{n})^{-1-ir}$$

is introduced. The four-vector $n = (1, \underline{n})$ belongs to the cone $n^2 = 0$. In the left term of eq. (1) we make use of the fact [9], that under the transformations (13) the function $K(u, v/m_g)$ in eq. (15) transforms by the irreducible representation:

$$T_a^{[mg]} K(u, v/mg) = \overline{\alpha_{mg}(Ka, v)} \, K(u, V_{a,v}/mg). \tag{43}$$

To this one at $a = H_K$ there corresponds the transformation in the configurational representation of the following type:

$$T_{H_K}^{[\nu, r]} \psi_\gamma^{(s)}(\underline{r}) = (\kappa n)^{-1-ir} \psi_\gamma^{(s)}(\underline{r}_K), \tag{44}$$

where $\underline{r}_K = r \, \underline{n}_K$, and the unit vector $\underline{n}_K$ is

$$\underline{n}_K = \frac{1}{\kappa n} \left(\underline{n} - \underline{K} \frac{1 + \kappa n}{1 + K^0} \right). \tag{45}$$

Next,

$$T_{H_k}\, \psi^{(s)}_{\sigma}(\underline{p}) = \sum_{\sigma'=-s}^{s} \int_0^{\infty} dr\,(r^2+\sigma'^2) \int d\omega_{\underline{n}}\, \xi^{+}_{\sigma\sigma'}(\underline{p},\underline{r})\, \xi(\underline{k},\underline{r})\, \psi^{(s)}_{\sigma'}(\underline{r}_k) \tag{46}$$

and after appropriate substitutions:

$$\begin{aligned}\sum_{\sigma'=-s}^{s} \int d\omega_{\underline{n}}\, \xi^{+}_{\sigma\sigma'}(\underline{p},\underline{r})\, \xi_{\sigma'\sigma'''}(\underline{k},\underline{r}) &= \\ = \sum_{\sigma',\sigma''=-s}^{s} D^{(s)}_{\sigma\sigma'}\{V(H_k,p)\} \int d\omega_{\underline{n}}\, \xi^{+}_{\sigma'\sigma''}(\underline{p(-)k},\underline{r})\, D^{(s)}_{\sigma''\sigma'''}(u_r),\end{aligned} \tag{47}$$

from where we can easily derive the addition theorem for the plane waves with spin. Since according to the results of the kernel of eq. (28) automatically performs a transformation from the canonical basis to helicity one, then we can deal with eq. (37) for the wave function $\psi^{(s)}_{\sigma}(p)$ in the canonical basis. In this case for constructing local quasipotential in Γ- space we need the addition theorem just in the form (46).

IV.

It follows from eq. (47), that for a quasipotential to be local in the Γ- space it should be in the momentum representation of the form:

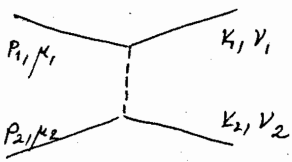
$$V_{\sigma\mu}(\underline{p},\underline{k};E_q) = \sum_{\sigma'=-s}^{s} D^{(s)}_{\sigma\sigma'}\{V(H_k,p)\}\, V_{\sigma'\mu}\left(\underline{p(-)k}\,;E_q\right) \tag{48}$$

Therefore the question arises whether it is possible to construct a quasipotential of such a type from any field theory matrix element.

For example, we consider an interaction of two particles with spin 1/2 in the case:

$$H_{int}(x) = g : \bar{\Psi}(x)\gamma^5\Psi(x)\,\varphi(x): ,$$

where the spinor field $\Psi(x)$ can describe the nucleons and antinucleons with mass m, and $\varphi(x)$ corresponds to a pseudoscalar meson with mass μ. In accordance with the general rules for constructing the quasipotential given on [3,11] in second order in g, the quasipotential is proportional to the diagram



In the c.m.s. $\underline{p}_1 = -\underline{p}_2 = \underline{p}$, $\underline{k}_1 = -\underline{k}_2 = \underline{k}$

the diagram gives

$$V^{\mu_1\nu_1,\mu_2\nu_2}(\underline{\mathbf{p}},\underline{\mathbf{k}};E_q) = g^2\,\frac{[\bar{u}^{\mu_1}(p_1)\gamma^5 u^{\nu_1}(k_1)][\bar{u}^{\mu_2}(p_2)\gamma^5 u^{\nu_2}(k_2)]}{(p_1-k_1)^2-\mu^2}. \quad (49)$$

All the momenta are on the mass shell $p^2 = m^2$ but are off the energy shell $p^0 \neq k^0$. Therefore, the denominator eq. (49) can be transformed to the form:

$$(p-k)^2-\mu^2 = 2m^2-\mu^2-2pk = \qquad (50)$$
$$= 2m^2-\mu^2-m\sqrt{m^2+(\underline{p}(-)\underline{k})^2}$$

i.e. it is a function only of the distance on hyperboloid. The spinor part of eq. (48) can be written in the form:

$$\bar{u}(\underline{p})\gamma^5 u(\underline{k}) = \bar{u}(0)\gamma^5 S_p^{-1} S_k\, u(0) = \qquad (51)$$
$$= \bar{u}(0)\gamma^5 D^{(1)}\{V(H_k,p)\} S_{p(-)k}\, u(0) \;,$$

where

$$\gamma^5 = -i\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

$$S_p = \begin{pmatrix} H_p & 0 \\ 0 & H_p^{+\,-1} \end{pmatrix} \text{ for } \gamma^\mu = \begin{pmatrix} 0 & g^{\mu\mu}\sigma^\mu \\ \sigma^\mu & 0 \end{pmatrix} .$$
$$(\mu = 0,1,2,3)$$

Thus we see that the expressions (50) and (51) permit to obtain from eq. (49) a quasipotential of the form (48), on passing to the two-component spinors χ_σ and ξ_μ (see [11]).

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Appendix

For completeness we give the orthogonality relations for kernel $\xi_{\sigma\sigma'}(p,r)$ in both momentum and configurational spaces:

$$\frac{1}{(2\pi)^3}\sum_{\sigma'=-s}^{s}\int_0^\infty dr\,(r^2+\sigma'^2)\int d\omega_{\underline{n}}\;\xi^{+}_{\sigma\sigma'}(p,r)\,\xi_{\sigma'\sigma''}(p',r)=p^0\delta(\underline{p}-\underline{p}')\,\delta_{\sigma\sigma'}$$

$$\frac{1}{(2\pi)^3}\sum_{\sigma'=-s}^{s}\int\frac{d^3p}{p^0}\;\xi_{\sigma\sigma'}(p,r)\,\xi^{+}_{\sigma'\sigma''}(p,r')=\frac{r^2}{\sigma^2+r^2}\,\delta(\underline{r}-\underline{r}')\,\delta_{\sigma\sigma''}$$

which follow from the Plancherel formula in [9] and (29) and (32). It is obvious that our formulae in the zero spin case are the same as the formulae (1.6) and (1.7) in [1].

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