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ON AUTOMODEL ASYMPTOTIC IN QUANTUM FIELD THEORY II

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Section 1

At present, in a number of experiments on deeply inelastic scattering of leptons on nucleons, one has observed a point-like behaviour of the form factors at high energies and large momentum transfers. A possibility of such a behaviour was first anticipated in the work by M.A.Markov/1/, nevertheless up to the present time there was no correct theoretical grounds for this phenomenon in the framework of the general principles of local quantum field theory.

Investigations of this kind are indeed very important for studying the behaviour of the matrix elements off the energy shell while in the case of elastic scattering we are dealing with the matrix elements only on the energy shell. Therefore, the theoretical explanation of the experimental results on deeply inelastic scattering on the basis of the general principles of local quantum field theory would lead to an additional check of these principles in the same way as in the case of dispersion relations for elastic scattering.

In the present paper, taking as an example the deeply inelastic electron-nucleon scattering, we develop the method of studying the asymptotic behaviour of the form factors which has been suggested in our paper/14/. This method is directly generalized to the cases of deeply inelastic lepton-hadron scattering processes.

Thus, we consider a deeply inelastic electron-nucleon scattering process:

$$e + N \longrightarrow e + \cdots$$

As is known, the appropriate cross sections are determined by means of the Fourier transform of the commutators

$$W_{\mu\nu} = \frac{1}{8\pi} \sum_{(\sigma)} \int \langle p,\sigma | [j_\mu(x), j_\nu(0)] | p,\sigma \rangle \, e^{iqx} dx \,, \qquad (1.1)$$

in which j_ν are the electromagnetic current components, q is the four-momentum of a virtual photon; $q^2 < 0$, and the matrix elements are taken between identical one-nucleon states $|p,\sigma\rangle$ with the four-momentum p of mass $1 (p^2 = 1)$ and spin $\sigma \left(\sigma = \pm\frac{1}{2}\right)$. We use here the usual relativistic normalization of the nucleon state

$$\langle p,\sigma | p',\sigma' \rangle = 2p^0 (2\pi)^3 \delta(\vec{p} - \vec{p}') \, \delta_{\sigma\sigma'} \,.$$

The asymptotic properties are studied in the domain

$$|q^2| \rightarrow +\infty \,, \quad \nu = 2qp \rightarrow +\infty \,; \quad \xi = \frac{-q^2}{2qp} = \text{const.} \qquad (1.2)$$

(to the physical domain (1.2) there corresponds $\xi > 0$).

The first results in these lines were obtained by Bjorken/2/ on the basis of the current algebra, and then were developed in refs./3/ In refs./4/ deeply inelastic scattering of leptons by hadrons and hadrons by hadrons was studied on the basis of the generalized automodelity principle and the dimension analysis.

Here we investigate the asymptotic behaviour of the form factors for the expressions (1.1) on the basis of the general principles of local quantum field theory[7,8]. Conditions will be indicated under which, in the asymptotic domain (1.2) there is observed the automodel behaviour of the form factors. There will also be established a relationship between the character of the automodel behaviour and the asymptotic in the vicinity of the light cone. Note that in refs.[5] this problem has been studied by an analogous approach, under very restricted subsidiary conditions.

It should be noted that the study of the asymptotic behaviour of inclusive processes on the basis of the general principles of local quantum field theory has been performed by A.A.Logunov and co-workers[9].

The tensor $W_{\mu\nu}(q,p)$ is usually expressed, taking into account the conservation law of electromagnetic current, in terms of two invariant functions, form factors $W_1(q,p)$ and $W_2(q,p)$:

$$W_{\mu\nu}=\left(-g_{\mu\nu}+\frac{q_\mu q_\nu}{q^2}\right)W_1+\left(p_\mu-\frac{qp}{q^2}q_\mu\right)\left(p_\nu-\frac{qp}{q^2}q_\nu\right)W_2 \quad (1.3)$$

The problem is to find the asymptotic of the form factors W_j in the domain (1.2) and relate it to the behaviour of the current commutators (1.1) in the vicinity of the light cone.

To this end, using new form factors*:

$$V_1 = \frac{1}{q^2}\left[W_1 + \frac{(qp)^2}{q^2} W_2\right] , \quad V_2 = \frac{W_2}{q^2} \tag{1.4}$$

we write the representation (1.3) in a "local" form:

$$\begin{aligned} W_{\mu\nu} = & (-g_{\mu\nu} q^2 + q_\mu q_\nu) V_1 + \\ & + \left[p_\mu p_\nu q^2 - (p_\mu q_\nu + q_\mu p_\nu) qp + g_{\mu\nu} (qp)^2\right] V_2 \ . \end{aligned} \tag{1.5}$$

In so doing, one usually starts from the assertion that the introduced functions $V_j(q,p)$ are causal[5], i.e. their Fourier transforms vanish for $x^2 < 0$. However, this assertion has not been reliably proved yet[6]. In particular, in ref.[6] one uses commutation relations between currents which do not follow from the general principles of quantum field theory. In our paper we prove that the functions V_j and, consequently, the functions W_j are causal (Appendix I).

To solve our problems it is convenient to introduce new invariant causal functions

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*For the exact definition of the functions V_1 and V_2 see Appendix I.

$$F_1(q,p) = \sum g_{\mu\mu} g_{\nu\nu} p_\mu p_\nu W_{\mu\nu} = p^\mu p^\nu W_{\mu\nu} , \tag{1.6}$$

$$F_2(q,p) = F_1(q,p) - \sum g_{\mu\mu} W_{\mu\mu} \qquad (g^{00} = +1) \tag{1.7}$$

Taking into consideration relativistic invariance of the functions $F_j(q,p)$, $V_j(q,p)$ and $W_j(q,p)$ we pass to a frame of reference in which $p = (1,\vec{0})$. Then, on the basis of (1.3)-(1.7), we have the following relations between the functions introduced

$$W_1 = q^2 V_1 - q_0^2 V_2 , \qquad W_2 = q^2 V_2 ; \tag{1.8}$$

$$F_1 = \frac{\vec{q}^2}{q^2}\left(W_1 + \frac{\vec{q}^2}{q^2} W_2\right), \quad F_2 = \left(3 + \frac{\vec{q}^2}{q^2}\right) W_1 + \frac{q_0^2 \vec{q}^2}{(q^2)^2} W_2 ; \tag{1.9}$$

$$W_1 = -\frac{q_0^2}{2\vec{q}^2} F_1 + \frac{F_2}{2} , \quad W_2 = \frac{q^2}{2\vec{q}^2}\left[\left(\frac{q_0^2}{\vec{q}^2} + 2\frac{q^2}{\vec{q}^2}\right) F_1 - F_2\right] ; \tag{1.10}$$

$$F_1 = \vec{q}^2 (V_1 - V_2), \quad F_2 = 3 q_0^2 (V_1 - V_2) - 2\vec{q}^2 V_1 ; \tag{1.11}$$

$$V_1 = \frac{1}{2\vec{q}^2}\left(\frac{3q_0^2}{\vec{q}^2} F_1 - F_2\right), \quad V_2 = V_1 - \frac{F_1}{\vec{q}^2} ; \tag{1.12}$$

$$W_{0\nu} = \frac{q_0 q_\nu}{\vec{q}^2} F_1 = q_0 q_\nu (V_1 - V_2), \quad \nu = 1,2,3 ; \tag{1.13}$$

$$W_{ij} = \left(\delta_{ij} + \frac{q_i q_j}{q^2}\right) W_1 + \frac{q_0^2}{q^2}\frac{q_i q_j}{q^2} W_2 , \quad i,j = 1,2,3 ; \tag{1.14}$$

$$W_{ij} = \frac{q_i q_j}{2\vec{q}^2}\left(\frac{3q_0^2}{\vec{q}^2} F_1 - F_2\right) = q_i q_j V_1 , \quad i \neq j ; \tag{1.15}$$

$$F_1 = W_{00} , \quad F_2 = \sum_{i=1}^{3} W_{ii} . \tag{1.16}$$

In what follows, $\widetilde{F}$ will denote the Fourier transform F ,

In an arbitrary frame of reference, the functions W_j and V_j are expressed in terms of the functions F_j by formulas:

$$W_1 = \frac{F_1}{2(1+4\xi/\nu)} + \frac{F_2}{2}, \tag{1.17}$$

$$W_2 = \frac{1}{2(1+\nu/4\xi)^2}\left[\left(2-\frac{\nu}{4\xi}\right)F_1 + \left(1+\frac{\nu}{4\xi}\right)F_2\right];$$

$$V_1 = \frac{2}{\nu^2(1+4\xi/\nu)}\left(\frac{3}{1+4\xi/\nu}F_1 - F_2\right), \quad V_2 = V_1 - \frac{4F_1}{\nu^2(1+4\xi/\nu)}. \tag{1.18}$$

We enumerate the main particular features of the functions introduced. We denote $F_j(q) = F_j(q,1,\vec{0})$, $W_j(q) = W_j(q,1,\vec{0})$ and $V_j(q) = V_j(q,1,\vec{0})$. It follows from (1.16), (1.10) and (1.12) that these functions are odd with respect to q_0 , radially symmetric, i.e. they depend only upon q_0 and $|\vec{q}|$, and vanish in the domain $-q^2/2|q_0| > 1$ (spectrality condition); in addition

$$\epsilon(q_0)\,F_j(q) \geqslant 0 \quad \text{provided that } q^2<0 \quad (j=1,2), \tag{1.19}$$

$$\epsilon(q_0)\,W_1(q) \geqslant 0 \quad \text{provided that } q^2<0 . \tag{1.20}$$

The inequality (1.20) follows from the representation

$$W_1 = \sum_{i,j} e_i e_j W_{ij} = \frac{1}{8\pi}\sum_{(\sigma)}\int \langle\sigma|[\vec{e}\vec{j}(x), \vec{e}\vec{j}(0)]\sigma\rangle\, e^{iqx}\,dx,$$

which is valid owing to (1.14) for any unit vector orthogonal to the vector $\vec{q}$.

From the inequality (1.20), due to (1.19) we have

$$\epsilon(\nu)\left(F_2 - \frac{F_1}{1+4\xi/\nu}\right) \geqslant 0 \quad \text{if} \quad \xi > 0 . \tag{1.21}$$

Further information on these functions will be extracted from analogies with the free field (see Appendix III). In this case, for $\nu > 0$ and $\xi > 0$ we have:

$$W_1^0 = \frac{\nu}{4\xi} W_2^0 , \quad W_2^0 = \frac{1}{2}\delta(1-\xi), \quad V_1^0 = 0, \quad V_2^0 = -\frac{4}{\nu^2} W_1^0 ,$$
$$F_1^0 = \left(1+\frac{4\xi}{\nu}\right) W_1^0 , \quad F_2^0 = 3W_1^0 , \quad F_2^0 - F_1^0 = \left(2-\frac{4\xi}{\nu}\right) W_1^0 . \tag{1.22}$$

Now we assume that in the case of interacting field currents, just as in the case of free fields, the quantities F_1 and $F_2 - F_1$ in the physical domain (1.2) tend to finite and nonzero limit functions (distributions)*. Then from (1.7), in the domain (1.2) we get asymptotic equalities

$$W_1 \sim \frac{1}{2}(F_2 - F_1), \quad W_2 \sim \frac{2\xi}{\nu}(F_2 - F_1), \quad W_1 \sim \frac{\nu}{4\xi} W_2 , \tag{1.23}$$

which are analogous to the exact equalities (1.22) for free fields. Denoting in the domain (1.2)

$$F_j(\xi,\nu) \sim F_j(\xi) , \qquad j = 1,2 , \tag{1.24}$$

*In the case $F_2 - F_1 \sim 0$, owing to (1.17), $W_1 \sim 0$ in the domain (1.2); this case corresponds to the boson current structure.

and taking into aoooount (1.23), we are lead to the automodel Bjorken formulas in the physical part of (1.2)

$$W_1(\xi,\nu)\sim f_1(\xi)\,, \qquad \frac{\nu}{4}W_2(\xi,\nu)\sim f(\xi)\,, \tag{1.25}$$

where, in virtue of (1.23) and (1.21)

$$f_1(\xi)=\frac{1}{2}\left[F_2(\xi)-F_1(\xi)\right], \quad f_2(\xi)=\xi f_1(\xi)\geqslant 0. \tag{1.26}$$

It should be noted that the automodel relations (1.25) and (1.26) have been experimentally proved.

As we have seen above, for the asymptotio formulas (1.25) - (1.26) to be valid it is sufficient to require that the functions F_1 and F_2-F_1 tend to finite and nonzero limits in the physioal part of the domain (1.2). Therefore, there arises naturally the question as to what additional requirements following from the dynamios of the prooess should be imposed on the functions F_1 and F_2 which would provide the indioated behaviour of the funotions F_1 and F_2-F_1, and oonsequently, the validity of the automodel formulas (1.25) - (1.26).

On the basis of the analogy with the oase of free field (Appendix III), we formulate these additional requirements in terms of the weight functions Ψ_j whioh oorrespond to the functions $F_j\ (j=1,2)$ in the integral representation (2.3), namely we require that the funotions Ψ_j should satisfy the oondition (2.12) at $\kappa=0$, i.e.+

+It is worth noting that in perturbation theory the requirement (1.22) is not always fulfilled due to the presence of logarithmic divergences with respect to λ^2 in some terms.

$$\psi_j(|\vec{u}|, \chi^2) \longrightarrow \psi_j(|\vec{u}|), \quad \chi^2 \longrightarrow +\infty . \tag{1.27}$$

In Section 2 we show that under the condition (1.27) the functions F_j in the domain (1.2) have automodel behaviour like

$$F_j(\xi,\nu) \sim 2\pi \int_{\xi}^{1} \psi_j(|\rho|)(\rho-\xi)\,\rho\, d\rho \equiv F_j(\xi), \tag{1.28}$$

where the functions $\rho\,\psi_j(|\rho|)$ are odd continuations of the functions $\rho\psi_j(\rho)$ defined by the formula (2.21). Note that from (1.28) it follows that the functions $F_j(\xi)$ vanish when $\xi > 1$, are nonnegative when $\xi > 0$, and constant when $\xi < -1$:

$$F_j(\xi) = 2\pi \int_{-1}^{1} \psi_j(|\rho|)\, \rho^2 d\rho\,, \qquad \xi < -1. \tag{1.29}$$

Now we suppose that the functions $\psi_j(\rho)$ satisfy the condition (3.18)

$$\int_{0}^{1} \psi_j(\rho)\, \rho^2\, d\rho = 0\,, \tag{1.30}$$

i.e. in virtue of (1.29), $F_j(\xi) = 0$ at $\xi < -1$.

The equality (1.30) for the function F_1 is fulfilled (Appendix I, (I.10)); for the free field the equalities (1.29) are also fulfilled (Appendix III, formula (III.30).

The derived asymptotic behaviour (1.28) of the functions $F_j(\xi,\nu)$ in the physical part of the domain (1.2) describes unambiguously the asymptotic behaviour of their Fourier transforms $\widetilde{F}_j(x,p)$ in the vicinity of the light cone $x^2=0$.

In Section 3 we have obtained the asymptotic behaviour of the functions $\widetilde{F}_j(x,p)$ (see (3.25))

$$\widetilde{F}_j(x,p) \sim \frac{i}{\pi} Q_j(px)\frac{\partial}{\partial(px)}\mathcal{D}(x,0), \quad x^2\sim 0, \quad (1.31)$$

where, in virtue of (3.22) and (1.28)

$$Q_j(z)=2\int_0^1 F_j(\xi)\sin z\xi \, d\xi \ . \quad (1.32)$$

It follows immediately from (1.18) and (1.28) the asymptotic of the functions V_j in the physical part of the domain (1.2):

$$V_1(\xi,\nu)\sim\frac{2}{\nu^2}\left[3F_1(\xi)-F_2(\xi)\right], \quad V_2(\xi,\nu)\sim\frac{2}{\nu^2}\left[F_1(\xi)-F_2(\xi)\right] \quad (1.33)$$

Now we assume that the weight functions $\psi^j(\vec{u},\lambda^2)$ corresponding to the functions V_j obey the condition (2.28) for a certain s. Then the comparison of the asymptotics (1.33) and (2.29) yields $s=1$ and

$$\pi\xi\psi_0^1(\xi)=3F_1(\xi)-F_2(\xi), \quad \pi\xi\psi_0^2(\xi)=F_1(\xi)-F_2(\xi). \quad (1.34)$$

The appropriate behaviour of $\widetilde{V}_j$ in the vicinity of the light cone is given by the formula (3.27) at $S=1$:

$$\widetilde{V}_j(x,p) \sim \frac{i}{16\pi^2}\,\epsilon(px)\,\Theta(x^2)\,x^2\,\mathcal{G}_j(px), \qquad (1.35)$$

where the functions $\mathcal{G}_j(\mathfrak{X})$ are expressed in terms of $\psi_0^j(\xi)$ by the formula (3.14):

$$\mathcal{G}_j(\mathfrak{X}) = 4\pi \int_0^1 \Phi_1^j(\xi)\cos \mathfrak{X}\xi\, d\xi\,, \quad \Phi_2^j(\xi) = \int_\xi^1 \rho\, \psi_0^j(\rho)\, d\rho\,. \qquad (1.36)$$

The behaviour of $\widetilde{W}_{\mu\nu}$ in the vicinity of $x^2=0$, owing to (1.5) and (1.35), is as follows

$$\widetilde{W}_{\mu\nu}(x,p) \sim \frac{i}{16\pi^2}\left(g_{\mu\nu}\Box - \frac{\partial^2}{\partial x_\mu \partial x_\nu}\right)\mathcal{G}_1(px)\,\epsilon(px)\,\Theta(x^2)\,x^2 +$$
$$+ \frac{i}{16\pi^2}\left[-p_\mu p_\nu \Box + \left(p_\mu \frac{\partial}{\partial x_\nu} + p_\nu \frac{\partial}{\partial x_\mu}\right)\left(p\frac{\partial}{\partial x}\right) - \qquad (1.37)$$
$$- g_{\mu\nu}\left(p\frac{\partial}{\partial x}\right)^2\right]\mathcal{G}_2(px)\,\epsilon(px)\,\Theta(x^2)\,x^2\,.$$

<u>Section 2</u>

Here we shall investigate the asymptotic behaviour of the tempered distribution F in the domain

$$|q^2| \to +\infty,\ \nu = 2qp \to +\infty;\ \frac{-q^2}{2qp} = \xi = \text{const}, \qquad (2.1)$$

satisfying the following conditions (see Section 1)

I. $F(q,p) = -F(-q,p)$,

II. $F(q,p) = 0$ if $-q^2/|2qp| > 1 \quad (p^2 = 1)$,

III. $\widetilde{F}(x,p) = 0$ if $x^2 < 0$,

IV. $F(\Lambda q, \Lambda p) = F(q,p)$ for all $\Lambda \in L_+^\uparrow$.

In virtue of property IV, it is sufficient to consider our problem in the rest system, where $p = (1,\vec{0})$. In this frame of reference, the functions

$$\widetilde{F}(x) \equiv \widetilde{F}(x,1,\vec{0}) \quad \text{and} \quad F(q) \equiv F(q,1,\vec{0})$$

depend only upon $x_0, |\vec{x}|$ and $q_0, |\vec{q}|$, respectively, and obey the following conditions:

I'. $F(q) = -F(-q)$,

II'. $F(q) = 0$ if $-q^2/|2q_0| > 1$,

III'. $\widetilde{F}(x) = 0$ if $x^2 < 0$,

IV'. $F(q) = F(q_0, |\vec{q}|)$.

The asymptotic domain (2.1) assumes the form

$$|q^2| \to +\infty, \; \nu = 2q_0 \to +\infty; \; \frac{-q^2}{2q_0} = \xi; \; q_0 \sim |\vec{q}|. \tag{2.2}$$

For the function $F(q)$ satisfying conditions I', II' and III' there exists the unique tempered distrubition $\Psi(\vec{u},\varkappa^2)$ (see Appendix II) such that the well-known integral Jost-Lehmann-Dyson representation/10/ (see also ref./11/) is valid

$$F(q)=\int \epsilon(q_0)\,\delta[q_0^2-(\vec{q}-\vec{u})^2-\varkappa^2]\,\psi(\vec{u},\varkappa^2)\,d\vec{u}\,d\varkappa^2\,, \tag{2.3}$$

the support ψ being contained in the manifold

$$\left[(\vec{u},\varkappa^2):\ |\vec{u}|\leq 1,\ \varkappa^2 \geqslant (1-\sqrt{1-\vec{u}^2})^2\right]. \tag{2.4}$$

It follows from the property IV' that the weight function $\psi(\vec{u},\varkappa^2)$ depends on $\vec{u}$ only via $|\vec{u}|$ so that

$$\psi(\vec{u},\varkappa^2)\equiv\psi(|\vec{u}|,\varkappa^2).$$

The representation (2.3) is now rewritten in the form

$$F(q)=2\pi\epsilon(q_0)\int_0^\infty d\varkappa^2\int_0^1 \rho^2 d\rho\,\psi(\rho,\varkappa^2)\int_{-1}^{1} d\mu\,\delta(q^2-\rho^2+2\rho|\vec{q}|\mu-\varkappa^2).$$

In terms of the variables (ξ,ν) this representation takes the form

$$F(\xi,\nu)=\frac{2\pi}{\nu}\int_0^\infty d\varkappa^2\int_0^1 \rho^2 d\rho\,\psi(\rho,\varkappa^2)\int_{-1}^{1} d\mu\,\delta\left(\xi+\frac{\rho^2+\varkappa^2}{\nu}-\rho\mu\sqrt{1+\frac{4\xi}{\nu}}\right). \tag{2.5}$$

It follows from the condition II' that

$$F(\xi,\nu)=0 \quad \text{if} \quad \xi>1\,. \tag{2.6}$$

We stress that the above properties of the weight function $\psi(\rho,\varkappa^2)$ do not provide, generally speaking, a definite asymptotic behaviour of the r.h.s. of (2.5) in the domain (2.2). Our task is to find the conditions on this function which would ensure a definite asymptotic behaviour of the function $F(\xi,\nu)$ in the domain (2.2).

As the example of free field shows (Appendix III) this asymptotic may be expressed in terms of the distributions of ξ . Therefore we shall look for a **weak** limit of the sequence $F(\xi,\nu)$, $\nu \to +\infty$, i.e. the limit of "integrals" $\int F(\xi,\nu) f(\xi) d\xi$ at $\nu \to +\infty$ for all the finite test functions $f(\xi)$. To solve this problem we shall extensively use the technique of the small parameter.

Let $f(\xi)$ be an arbitrary infinitely differentiable finite function. Multiplying the equality (2.5) by $f(\xi)$ and "integrating" over ξ for all $\nu > 0$ we get*

$$\int F(\xi,\nu) f(\xi) d\xi =$$

$$= \frac{2\pi}{\nu} \int_0^\infty d\lambda^2 \int_0^1 \rho^2 d\rho \, \Psi(\rho,\lambda^2) \int_{-1}^1 d\mu \int d\xi \, f(\xi) \, \delta\left(\xi - \rho\mu + \frac{\lambda^2}{\nu} + \frac{\chi}{\nu}\right), \qquad (2.7)$$

where

$$\chi = \chi(\rho,\mu,\xi,\nu) = \rho^2 - \rho\mu\nu\left(\sqrt{1+4\xi/\nu} - 1\right). \qquad (2.8)$$

For rather large ν the function χ analytic in all the arguments is expanded into a series in inverse powers ν which converges uniformely together with all the derivatives with respect to all the considered arguments (under the condition that ρ, μ, ν changes in any bounded

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* The equality (2.7) may be regarded as the definition of the distribution $F(\xi,\nu)$ for each $\nu > 0$.

domain). We note that the integration over λ^2 in (2.8) is actually performed over the finite interval which is determined by the support of the function $f(\xi)$ from the condition $\xi = \rho\mu - \lambda^2/\nu - \chi/\nu > a$, so that $0 \leq \lambda^2 \leq (1+a)\nu + \alpha$, where $\alpha = \sup(-\chi)$.

In the internal integral in (2.7) we make a replacement of the integration variable ξ by the formula:

$$\mathfrak{x} = \xi - \rho\mu + \chi/\nu\,, \quad \xi = \mathfrak{x} + \rho\mu + \chi_1/\nu\,, \quad \frac{d\xi}{d\mathfrak{x}} = 1 + \chi_2/\nu\,. \qquad (2.9)$$

It follows from the properties of the function χ that the functions $\chi_j(\rho,\mu,\mathfrak{x},\nu)$, $j=1,2$ defined by (2.9), at sufficiently large ν, possess the same properties as the function χ. Now the equality (2.7) takes the form

$$\int F(\xi,\nu) f(\xi) d\xi =$$

$$= \frac{2\pi}{\nu}\int_0^\infty d\lambda^2 \int_0^1 \rho^2 d\rho\, \psi(\rho,\lambda^2) \int_{-1}^1 d\mu \int d\mathfrak{x}\, f\!\left(\mathfrak{x}+\rho\mu+\frac{\chi_1}{\nu}\right)\left(1+\frac{\chi_2}{\nu}\right)\delta\!\left(\mathfrak{x}^2+\frac{\lambda^2}{\nu}\right). \qquad (2.10)$$

Eliminating in (2.10) the δ- function we obtain

$$\int F(\xi,\nu) f(\xi) d\xi =$$

$$= \frac{2\pi}{\nu}\int_0^\infty d\lambda^2 \int_0^1 \rho^2 d\rho\, \psi(\rho,\lambda^2) \int_{-1}^1 d\mu\, f\!\left(-\frac{\lambda^2}{\nu}+\rho\mu+\frac{\chi_1}{\nu}\right)\left(1+\frac{\chi_2}{\nu}\right). \qquad (2.11)$$

Note that in (2.11) the functions χ_j depend upon λ^2 only via the ratio $\lambda^2/\nu = \tau$ which changes in a bounded interval since $0 \leq \lambda^2 \leq (1+a)\nu + \alpha_1$ where $\alpha_1 = \sup X_1$.

Now we restrict the class of the weight functions $\psi(\varrho,\lambda^2)$ by assuming that at sufficiently large λ^2 the distribution $\psi(\varrho,\lambda^2)$ is the ordinary function with respect to λ^2 and for a certain $\kappa > -1$ there exists a non zero limit (in the sense of the distributions with respect to ϱ)

$$\lim_{\lambda^2 \to +\infty} \frac{\psi(\varrho,\lambda^2)}{\lambda^{2\kappa}} = \psi_0(\varrho) , \quad \psi_0(\varrho) \not\equiv 0 . \tag{2.12}$$

The condition (2.12) implies that for any $\varphi(\varrho)$ there exists the limit relation

$$\frac{1}{\lambda^{2\kappa}} \int \psi(\varrho,\lambda^2)\varphi(\varrho)\varrho^2 d\varrho \to \int \psi_0(\varrho)\varphi(\varrho)\varrho^2 d\varrho , \quad \lambda^2 \to +\infty . \tag{2.13}$$

The limit relation (2.12) means also that

$$\psi(\varrho,\lambda^2) = \Theta(\lambda^2)\, \lambda^{2\kappa} \psi_0(\varrho) + \psi_1(\varrho,\lambda^2) , \tag{2.14}$$

in this case

$$\varepsilon(\varrho,\lambda^2) = \frac{\psi_1(\varrho,\lambda^2)}{\lambda^{2\kappa}} \to 0 , \quad \lambda^2 \to +\infty . \tag{2.15}$$

Inserting the expressions (2.14) and (2.15) in (2.11) we get

$$\int F(\xi,\nu) f(\xi) d\xi = J_1(\nu) + J_2(\nu) , \tag{2.16}$$

where (for a certain sufficiently large N)

$$J_1(\nu)=\frac{2\pi}{\nu}\int_0^\infty d\lambda^2\,\lambda^{2K}\int_0^1\rho^2d\rho\,\psi_0(\rho)\int_{-1}^1 d\mu\, f\left(-\frac{\lambda^2}{\nu}+\rho\mu+\frac{\chi_1}{\nu}\right)\left(1+\frac{\chi_2}{\nu}\right), \tag{2.17}$$

$$J_2(\nu)=\frac{2\pi}{\nu}\int_N^\infty d\lambda^2\,\lambda^{2K}\int_0^1\rho^2d\rho\,\varepsilon(\rho,\lambda^2)\int_{-1}^1 d\mu\, f\left(-\frac{\lambda^2}{\nu}+\rho\mu+\frac{\chi_1}{\nu}\right)\left(1+\frac{\chi_2}{\nu}\right)+$$
$$+\frac{2\pi}{\nu}\int_0^N d\lambda^2\int_0^1\rho^2d\rho\,\psi_1(\rho,\lambda^2)\int_{-1}^1 d\mu\, f\left(-\frac{\lambda^2}{\nu}+\rho\mu+\frac{\chi_1}{\nu}\right)\left(1+\frac{\chi_2}{\nu}\right). \tag{2.18}$$

In the integral $J_1(\nu)$ we replace the integration variable $\lambda^2=\nu\tau$,

$$J_1(\nu)=2\pi\nu^K\int_0^\infty d\tau\,\tau^K\int_0^1\rho^2d\rho\,\psi_0(\rho)\int_{-1}^1 d\mu\, f\left(-\tau+\rho\mu+\frac{\chi_1}{\nu}\right)\left(1+\frac{\chi_2}{\nu}\right)= \tag{2.19}$$
$$=2\pi\nu^K\int_0^1\rho^2d\rho\,\psi_0(\rho)\int_0^\infty d\tau\,\tau^K\int_{-1}^1 d\mu\, f\left(-\tau+\rho\mu+\frac{\chi_1}{\nu}\right)\left(1+\frac{\chi_2}{\nu}\right).$$

From the properties of the functions f, χ_1 and χ_2 it follows a limiting relation

$$\int_0^\infty d\tau\,\tau^K\int_{-1}^1 d\mu\, f(-\tau+\rho\mu+\chi_1/\nu)(1+\chi_2/\nu)\longrightarrow$$
$$\longrightarrow\frac{1}{\rho(K+1)}\int_{-\infty}^{\rho} f(\xi)(\rho-\xi)^{K+1}d\xi-\frac{1}{\rho(K+1)}\int_{-\infty}^{-\rho} f(\xi)(-\rho-\xi)^{K+1}d\xi,\quad \nu\to+\infty$$

together with all the derivatives with respect to ρ (in the sense of uniform convergence with respect to ρ in each finite interval). Therefore in (2.18) a transition to the limit under the integral sign is possible and we obtain, at $\nu \to +\infty$,

$$\nu^{-K} J_1(\nu) \longrightarrow \frac{2\pi}{K+1} \int_0^1 \rho \, d\rho \, \psi_0(\rho) \times$$

$$\times \left[\int_{-\infty}^{\rho} f(\xi)(\rho-\xi)^{K+1} d\xi - \int_{-\infty}^{-\rho} f(\xi)(-\rho-\xi)^{K+1} d\xi \right]. \tag{2.20}$$

We denote by $\rho\psi_0(|\rho|)$ an odd continuation of the distribution $\rho\psi_0(\rho)$ to $\rho<0$. The distribution $\rho\psi_0(|\rho|)$ is determined according to the rule

$$\int_{-1}^{1} \rho\psi_0(|\rho|)\varphi(\rho)d\rho = \int_0^1 \psi_0(\rho)[\varphi(\rho)-\varphi(-\rho)]\rho \, d\rho \tag{2.21}$$

for all the test functions $\varphi(\rho)$. In virtue of (2.21) the asymptotic equality (2.20) takes the form:

$$\nu^{-K} J_1(\nu) \sim \frac{2\pi}{K+1} \int_{-\infty}^{1} \rho \, d\rho \, \psi_0(|\rho|) \int_{-\infty}^{\rho} d\xi f(\xi)(\rho-\xi)^{K+1} =$$

$$= \frac{2\pi}{K+1} \int f(\xi) \left[\int_{\xi} \psi_0(|\rho|)(\rho-\xi)^{K+1} \rho \, d\rho \right] d\xi. \tag{2.22}$$

Now we consider the integral $J_2(\nu)$. We have

$$\tau^K \int_{-1}^{1} d\mu f\left(-\tau+\rho\mu+\frac{x_1}{\nu}\right)\left(1+\frac{x_2}{\nu}\right) = \int_{-\infty}^{\infty} R\left(\omega,\rho;\frac{1}{\nu}\right) e^{i\omega\tau} d\omega, \tag{2.23}$$

where $R(\omega,\rho;\frac{1}{\nu})$ is a uniformely bounded function for all sufficiently large ν in the topology of the space of rapidly decreasing infinitely differentiable functions with respect to ω and infinitely differentiable functions with respect to ρ. Taking into account the fact that the upper limit of integration over λ^2 in (2.18) is finite and equal to $(1+a)\nu+d_1$, inserting (2.23) for $\tau = \lambda^2/\nu$ in the first term of (2.18) and taking into consideration the fact that the second term in (2.18) is of the order of $1/\nu$, we get

$$J_2(\nu) = 2\pi\nu^{K-1}\int_{-\infty}^{\infty} d\omega \int_N^{(1+a)\nu+d_1} d\lambda^2\, e^{i\lambda^2/\nu\,\omega} \int_0^1 \rho^2 d\rho\, \varepsilon(\rho,\lambda^2) R(\omega,\rho;\tfrac{1}{\nu}) + O(\tfrac{1}{\nu}),$$

from where we derive

$$\left|\nu^{-K} J_2(\nu)\right| \leqslant 2\pi \int_{-\infty}^{\infty} \frac{d\omega}{1+\omega^2}\, \frac{1}{\nu} \int_N^{(1+a)\nu+d_1} d\lambda^2 \,\times$$

$$\times \left| \int_0^1 \rho^2 d\rho\, \varepsilon(\rho,\lambda^2)(1+\omega^2) R(\omega,\rho;\tfrac{1}{\nu}) \right| + O\left(\frac{1}{\nu^{K+1}}\right) \to 0, \quad \nu \to +\infty .$$

From the above result and eqs. (2.22) and (2.16) it follows an asymptotic equality in the domain (2.1)

$$F(\xi,\nu) \sim 2\pi\,\Gamma(K+1)\,\nu^K\,\Phi_{K+2}(\xi) = \nu^K F(\xi), \tag{2.24}$$

where we have the following notation

$$\Phi_K(\xi) = \frac{1}{\Gamma(K)} \int_\xi^1 \psi_0(|\rho|)(\rho-\xi)^{K-1} \rho\, d\rho ,$$
$$\Gamma(\xi) = 2\pi\,\Gamma(K+1)\,\Phi_{K+2}(\xi) . \tag{2.25}$$

The "integral" in (2.25) should be viewed as a k-th primitive* of the distribution $\rho\psi_0(|\rho|)$ vanishing for $\xi > 1$. It follows from (2.6) that

* For noninteger K, the definition of the primitive is given, e.g., in ref. /12/.

$$F(\xi)=0, \quad \xi>1 . \tag{2.26}$$

Next, noticing that the function $\varrho\Psi_0(|\varrho|)$ vanishes for $|\varrho|>1$, we derive from eqs. (2.24) and (2.25) that

$$F(\xi,\nu)\sim\frac{2\pi}{K+1}\nu^K\int_{-1}^{1}\Psi_0(|\varrho|)(\varrho-\xi)^{K+1}\varrho\,d\varrho \qquad \xi<-1 . \tag{2.27}$$

For integer $K\geqslant0$ eq. (2.27) turns to a polynomial in ξ.

Thus, under the condition (2.12), the automodel behaviour (2.24) holds.

Now we consider the case of weight functions $\Psi(\varrho,\lambda^2)$ such that their s-th primitive $\Psi_{(s)}(\varrho,\lambda^2)$ $(s>-1)$ with respect to λ^2 is integrable over λ^2. This means that for any test function $\varphi(\varrho)$ the expression $\int\Psi_{(s)}(\varrho,\lambda^2)\,\varphi(\varrho)\,\varrho^2 d\varrho$ is absolutely integrable over λ^2 in the interval $(N,+\infty)$ for a certain $N>0$. We denote

$$\int_{-0}^{\infty}\Psi_{(s)}(\varrho,\lambda^2)\,d\lambda^2=\Psi_0(\varrho) . \tag{2.28}$$

It follows from the properties of the functions f,χ_1, χ_2 that the sequence of the functions

$$(-1)^s \frac{\partial^s}{\partial \lambda^{2s}} \int_{-1}^{1} d\mu\, f\left(-\frac{\lambda^2}{\nu} + \rho\mu + \frac{x_1}{\nu}\right)\left(1 + \frac{x_2}{\nu}\right), \quad \nu \to +\infty$$

is uniformely bounded in all (ρ, λ^2) and tends to the function $\frac{1}{\nu^s}\int_{-1}^{1} d\mu\, f^{(s)}(\rho\mu) = \frac{1}{\nu^s \rho}\int_{-\rho}^{\rho} f^{(s)}(\xi)\,d\xi = \frac{f^{(s-1)}(\rho) - f^{(s-1)}(-\rho)}{\nu^s \rho}$

uniformely in these arguments in each finite domain together with all the derivatives. Therefore, it is possible a transition to the limit at $\nu \to +\infty$ under the integral sign in (2.11), and with the account of (2.21) and (2.28) we get

$$\nu^{s+1}\int F(\xi,\nu) f(\xi)\,d\xi = 2\pi(-\nu)^s \int_0^\infty d\lambda^2 \int_0^1 \rho^2 d\rho\, \Psi_{(s)}(\rho, \lambda^2) \times$$

$$\times \frac{\partial^s}{\partial \lambda^{2s}} \int_{-1}^{1} d\mu\, f\left(-\frac{\lambda^2}{\nu} + \rho\mu + \frac{x_1}{\nu}\right)\left(1 + \frac{x_2}{\nu}\right) \longrightarrow$$

$$\longrightarrow 2\pi \int_0^1 \Psi_0(\rho)\left[f^{(s-1)}(\rho) - f^{(s-1)}(-\rho)\right]\rho\, d\rho =$$

$$= 2\pi \int_{-1}^{1} \rho\, \Psi_0(|\rho|) f^{(s-1)}(\rho)\, d\rho = 2\pi(-1)^{s-1}\int_{-1}^{1}\left[\rho\Psi_0(|\rho|)\right]^{(s-1)} f(\rho)\,d\rho,$$

i.e.

$$F(\xi,\nu) \sim \frac{2\pi(-1)^{s+1}}{\nu^{s+1}}\left[\xi\, \Psi_0(|\xi|)\right]^{(s-1)} . \qquad (2.29)$$

Thus, under the condition (2.28) the automodel behaviour (2.29) is valid.

We write down two particular cases of eq. (2.29): for $s = 0$, using the notation (2.25)

$$F(\xi,\nu) \sim \frac{2\pi}{\nu}\, \Phi_1(\xi) \quad ; \tag{2.30}$$

for s=1

$$F(\xi,\nu) \sim \frac{2\pi}{\nu^2}\, \xi\, \Psi_0(|\xi|) \, . \tag{2.31}$$

We note that the r.h.s. of eq. (2.29) vanishes at $\xi > 1$; for integer $S \geqslant 0$ it vanishes at $\xi < -1$, as well.

Section 3

In this Section we study the asymptotic behaviour near the light cone $x^2=0$ of the function $\widetilde{F}(x,p)$ satisfying the conditions I-IV of Section 2.

Applying the inverse Fourier transformation to the representation (2.3) we get

$$\widetilde{F}(x) = \frac{1}{(2\pi)^4}\int F(q)\, e^{-iqx}\, dq = $$
$$= -\frac{i}{2\pi}\int_0^\infty \mathcal{D}(x,\lambda^2)\, \Delta(\vec{x},\lambda^2)\, d\lambda^2 \, , \tag{3.1}$$

where $\mathcal{D}(x,\lambda^2)$ is the well-known commutation function for free scalar fields of particles with mass λ /13/:

$$\mathcal{D}(x,\lambda^2) = \frac{i}{(2\pi)^3}\int e^{-iqx}\, \epsilon(q_0)\, \delta(q^2-\lambda^2)\, dq = $$
$$= \frac{1}{2\pi}\, \epsilon(x_0)\, \frac{\partial}{\partial x^2}\left[\Theta(x^2)\, J_0(\lambda\sqrt{x^2})\right] ; \tag{3.2}$$

the spectral function $\Delta(\vec{x},\lambda^2) \equiv \Delta(|\vec{x}|,\lambda^2)$ is the Fourier transform of the weight function $\psi(|\vec{u}|,\lambda^2)$

$$\Delta(z,\lambda) = 4\pi \int_0^1 \psi(\rho,\lambda^2) \frac{\sin \rho z}{z} \rho \, d\rho . \qquad (3.3)$$

It follows from (3.3) that the function $\Delta(x,\lambda^2)$ is entire analytic in z polynomially bounded together with all the derivatives on the real axis.

We first consider the case when the condition (2.12) holds. In this case, in virtue of (2.14) and (2.15), the function $\Delta(z,\lambda^2)$ is represented in the form

$$\Delta(z,\lambda^2) = 4\pi\, \Theta(\lambda^2) \lambda^{2K} \int_0^1 \psi_0(\rho) \frac{\sin \rho z}{z} \rho \, d\rho + \Delta_1(z,\lambda^2), \qquad (3.4)$$

where we have the following notation

$$\Delta_1(z,\lambda^2) = 4\pi \int_0^1 \psi_1(\rho,\lambda^2) \frac{\sin \rho z}{z} \rho \, d\rho \; ;$$

in this case

$$\frac{|\Delta_1(z,\lambda^2)|}{\lambda^{2K}} \longrightarrow 0 , \quad \lambda^2 \longrightarrow +\infty \qquad (3.5)$$

uniformely over all z together with all the derivatives with respect to z .

In an arbitrary frame of reference, eq. (3.1) takes the following form

$$\widetilde{F}(x,p) = \frac{-i}{2\pi} \int \mathcal{D}(x,\lambda^2)\, \Delta\left[\sqrt{(px)^2 - x^2}, \lambda^2\right] d\lambda^2 . \qquad (3.6)$$

REMARK

The integral in (3.6) converges rapidly in the sense of the theory of distributions. In fact, $\Delta(z,\lambda^2)$ is of the polynomial growth in λ^2, and for any test function $\varphi(x)$ (from S) the estimate

$$\left| \int \mathcal{D}(x,\lambda^2)\varphi(x)dx \right| \leq \frac{K_n(\varphi)}{1+\lambda^{2n}}$$

is valid for any $n \geqslant 0$.

To calculate the main term of the asymptotic of the function $\widetilde{F}(x,p)$ in the vicinity $x^2=0$ we represent it, according to (3.4) and (3.1), as a sum of two terms:

$$\widetilde{F}(x,p) = I_1(x,p) + I_2(x,p), \qquad (3.7)$$

where

$$I_1(x,p) = -\frac{i}{2\pi}4\pi\int_0^\infty d\lambda^2\lambda^{2K}\mathcal{D}(x,\lambda^2)\int_0^1 \Psi_0(\rho) \times \qquad (3.8)$$

$$\times \frac{\sin\rho\sqrt{(px)^2-x^2}}{\sqrt{(px)^2-x^2}}\rho d\rho ,$$

$$I_2(x,p) = -\frac{i}{2\pi}\int_0^\infty d\lambda^2 \mathcal{D}(x,\lambda^2)\,\Delta_1\left[\sqrt{(px)^2-x^2}, \lambda^2\right]. \qquad (3.9)$$

To derive the main term of the asymptotic of the quantity I_1 we calculate

$$\int_0^\infty d\lambda^2 \lambda^{2K} \mathcal{D}(x,\lambda^2) = \frac{i}{(2\pi)^3}\int_0^\infty d\lambda^2\lambda^{2K}\int \epsilon(q_0)\delta(q^2-\lambda^2)e^{-iqx}dq =$$

$$= \frac{i}{(2\pi)^3}\int \epsilon(q_0)(q^2)^K e^{-iqx}dq = -4(-\Box)^K\left[\frac{\mathcal{D}(x,0)}{x^2}\right]. \quad (3.10)$$

Here we have used the formula/11/

$$\Box\left[\epsilon(q_0)\Theta(q^2)\right] = 4\epsilon(q_0)\,\delta(q^2)\,. \quad (3.11)$$

Therefore

$$I_1(x,p) \sim \frac{2i}{\pi}\,G(px)(-\Box)^K\left[\frac{\mathcal{D}(x,0)}{x^2}\right], \quad x^2\sim 0\,, \quad (3.12)$$

where we denote as follows

$$G(z) = 4\pi\int_0^1 \psi_0(\rho)\,\frac{\sin z\rho}{z}\,\rho\, d\rho\,. \quad (3.13)$$

In particular, if $\rho\psi_0(|\rho|)$ is a measure, then

$$G(z) = 4\pi\int_0^1 \Phi_1(\xi)\cos z\xi\, d\xi\,. \quad (3.14)$$

Now we show that the quantity I_2 has a weaker singularity on the light cone than I_1. We have from (3.9):

$$I_2(x,p) = \frac{1}{(2\pi)^4}\int_0^\infty d\lambda^2\int e^{-iqx}\epsilon(qp)\delta(q^2-\lambda^2)dq\,\Delta_1\left[\sqrt{(px)^2-x^2},\lambda^2\right] =$$

$$= \frac{1}{(2\pi)^4}\int \epsilon(qp)\,\Theta(q^2)\,\Delta_1\left[\sqrt{(px)^2-x^2},q^2\right]e^{-iqx}dq\,. \quad (3.15)$$

Hence, owing to the properties of the function Δ_1, it follows that in the vicinity of the light cone, we have an asymptotic equality

$$I_2(x,p) \sim \frac{1}{(2\pi)^4} \int \epsilon(qp)\Theta(q^2)\Delta_1(px,q^2)\, e^{-iqx} dq . \qquad (3.16)$$

We introduce a pseudo-differential operator $\Delta_1(px,-\Box)$, and define its action on the test functions $\varphi(x)$ from S according to the rule

$$\Delta_1(px,-\Box)\varphi(x) = \frac{1}{(2\pi)^4} \int \tilde{\varphi}(q)\Delta_1(px,q^2)\, e^{-iqx} dq .$$

We extend the operator $\Delta_1(px,-\Box)$ on the distributions in a usual manner. Then the r.h.s. of (3.16) is rewritten in the form

$$I_2(x,p) \sim \Delta_1(px,-\Box) \int \epsilon(qp)\Theta(q^2)\, e^{-iqx} dq =$$

$$= \frac{2i}{\pi} \Delta_1(px,-\Box)\left[\frac{\mathcal{D}(x,0)}{x^2}\right], \quad x^2 \sim 0;$$

here we have used eq. (3.11). Because of (3.5) and (3.12) the quantity $I_2(x,p)$ will have weaker singularities in the vicinity $x^2=0$, than the quantity $I_1(x,p)$, since, due to (3.13), $G(px) \not\equiv 0$.

Thus, due to (3.7) and (3.12), in the vicinity $x^2=0$, we have the following asymptotic behaviour of the function

$$\tilde{F}(x,p) \sim \frac{2i}{\pi} G(px)(-\Box)^K \left[\frac{\mathcal{D}(x,0)}{x^2}\right], \qquad (3.17)$$

where the function $G(px)$ is determined from eq. (3.13).

Now we assume that the condition

$$\int_0^1 \psi_0(\rho)\rho^2 d\rho = \frac{1}{4\pi}\int_{|\vec{u}|<1} \psi_0(|\vec{u}|)\, d\vec{u} = 0. \qquad (3.18)$$

is fulfilled.

REMARK

The condition (3.18) is fulfilled if the function obeys the condition

$$F(q_0,\vec{0}) \to 0, \quad q_0 \to +\infty. \qquad (3.19)$$

Indeed, from the representation (2.3) we get

$$F(q_0,\vec{0}) = 4\pi \int_0^1 \epsilon(q_0)\psi(\rho, q_0^2-\rho^2)\,\rho^2 d\rho. \qquad (3.20)$$

Letting q_0 tend to infinity in eq. (3.20) and employing the limit relations (3.19) and (2.12) (for $\kappa=0$) we get the condition (3.19)*.

Considering the function $\rho\psi(|\rho|)$ as a measure, we make ourselves sure from (2.25) and (3.18) that the primitive $\Phi_2(\xi)$ vanishes for $\xi=0$. Then from (3.14) it follows immediately that

* It is supposed that we can make a transition to the limit under the integral sign in (3.20).

$$\mathcal{G}(z) = -4\pi z \int_0^1 \Phi_2(\xi) \sin z\xi \, d\xi = -zQ(z), \tag{3.21}$$

where

$$Q(z) = 4\pi \int_0^1 \Phi_2(\xi) \sin z\xi \, d\xi . \tag{3.22}$$

In this case, for $\kappa = 0$, eq. (3.17) is simplified as follows

$$\widetilde{F}(x,p) = \frac{i}{2\pi} Q(px)\, px \frac{i}{(2\pi)^3} \int \epsilon(qp)\Theta(q^2) e^{-iqx} dq =$$
$$= \frac{1}{2\pi} Q(xp) \frac{i}{(2\pi)^3} \int \frac{\partial}{\partial(qp)} [\epsilon(qp)\Theta(q^2)] e^{-iqx} dq . \tag{3.23}$$

Taking into consideration the equality

$$\frac{\partial}{\partial q_0} [\epsilon(q_0)\Theta(q^2)] = 2q_0 \epsilon(q_0) \delta(q^2) , \tag{3.24}$$

we rewrite (3.23)

$$\widetilde{F}(x,p) \sim \frac{1}{\pi} Q(xp) \frac{i}{(2\pi)^3} \int qp\, \epsilon(qp) \delta(q^2) e^{-iqx} dq ,$$

i.e. finally

$$\widetilde{F}(x,p) \sim \frac{i}{\pi} Q(px) \frac{\partial}{\partial(px)} \mathcal{D}(x,0) , \quad x^2 \sim 0 , \tag{3.25}$$

where the coefficient $Q(px)$ is related by eq. (3.22) to the function $\Phi_2(\xi)$ which defines the automodel behaviour (2.24) of the function $F(\xi,\nu)$.

Now we consider the case when the s-th primitive $\psi_{(s)}(\varrho,\lambda^2)$ of the weight function $\psi(\varrho,\lambda^2)$ is integrable over λ^2. In this case the s-th primitive $\Delta_{(s)}(\tau,\lambda^2)$ of the spectral function $\Delta(\tau,\lambda^2)$ is also integrable over λ^2.

Since the principal singularity of the function $\frac{\partial^s}{\partial\lambda^{2s}}\mathcal{D}(x,\lambda^2)$ in the vicinity $x^2=0$ (Appendix IV) is $\mathcal{D}(x,0)$ for $s=0$ and

$$\frac{(-1)^s}{2\pi 4^s\,\Gamma(s)}\,\epsilon(x_0)\,\Theta(x^2)\,x^{2(s-1)} \qquad \text{for} \quad s\geqslant 1, \tag{3.26}$$

then, using (3.6), (3.3) and (2.28) we get for $s\geqslant 1$

$$\tilde{F}(x,p)=\frac{i(-1)^{s+1}}{2\pi}\int\limits_0^\infty \frac{\partial^s}{\partial\lambda^{2s}}\,\mathcal{D}(x,\lambda^2)\,\Delta_{(s)}\left[\sqrt{(px)^2-x^2},\lambda^2\right]d\lambda^2\sim$$

$$\sim\frac{i(-1)^{s+1}}{2\pi}\,\frac{(-1)^s}{2\pi 4^s\,\Gamma(s)}\,\epsilon(px)\,\Theta(x^2)\,x^{2(s-1)}\int\limits_{-0}^{\infty}\Delta_{(s)}(px,\lambda^2)\,d\lambda^2=$$

$$=-\frac{i}{4^{s+1}\pi^2\,\Gamma(s)}\,\epsilon(px)\,\Theta(x^2)\,x^{2(s-1)}\,4\pi\int\limits_{-0}^{\infty}d\lambda^2\int\limits_0^1\varrho\,d\varrho\,\psi_{(s)}(\varrho,\lambda^2)\,\frac{\sin\varrho px}{px},$$

i.e.

$$\tilde{F}(x,p)\sim\frac{-i}{4^{s+1}\pi^2\,\Gamma(s)}\,G(px)\,\epsilon(px)\,\Theta(x^2)\,x^{2(s-1)}, \quad x^2\sim 0. \tag{3.27}$$

For $s=0$ we obtain in a similar way

$$\tilde{F}(x,p)\sim-\frac{i}{2\pi}\,G(px)\,\mathcal{D}(x,0), \quad x^2\sim 0. \tag{3.28}$$

In eqs. (3.27) and (3.28) the function $G(z)$ is determined by the formulas (3.13), (2.28) and (2.21). If the function $\rho\psi_0(|\rho|)$ is a measure, then

$$G(z) = 4\pi \int_0^1 \Phi_1(\xi) \cos z\xi \, d\xi \, . \qquad (3.29)$$

Thus, again the coefficient for the main part of the singularity (3.27) and (3.28), in the vicinity of the light cone, is expressed by the formula (3.29) via the function $\Phi_1(\xi)$ which defines the automodel behaviour (2.29).

It is worth noting that when the condition (3.18) is satisfied the coefficient function $G(z)$ is expressed by means of the formulas (3.21) and (3.22).

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APPENDIX I. On Causality of the Functions V_1 and V_2

The tempered distribution $V(q)$ is called causal if its Fourier transform $\tilde{V}(x)$ vanishes at $x^2<0$.

Let V be a causal function and $T(\vec{x}) \in C^\infty(R^3)$. We introduce the distribution

$$\tilde{V}_T(x_0) = \int \tilde{V}(x_0,\vec{x})\, T(\vec{x})\, d\vec{x},$$

"integrable" by definition with the test functions $\varphi(x_0)$ according to the rule

$$\int \tilde{V}_T(x_0)\varphi(x_0)\,dx_0 = \int \tilde{V}(x)\varphi(x_0) T(\vec{x})\, dx .$$

Let $\tilde{f}(\vec{x})$ be a finite distribution. The potential* with density $\tilde{f}$, i.e. the contraction $\tilde{f}$ with $\frac{1}{|\vec{x}|}$ is denoted as

$$\left(\tilde{f} * \frac{1}{|\vec{x}|}\right)(\vec{\xi}) = \int \frac{f(\vec{x})}{|\vec{x}-\vec{\xi}|}\, d\vec{x} .$$

In this Appendix we prove that the functions V_1 and V_2 from Section 1 are causal.

L e m m a I. Let the distribution $\tilde{f}(\vec{x})$ be radially symmetric, $\tilde{f}(\vec{x}) = \tilde{f}(|\vec{x}|)$, and vanish when $|\vec{x}|>R$. Then

$$\int \frac{\tilde{f}(|\vec{x}|)}{|\vec{x}-\vec{\xi}|}\, d\vec{x} = \frac{1}{|\vec{\xi}|}\int \tilde{f}(|\vec{x}|)\, d\vec{x}, \qquad |\vec{\xi}|>R. \tag{I.1}$$

* The main notions of the theory of potential used here is given, e.g. in ref./12/

If, in addition

$$\int \tilde{f}(|\vec{x}|)\,d\vec{x} = 0, \tag{I.2}$$

then

$$\int \left(\tilde{f} * \frac{1}{|\vec{x}|}\right)(\vec{\xi})\,d\vec{\xi} = -\frac{2\pi}{3}\int \tilde{f}(|\vec{x}|)\,x^2\,d\vec{x}. \tag{I.3}$$

<u>P r o o f</u>. Let $\varphi(\vec{\xi})$ be an arbitrary test function. Then

$$\int \left(\tilde{f} * \frac{1}{|\vec{x}|}\right)\varphi(\vec{\xi})\,d\vec{\xi} = \int_0^\infty d\varrho\,\tilde{f}(\varrho)\int_{|\vec{x}|=\varrho} dS_{\vec{x}}\int d\vec{\xi}\,\frac{\varphi(\vec{\xi})}{|\vec{x}-\vec{\xi}|}. \tag{I.4}$$

But

$$\int_{|\vec{x}|=\varrho} \frac{dS_{\vec{x}}}{|\vec{x}-\vec{\xi}|} = 2\pi\varrho^2\int_{-1}^{1}\frac{d\mu}{\sqrt{\varrho^2+\vec{\xi}^2-2\varrho|\vec{\xi}|\mu}} =$$

$$= \frac{2\pi\varrho}{|\vec{\xi}|}\left(\varrho+|\vec{\xi}|-|\varrho-|\vec{\xi}||\right). \tag{I.5}$$

Then assuming that the support $\varphi(\vec{\xi})$ lies outside the sphere $|\vec{\xi}|\leqslant R$ and changing the order of integration in (I.4) we get

$$\int \left(\tilde{f} * \frac{1}{|\vec{x}|}\right)(\vec{\xi})\,d\vec{\xi} = 4\pi\int_0^\infty \varrho^2\,d\varrho\,\tilde{f}(\varrho)\int d\vec{\xi}\,\frac{\varphi(\vec{\xi})}{|\vec{\xi}|}$$

from where the equality (I.1) follows.

Let now the condition (I.2) be fulfilled. Then, in virtue of (I.1), the potential $\left(f * \frac{1}{|\vec{x}|}\right)(\vec{\xi})$ vanishes at $|\vec{\xi}| > R$. Let $\varphi(\vec{\xi})$ be a test function equal to unity for $|\vec{\xi}|\leq R+\varepsilon$ $(\varepsilon>0)$ and equal to zero for $|\vec{\xi}|>R+2\varepsilon$. Then, on account of (I.5), (I.4) and (I.2) we have

$$\int\left(\tilde{f}*\frac{1}{|\vec{x}|}\right)(\vec{\xi})d\vec{\xi}=\lim_{\varepsilon\to+0}2\pi\int_0^\infty d\rho\tilde{f}(\rho)\int d\xi\,\varphi(\vec{\xi})\frac{\rho}{|\vec{\xi}|}(\rho+|\vec{\xi}|-$$

$$-|\rho-|\vec{\xi}||)=16\pi^2\int_0^\infty\tilde{f}(\rho)\left[\rho\int_0^\rho\vec{\xi}^2d|\vec{\xi}|+\rho^2\int_\rho^R|\vec{\xi}|d|\vec{\xi}|\right]d\rho=$$

$$=8\pi^2\int_0^\infty\tilde{f}(\rho)\left(R^2-\frac{\rho^2}{3}\right)\rho^2d\rho=-\frac{2\pi}{3}\int\tilde{f}(|\vec{x}|)\vec{x}^2d\vec{x},$$

so that the eqiuality (I.3) is valid. The lemma has been proved.

Let the Fourier transform $\tilde{f}(\vec{x})$ of the distribution $f(\vec{q})$ vanish when $|x|>R$. We define the distribution $f(\vec{q})/\vec{q}^2$ so that its Fourier transform vanishes at infinity.

It follows from this definition that the Fourier transform $f(\vec{q})/\vec{q}^2$ is a potential with density $\tilde{f}/4\pi$ i.e.

$$\widetilde{\frac{f(\vec{q})}{\vec{q}^2}}(\vec{\xi})=\frac{1}{4\pi}\left(\tilde{f}*\frac{1}{|\vec{x}|}\right)(\vec{\xi}). \tag{I.6}$$

L e m m a 2. Let the distributions $\tilde{f}_1(\vec{x})$ and $\tilde{f}_2(\vec{x})$ vanish for $|\vec{x}|>R$ and obey the relation

$$\vec{q}^2f_1(\vec{q})=q_1^{\alpha_1}q_2^{\alpha_2}q_3^{\alpha_3}f_2(\vec{q}). \tag{I.7}$$

Let then $\tilde{f}(\vec{x})$ be radially symmetric, $\tilde{f}_2(\vec{x})=\tilde{f}_2(|\vec{x}|)$. Then the Fourier transform of the function $f_2(\vec{q})/\vec{q}^2$ is also radially symmetric and vanishes for $|\vec{x}|>R$.

P r o o f. In virtue of (I.6) and (I.1), for $|\vec{\xi}|>R$, we have

$$\frac{\widetilde{f_2(\vec{q})}}{\vec{q}^2}(\vec{\xi}) = \frac{1}{4\pi}\left(\tilde{f} * \frac{1}{|\vec{x}|}\right)(\vec{\xi}) = \frac{1}{4\pi|\vec{\xi}|}\int \tilde{f}_2(|\vec{x}|)\,d\vec{x}. \tag{I.8}$$

On the other hand, from (I.7) we deduce

$$\tilde{f}_1(\vec{\xi}) = (-i)^{\alpha_1+\alpha_2+\alpha_3} \frac{\partial^{\alpha_1+\alpha_2+\alpha_3}}{\partial\xi_1^{\alpha_1}\partial\xi_2^{\alpha_2}\partial\xi_3^{\alpha_3}} \frac{\widetilde{f_2(\vec{q})}}{\vec{q}^2}(\vec{\xi}).$$

Hence, on account of (I.8), for $|\vec{\xi}| > R$ follows the equality

$$\int \tilde{f}_2(|\vec{x}|)\,d\vec{x} \; \frac{\partial^{\alpha_1+\alpha_2+\alpha_3}}{\partial\xi_1^{\alpha_1}\partial\xi_2^{\alpha_2}\partial\xi_3^{\alpha_3}} \frac{1}{|\vec{\xi}|} = 0,$$

i.e.

$$\int \tilde{f}_2(|\vec{x}|)\,d\vec{x} = 0, \tag{I.9}$$

which is to be proved.

Now we consider the causal function $f(q)$. Since $f(x_0, \vec{x}) = 0$ for $|\vec{x}| > |x_0|$, then applying lemma 2 at $R = |x_0|$ we see that the following lemma is valid.

L e m m a 3. *Let the functions $f_1(q)$ and $f_2(q)$ be causal and obey the relation*

$$\vec{q}^2 f_1(q) = q_1^{\alpha_1} q_2^{\alpha_2} q_3^{\alpha_3} f(q);$$

let then the function $f_2(q)$ be radially symmetric $f_2(q) = f_2(q_0, |\vec{q}|)$. Then the function $f_2(q)/\vec{q}^2$ is also a radially symmetric and causal function.

It follows immediately from lemmas 1 and 3 that the functions V_1 and V_2 may always be chosen causal and radially symmetric. Indeed, applying lemma 3 to the equality (I.13):

$$\vec{q}^2 W_{0\nu} = q_0 q_\nu F_1$$

we conclude that the function $q_0 F_1/\vec{q}^2$ is causal and radially symmetric and, consequently, in vitue of lemma I, the equality

$$\frac{\partial}{\partial x_0} \int F_1(x_0, \vec{x})\, d\vec{x} = 0.$$

holds.

Hence, because of the fact that the function $\widetilde{F}_1(x)$ is odd, with respect to x_0 it follows the equality

$$\int \widetilde{F}_1(x_0, \vec{x})\, d\vec{x} = 0 . \qquad \text{(I.10)}$$

Then, due to (I.11)

$$V_1 - V_2 = \frac{F_1}{\vec{q}^2} . \qquad \text{(I.11)}$$

According to lemma I, for $R = |x_0|$, from (I.10) it follows that the function $V_1 - V_2$ is causal and radially symmetric.

Then, owing to (I.15) and (I.11) we have

$$2\vec{q}^2 W_{ij} = q_i q_j \left(\frac{3q_0^2}{\vec{q}^2} F_1 - F_2 \right) =$$

$$= q_i q_j \left[3q_0^2 (V_1 - V_2) - F_2 \right] . \qquad \text{(I.12)}$$

Since the functions W_{ij} are causal and the function $3q_0^2(V_1-V_2)-F_2$ is causal and radially symmetric, on the basis of lemma 3, from (I.12) and (I.12) it follows that the function

$$V_1=\frac{1}{2\vec{q}^2}\left[3q_0^2(V_1-V_2)-F_2\right]=\frac{1}{2\vec{q}^2}\left(3q_0^2\frac{F_1}{\vec{q}^2}-F_2\right)$$

is causal and radially symmetric and the equality

$$0=\int\widetilde{\left(3q_0^2\frac{F_1}{\vec{q}^2}-F_2\right)}(x_0,\vec{x})\,dx=\int\left(-3\frac{\partial^2}{\partial x_0^2}\frac{\widetilde{F}_1}{\vec{q}^2}-\widetilde{F}_2\right)d\vec{x}\,.$$

is fulfilled.

According to (I.6) and (I.3) the latter condition takes the form

$$0=\int\left[\frac{3}{4\pi}\frac{\partial^2}{\partial x_0^2}\left(\widetilde{F}_1*\frac{1}{|\vec{\xi}|}\right)+\widetilde{F}_2\right]d\vec{x}=$$

$$=\int\left[-\frac{1}{2}\frac{\partial^2}{\partial x_0^2}\widetilde{F}_1(x_0,|\vec{x}|)\vec{x}^2+\widetilde{F}_2(x_0,|\vec{x}|)\right]d\vec{x}\,,$$

i.e.

$$\int\left[\frac{\partial^2}{\partial x_0^2}F_1(x_0,|\vec{x}|)\vec{x}^2-2F_2(x_0,|\vec{x}|)\right]d\vec{x}=0. \qquad \text{(I.13)}$$

APPENDIX II. On the Calculation of the Weight Function $\Psi(\vec{u},\lambda^2)$.

In the Jost-Lehmann-Dyson representation (2.3) for the odd function $F(q)$, the weight function $\Psi(\vec{u},\lambda^2)$ can be presented as follows (see Jost-Lehmann/10/)

$$\Psi(\vec{u},\lambda^2)=\frac{i}{2\pi}\frac{\partial}{\partial\lambda^2}\left[\theta(\lambda^2)\int_0^\infty d\varkappa^2 J_0(\varkappa\lambda)\int d\vec{\eta}\, e^{-i\vec{\eta}\vec{u}}\,\Phi(\vec{\eta},\varkappa^2)\right], \quad (II.1)$$

where

$$\Phi(\vec{\eta},\varkappa^2)=\mathcal{F}(\varkappa^2+\vec{\eta}^2,\vec{\eta}), \qquad \mathcal{F}(x_0^2,\vec{x})=\epsilon(x_0)\widetilde{F}(x). \quad (II.2)$$

If the function $\widetilde{F}(x)$ is radially symmetric in $\vec{x}$ then $\Phi(\vec{\eta},\varkappa^2)=\Phi(|\vec{\eta}|,\varkappa^2)$, and formula (A.I) takes the following form*:

$$\Psi(z,\lambda^2)=\frac{2i}{z}\frac{\partial}{\partial\lambda^2}\left[\theta(\lambda^2)\int d\varkappa^2 J_0(\varkappa\lambda)\times\right. \quad (II.3)$$

$$\left.\times\int\rho\, d\rho\,\sin z\rho\;\Phi(\rho,\varkappa^2)\right].$$

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* The divergent integrals given below should be understood in a regularized sense, on the basis of the Fourier transformation technique for the tempered distributions.

APPENDIX III. The Free Field

For the free field $\psi(x)$ of fermions with spin 1/2

$$j_\nu(x) = \overline{\psi}(x)\,\gamma_\nu\,\psi(x). \tag{III.1}$$

From here, on the basis of (1.6) and (1.7), after simple calculations, we obtain

$$\begin{aligned} F_1^0(q,p) &= \epsilon(qp+1)(qp+2)\,\delta\left[(q+p)^2-1\right] - \\ &\quad -\epsilon(qp-1)(qp-2)\,\delta\left[(q-p)^2-1\right], \\ F_2^0(q,p) &= 3\epsilon(qp+1)\,qp\,\delta\left[(q+p)^2-1\right] - \\ &\quad -3\epsilon(qp-1)\,qp\,\delta\left[(q-p)^2-1\right]. \end{aligned} \tag{III.2}$$

Consequently, in the domain where $\nu>0$ and $\xi>0$ we have

$$F_1^0(\xi,\nu) = \frac{1}{2}\left(1+\frac{4}{\nu}\right)\delta(1-\xi), \qquad F_2^0(\xi,\nu) = \frac{3}{2}\,\delta(1-\xi). \tag{III.3}$$

Then, in virtue of (1.17), when $\nu>0$ and $\xi>0$ we deduce

$$W_1^0(\xi,\nu) = \frac{1}{2}\,\delta(1-\xi), \qquad W_2^0(\xi,\nu) = \frac{2}{\nu}\,\delta(1-\xi). \tag{III.4}$$

From (III.4) it follows the following relationship between the form factors of the free field

$$W_1^\circ = \frac{\nu}{4\xi} W_2^\circ , \quad \nu > 0, \ \xi > 0 . \tag{III.5}$$

By the way, we note that the relation (III.5) alone ensures, due to (1.9) the equalities

$$F_1^\circ = \left(1 + \frac{4\xi}{\nu}\right) W_1^\circ , \qquad F_2^\circ = 3 W_1^\circ . \tag{III.6}$$

We calculate the weight functions $\psi_1^\circ(z,\lambda^2)$ $\psi_2^\circ(z,\lambda^2)$, which correspond to the functions F_1° and F_2° in the (2.3) representation. To this end we first calculate the weight function $\psi(z,\lambda^2)$ which corresponds to the function

$$F(q) = \epsilon(q_0+1)\,\delta\left[(q_0+1)^2 - \vec{q}^{\,2} - 1\right] + \\ + \epsilon(q_0-1)\,\delta\left[(q_0-1)^2 - \vec{q}^{\,2} - 1\right] . \tag{III.7}$$

In this case

$$F(x) = -\frac{i}{\pi} \cos x_0 \ \mathcal{D}(x,1) , \tag{III.8}$$

so that, in virtue of (II.2) and (3.2)

$$\Phi(\rho,\varkappa^2) = \frac{-i}{2\pi^2} \cos\sqrt{\varkappa^2+\rho^2} \ \frac{\partial}{\partial\varkappa^2}\left[\Theta(\varkappa^2)\, J_0(\varkappa)\right] . \tag{III.9}$$

Inserting (III.8) in (II.3) we get

$$\psi(z,\lambda^2)=\frac{1}{\pi^2 z}\frac{\partial}{\partial\lambda^2}\left\{\Theta(\lambda^2)\int_0^\infty d\varkappa^2 J_0(\varkappa\lambda)\frac{\partial}{\partial\varkappa^2}\left[\Theta(\varkappa^2)J_0(\varkappa)\right]\times\right.$$
$$\left.\times\int_0^\infty \rho\,d\rho\,\sin z\rho\,\cos\sqrt{\varkappa^2+\rho^2}\right\}. \qquad \text{(III.10)}$$

But* (see Gradshtein-Ryzhik, 3.876.1).

$$\int_0^\infty \sin z\rho\,\cos\sqrt{\varkappa^2+\rho^2}\,\rho\,d\rho=\frac{\partial^2}{\partial z\,\partial\xi}\int_0^\infty \sin\left(\xi\sqrt{\varkappa^2+\rho^2}\right)\frac{\cos z\rho\,d\rho}{\sqrt{\varkappa^2+\rho^2}}\Bigg|_{\xi=1}=$$
$$\text{(III.11)}$$
$$=-\frac{\partial^2}{\partial z\,\partial\xi}\left[\frac{\pi}{2}\Theta(\xi-z)J_0\left(\varkappa\sqrt{\xi^2-z^2}\right)\right]\Bigg|_{\xi=1}=\pi\frac{\partial}{\partial z}\frac{\partial}{\partial z^2}\left[\Theta(1-z^2)J_0\left(\varkappa\sqrt{1-z^2}\right)\right]$$

and therefore

$$\psi(z,\lambda^2)=\frac{2}{\pi}\left(\frac{\partial}{\partial z^2}\right)^2\left(\Theta(1-z^2)\frac{\partial}{\partial\lambda^2}\left\{\Theta(\lambda^2)\int_0^\infty d\varkappa^2 J_0(\varkappa\lambda)J_0\left(\varkappa\sqrt{1-z^2}\right)\times\right.\right.$$
$$\left.\left.\times\frac{\partial}{\partial\varkappa^2}\left[\Theta(\varkappa^2)J_0(\varkappa)\right]\right\}\right)=\frac{2}{\pi}\left(\frac{\partial}{\partial z^2}\right)^2\left\{\Theta(1-z^2)\frac{\partial}{\partial\lambda^2}\left[\Theta(\lambda^2)-\right.\right.$$
$$\left.\left.-\Theta(\lambda^2)\int_0^\infty d\varkappa\,J_0(\varkappa\lambda)J_0\left(\varkappa\sqrt{1-z^2}\right)J_1(\varkappa)\right]\right\}= \qquad \text{(III.12)}$$
$$=\frac{2}{\pi}\left(\frac{\partial}{\partial z^2}\right)^2\left\{\Theta(1-z^2)\left[\delta(\lambda^2)-\delta(\lambda^2)\int_0^\infty d\varkappa\,J_0\left(\varkappa\sqrt{1-z^2}\right)J_1(\varkappa)-\right.\right.$$
$$\left.\left.-\Theta(\lambda^2)\frac{\partial}{\partial\lambda^2}\int_0^\infty d\varkappa\,J_0(\varkappa\lambda)J_0\left(\varkappa\sqrt{1-z^2}\right)J_1(\varkappa)\right]\right\}.$$

Taking into account that (see G.R., 6.578.3.4).

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* I.S.Gradshtein and I.M.Ryzhik, Tables of Integrals, Sums, Series and Products, Fizjatgiz, 1963.

$$\chi(z^2,\lambda^2) \equiv \int_0^\infty J_0(\varkappa\sqrt{1-z^2})\, J_0(\varkappa\lambda)\, J_0(\varkappa)\, d\varkappa =$$

$$= \begin{cases} 0 \ , & \lambda^2 > (1+\sqrt{1-z^2})^2 \ , \\ 1 \ , & \lambda^2 < (1-\sqrt{1-z^2})^2 \ , \end{cases} \qquad \text{(III.13)}$$

we get from (III.12)

$$\psi(z,\lambda^2) = -\frac{2}{\pi^2}\,\Theta(\lambda^2)\left(\frac{\partial}{\partial z^2}\right)^2 \frac{\partial}{\partial\lambda^2}\left[\Theta(1-z^2)\,\chi(z^2,\lambda^2)\right] , \qquad \text{(III.14)}$$

where $\chi(z^2,\lambda^2)$ is a continuous function at $0 \leqslant z^2 \leqslant 1$, $\lambda^2 \geqslant 0$ equal to unity at $\lambda^2 < (1-\sqrt{1-z^2})^2$ and equal to zero at $\lambda^2 > (1+\sqrt{1-z^2})^2$ (see(III.13).

In particular

$$\psi(z,\lambda^2) = 0 \ , \quad \lambda^2 > 4 \ \text{ or } \ \lambda^2 < (1-\sqrt{1-z^2})^2 \qquad \text{(III.15)}$$

Now we calculate the weight function $\psi^*(z,\lambda^2)$ corresponding to the function

$$F^*(q) = q_0\,\epsilon(q_0+1)\,\delta\left[(q_0+1)^2-\vec{q}^{\,2}-1\right] -$$
$$- q_0\,\epsilon(q_0-1)\,\delta\left[(q_0-1)^2-\vec{q}^{\,2}-1\right] . \qquad \text{(III.16)}$$

In this case, in virtue of (3.2) and (III.8)

$$\widetilde{F}^*(x) = -\frac{i}{\pi}\frac{\partial}{\partial x_0}\left[\sin x_0\, \mathcal{D}(x,1)\right] = -\frac{i}{\pi}\cos x_0\, \mathcal{D}(x,1) -$$

$$-\frac{i}{2\pi^2}\sin x_0 \epsilon(x_0)\frac{\partial}{\partial x_0}\frac{\partial}{\partial x^2}\left[\Theta(x^2) J_0(\sqrt{x^2})\right] = \qquad \text{(III.17)}$$

$$= \widetilde{F}(x) - \frac{i}{\pi^2}\epsilon(x_0) x_0 \sin x_0 \left(\frac{\partial}{\partial x^2}\right)^2 \left[\Theta(x^2) J_0(\sqrt{x^2})\right],$$

so that, due to (II.2) and (III.9)

$$\Phi^*(\rho,\varkappa^2) = \Phi(\rho,\varkappa^2) - \frac{i}{\pi^2}\sqrt{\varkappa^2+\rho^2}\sin\sqrt{\varkappa^2+\rho^2} \times \times\left(\frac{\partial}{\partial \varkappa^2}\right)^2\left[\Theta(\varkappa^2) J_0(\varkappa)\right]. \qquad \text{(III.18)}$$

Inserting (III.18) in (II.3) and taking into account (III.14), we get

$$\psi^*(z,\lambda^2) = \psi(z,\lambda^2) + \frac{2}{\pi^2 z}\frac{\partial}{\partial\lambda^2}\Big[\Theta(\lambda^2)\int_0^\infty d\varkappa^2 J_0(\varkappa\lambda) \times$$

$$\times\left(\frac{\partial}{\partial\varkappa^2}\right)^2\left[\Theta(\varkappa^2)J_0(\varkappa)\right]\int_0^\infty \rho\, d\rho \sin z\rho\sqrt{\varkappa^2+\rho^2}\,\sin\sqrt{\varkappa^2+\rho^2}\Big]. \qquad \text{(III.19)}$$

But (comp. (III.11))

$$\int_0^\infty \sin z\rho\sqrt{\varkappa^2+\rho^2}\,\sin\sqrt{\varkappa^2+\rho^2}\,\rho\, d\rho =$$

$$= \frac{\partial^3}{\partial z\,\partial \xi^2}\int_0^\infty \sin\left(\xi\sqrt{\varkappa^2+\rho^2}\right)\frac{\cos z\rho\, d\rho}{\sqrt{\varkappa^2+\rho^2}}\Bigg|_{\xi=1} = \qquad \text{(III.20)}$$

$$= \frac{\partial^3}{\partial z\,\partial\xi^2}\left[\frac{\pi}{2}\Theta(\xi - z) J_0(\varkappa\sqrt{\xi^2-z^2})\right]\Bigg|_{\xi=1} =$$

$$= 2\pi\frac{\partial^2}{\partial z\,\partial z^2}\left(\frac{\partial}{\partial z^2} - \frac{1}{2}\right)\left[\Theta(1-z^2)J_0(\varkappa\sqrt{1-z^2})\right]$$

and therefore, continuing the equality (III.19) we have

$$\Psi^*(z,\lambda^2)=\Psi(z,\lambda^2)+\frac{8}{\pi}\left(\frac{\partial}{\partial z^2}\right)^2\left(\frac{\partial}{\partial z^2}-\frac{1}{2}\right)\Bigg(\Theta(1-z^2)\frac{\partial}{\partial\lambda^2}\Big\{\Theta(\lambda^2)\times$$

$$\times\int_{-0}^{\infty}d\varkappa^2 J_0(\varkappa\lambda)J_0(\varkappa\sqrt{1-z^2})\left(\frac{\partial}{\partial\varkappa^2}\right)^2\left[\Theta(\varkappa^2)J_0(\varkappa)\right]\Big\}\Bigg)=\Psi(z,\lambda^2)+$$

$$+\frac{8}{\pi}\left(\frac{\partial}{\partial z^2}\right)^2\left(\frac{\partial}{\partial z^2}-\frac{1}{2}\right)\Bigg(\Theta(1-z^2)\frac{\partial}{\partial\lambda^2}\Big\{\Theta(\lambda^2)\int_{-0}^{\infty}d\varkappa^2 J_0(\varkappa\lambda)J(\varkappa\sqrt{1-z^2})\times$$

$$\times\left[\delta'(\varkappa^2)-\frac{1}{4}\delta(\varkappa^2)+\Theta(\varkappa^2)\left(\frac{\partial}{\partial\varkappa^2}\right)^2J_0(\varkappa)\right]\Big\}\Bigg)=\Psi(z,\lambda^2)+ \qquad \text{(III.21)}$$

$$+\frac{8}{\pi}\left(\frac{\partial}{\partial z^2}\right)^2\left(\frac{\partial}{\partial z^2}-\frac{1}{2}\right)\Bigg(\Theta(1-z^2)\frac{\partial}{\partial\lambda^2}\Big\{\Theta(\lambda^2)\int_{-0}^{\infty}d\varkappa^2\Big|\delta'(\varkappa^2)+\frac{\lambda^2-z^2}{4}\delta(\varkappa^2)+$$

$$+J_0(\varkappa\lambda)J_0(\varkappa\sqrt{1-z^2})\left(\frac{\partial}{\partial\varkappa^2}\right)^2J_0(\varkappa)\Big]\Big\}\Bigg)=\Psi(z,\lambda^2)+$$

$$+\frac{8}{\pi}\left(\frac{\partial}{\partial z^2}\right)^2\left(\frac{\partial}{\partial z^2}-\frac{1}{2}\right)\Big\{\Theta(1-z^2)\Big[\delta(\lambda^2)\frac{\lambda^2-z^2}{4}+\frac{\Theta(\lambda^2)}{4}+\delta(\lambda^2)\int_0^{\infty}d\varkappa^2 J_0(\varkappa\sqrt{1-z^2}$$

$$\times\left(\frac{\partial}{\partial\varkappa^2}\right)^2J_0(\varkappa)+\Theta(\lambda^2)\frac{\partial}{\partial\lambda^2}\int_0^{\infty}d\varkappa^2 J_0(\varkappa\lambda)J_0(\varkappa\sqrt{1-z^2})\left(\frac{\partial}{\partial\varkappa^2}\right)^2J_0(\varkappa)\Big]\Big\}.$$

With the account of the fact that (see G.R., 6.578.3.4).

$$4\int^{\infty}d\varkappa^2 J_0(\varkappa\lambda)J_0(\varkappa\sqrt{1-z^2})\left(\frac{\partial}{\partial\varkappa^2}\right)^2J_0(\varkappa)=$$

$$=-2\int_0^{\infty}J_0(\varkappa\lambda)J(\varkappa\sqrt{1-z^2})\frac{\partial}{\partial\varkappa}\left(\frac{J_1(\varkappa)}{\varkappa}\right)d\varkappa= \qquad \text{(III.22)}$$

$$=1-\chi^*(z^2,\lambda^2),$$

where one denotes

$$\chi^*(z^2,\lambda^2)=2\lambda\int_0^\infty J_1(\varkappa\lambda)\,J_0(\varkappa\sqrt{1-z^2})\,J_1(\varkappa)\frac{d\varkappa}{\varkappa}+2\sqrt{1-z^2}\int_0^\infty J_0(\varkappa\lambda)\times$$

$$\times J_1(\varkappa\sqrt{1-z^2})\,J_1(\varkappa)\frac{d\varkappa}{\varkappa}=\begin{cases}1-z^2+\lambda^2, & \lambda^2<(1-\sqrt{1-z^2})^2,\\ 1, & \lambda^2>(1-\sqrt{1-z^2})^2,\end{cases}\quad \text{(III.23)}$$

we obtain from (III.21)

$$\psi^*(z,\lambda^2)=\psi(z,\lambda^2)+\psi_0(z,\lambda^2), \qquad \text{(III.24)}$$

where $\psi_0(z,\lambda^2)=$

$$=\frac{2}{\pi}\Theta(\lambda^2)\left(\frac{\partial}{\partial z^2}\right)^2\left(\frac{\partial}{\partial z^2}-\frac{1}{2}\right)\left\{\Theta(1-z^2)\left[1-\frac{\partial}{\partial\lambda^2}\chi^*(z^2,\lambda^2)\right]\right\}. \qquad \text{(III.25)}$$

The function $\chi^*(z^2,\lambda^2)$ is continuous when $0\leqslant z^2\leqslant 1$, $\lambda^2\geqslant 0$, equal to unity when $\lambda^2<(1-\sqrt{1-z^2})^2$ and equal to $1-z^2+\lambda^2$ when $\lambda^2>(1+\sqrt{1-z^2})^2$ (see (III.23)). Therefore

$$\psi_0(z,\lambda^2)=\begin{cases}-\frac{2}{\pi}\left[\delta''(1-z^2)+\frac{1}{2}\delta'(1-z^2)\right]\equiv\psi_0(z), & \lambda^2>4,\\ 0, & \lambda^2<(1-\sqrt{1-z^2})^2.\end{cases}\qquad \text{(III.26)}$$

Now we calculate the weight functions $\psi_j^{\circ}(z,\lambda^2)$ which correspond to the functions $F_j^{\circ}(q)$ defined by the formulas (III.2). Owing to (III.7) and (III.16) we have

$$F_1^{\circ}(q) = 2F(q) + F^*(q), \quad F_2^{\circ}(q) = 3F^*(q). \tag{III.27}$$

Therefore, on the basis of (III.24) we can write

$$\psi_1^{\circ}(z,\lambda^2) = 3\,\psi(z,\lambda^2) + \psi_0(z,\lambda^2),$$

$$\psi_2^{\circ}(z,\lambda^2) = 3\,\psi(z,\lambda^2) + 3\,\psi_0(z,\lambda^2), \tag{III.28}$$

where the functions ψ and ψ_0 are defined by the equalities (III.14) and (III.25), respectively. When $\lambda^2 > 4$ the functions $\psi_j^{\circ}(z,\lambda^2)$ are no longer dependent on λ^2 and are equal due to (III.15) and (III.26)

$$\psi_1^{\circ}(z,\lambda^2) = \psi_0(z), \quad \psi_2^{\circ}(z,\lambda^2) = 3\,\psi_0(z). \tag{III.29}$$

Here we note that the weight functions $\psi_j^{\circ}(z,\lambda^2)$, in virtue of (III.15) and (III.26) vanish at $\lambda^2 < (1-\sqrt{1-z^2}\,)^2$ according to the general theorem (see (2.4)). Finally from (III.26) it follows that

$$\int_0^1 \psi_0(z)\, z^2\, dz = 0\,. \tag{III.30}$$

APPENDIX IV. Asymptotic of $\mathcal{D}$ Function

Here we study the asymptotic of the function $\frac{\partial^s}{\partial\lambda^{2s}}\mathcal{D}(x,\lambda^2)$ in the vicinity of the light cone $x^2=0$. Using formula (3.2) we have

$$\frac{\partial^s}{\partial\lambda^{2s}}\mathcal{D}(x,\lambda^2)=\frac{\epsilon(x_0)}{2\pi}\frac{\partial}{\partial x^2}\left[\Theta(x^2)\frac{\partial^s}{\partial\lambda^{2s}}J_0(\sqrt{\lambda^2x^2})\right].$$

Expanding the function J_0 in the vicinity of $x^2=0$ we get for

$$\frac{\partial^s}{\partial\lambda^{2s}}\mathcal{D}(x,\lambda^2)\sim\frac{(-1)^s}{2\pi\,4^s\,\Gamma(s+1)}\epsilon(x_0)\frac{\partial}{\partial x^2}\left[\Theta(x^2)\,x^{2s}\right],$$

and then

$$\frac{\partial^s}{\partial\lambda^{2s}}\mathcal{D}(x,\lambda^2)\sim\begin{cases}\frac{\epsilon(x_0)}{2\pi}\delta(x^2)=\mathcal{D}(x,0)\,, & s=0\,;\\ \frac{(-1)^s\epsilon(x_0)}{2\pi 4^s\,\Gamma(s)}\Theta(x^2)\,x^{2(s-1)}\,, & s\geqslant 1\,.\end{cases}\qquad\text{(IV.I)}$$

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