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SL(2,R) SYMMETRY OF THE DUAL TWO-PARTICLE AMPLITUDE

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1. Introduction

Many recent investigations show that duality is a consequence of some symmetry of the scattering amplitude. Starting with the papers [1,2,3] the real projective transformations, which form the group $SL(2,R)$, became a powerful tool for studying the dual amplitudes. It is most natural to place exactly these transformations in the basis of the dual symmetry. An additional reason for choosing the group $SL(2,R)$ is the possibility of defining a representations of this group in the space of coherent states [4,5], which allow the operatorial factorization of the dual N point amplitude [6,7].

In the present paper we find out a definite group sense of duality for the two-particle amplitude with usual analytical properties. In particular, we show, that duality follows as a consequence of the assumption for symmetry of the amplitude with respect to the unitary irreducible representations of $SL(2,R)$. This allows us to give an integral representation of the dual amplitude. It does not contradict the analyticity and could be exploited for combining both duality and unitarity of the amplitude.

2. Mellin transform of the scattering amplitude

Let $T(s,t,u)$ denote the $s \leftrightarrow u$ crossing symmetrical elastic scattering amplitude for two identical scalar particles. We assume, that $T(s,t,u)$ satisfies the dispersion relation without subtractions:

$$T(s,t,u)=\frac{1}{\pi}\int_{4\mu^2}^{\infty}\frac{\rho_s(s',t)}{s'-s}ds'+\frac{1}{\pi}\int_{4\mu^2}^{\infty}\frac{\rho_u(u',t)}{u'-u}du', \tag{1.1}$$

where μ is the particle mass, s, t, u are the Mandelstam variables and

$$s+t+u=4\mu^2.$$

It is convenient to introduce the following dimensionless variables:

$$\begin{aligned}\sigma &= a(s-4\mu^2) \\ \tau &= a(t+8\mu^2) \\ \omega &= a(u-4\mu^2).\end{aligned} \tag{1.2}$$

Here a is an arbitrary constant with dimensionality $1/\mu^2$. Obviously

$$\sigma+\tau+\omega=4\mu^2 a. \tag{1.3}$$

If we denote

$$A(\sigma,\tau,\omega)=T\left(\frac{\sigma}{a}+4\mu^2,\frac{\tau}{a}-8\mu^2,\frac{\omega}{a}+4\mu^2\right)$$

eq. (1.1) leads to

$$A(\sigma,\tau,\omega)=\frac{1}{\pi}\int_{0}^{\infty}\frac{\rho_\sigma(\sigma',\tau)}{\sigma'-\sigma}d\sigma'+\frac{1}{\pi}\int_{0}^{\infty}\frac{\rho_\omega(\omega',\tau)}{\omega'-\omega}d\omega', \tag{1.4}$$

where

$$\rho_\sigma(\sigma';\tau) = \rho_s\left(\frac{\sigma'}{a}+4\mu^2, \frac{\tau}{a}-8\mu^2\right) \tag{1.5}$$

$$\rho_\omega(\omega';\tau) = \rho_u\left(\frac{\omega'}{a}+4\mu^2, \frac{\tau}{a}-8\mu^2\right).$$

The relation (1.3) allows us to eliminate ω and ω' in eq. (1.4). Then we obtain

$$\mathcal{A}(\sigma,\tau) = \frac{1}{\pi}\int_0^\infty \frac{\rho_1(\sigma';\tau)}{\sigma'-\sigma}\,d\sigma' + \frac{1}{\pi}\int_{-\infty}^{4\mu^2 a-\tau} \frac{\rho_2(\sigma';\tau)}{\sigma-\sigma'}\,d\sigma'. \tag{1.6}$$

Here we have introduced the following notation

$$\begin{aligned} \mathcal{A}(\sigma,\tau) &= \mathcal{A}(\sigma,\tau,4\mu^2 a-\sigma-\tau) \\ \rho_1(\sigma';\tau) &= \rho_\sigma(\sigma';\tau) \\ \rho_2(\sigma';\tau) &= \rho_\omega(4\mu^2 a-\sigma'-\tau,\tau). \end{aligned} \tag{1.7}$$

Eq. (1.6) shows, that $\mathcal{A}(\sigma,\tau)$ has two cuts along the real axis in the complex σ plane. The first one goes from $\sigma=0$ to $\sigma=+\infty$ and the discontinuity of $\mathcal{A}(\sigma,\tau)$ on this cut is $\rho_1(\sigma,\tau)$. The second one goes from $\sigma=4\mu^2 a-\tau$ to $\sigma=-\infty$ and the corresponding disoontinuity is $\rho_2(\sigma,\tau)$.
As far as the position of the second branch-point depends on τ, we consider only those τ, which satisfy

$$4\mu^2 a-\tau<0, \tag{1.8}$$

i.e. we consider only the case, when both cuts do not overlap. Hence $\mathcal{A}(\sigma,\tau)$ is an analytical function on the real axis between both branch-points. If we consider those σ only, for which

$$4\mu^2 a-\tau < \sigma < 0, \tag{1.9}$$

then in eq. (1.6) we have in the first integral

$$Re\,(\sigma'-\sigma) > 0 \tag{1.10}$$

and in the second one

$$Re\,(\sigma-\sigma') > 0. \tag{1.11}$$

These last two inequalities allow us to replace both denominators with

$$\frac{1}{\sigma'-\sigma} = \int_0^1 x^{\sigma'-\sigma-1}\,dx \tag{1.12}$$

in the first integral and correspondingly with

$$\frac{1}{\sigma-\sigma'} = \int_0^1 x^{\sigma-\sigma'-1}\,dx \tag{1.B}$$

in the second one. Then instead of eq. (1.6) we get:

$$A(\sigma,\tau) = \int_0^1 \varrho_1(x,\tau)\,x^{-\sigma-1}\,dx + \int_0^1 \varrho_2(x,\tau)\,x^{\sigma-1}\,dx \quad , \tag{1.14}$$

where

$$\varrho_1(x,\tau) = \frac{1}{\pi}\int_0^\infty \varrho_1(\sigma',\tau)\,x^{\sigma'}\,d\sigma'$$

$$\varrho_2(x,\tau) = \frac{1}{\pi}\int_{-\infty}^{4\mu^2 a-\tau} \varrho_2(\sigma',\tau)\,x^{-\sigma'}\,d\sigma'. \tag{I.I5}$$

Now we recall, that $\rho_1(\sigma,\tau)$ and $\rho_2(\sigma,\tau)$ are the discontinuities of the amplitude along the cuts, i.e.

$$\rho_1(\sigma,\tau)=\frac{1}{2i}\left[A(\sigma+i0,\tau)-A(\sigma-i0,\tau)\right] \qquad \sigma>0\,,$$

$$\rho_2(\sigma,\tau)=\frac{-1}{2i}\left[A(\sigma+i0,\tau)-A(\sigma-i0,\tau)\right] \qquad \sigma<4\mu^2a-\tau\,.$$

The terms $x^{\sigma'}$ and $x^{-\sigma'}$ do not insert any new cuts in the integrand of eq. (1.15). Consequently, $\rho_1(x,\tau)$ and $\rho_2(x,\tau)$ can be represented as contour integrals

$$\rho_1(x,\tau)=\frac{1}{2\pi i}\int_{C_1}A(\sigma',\tau)x^{\sigma'}d\sigma' \tag{1.16}$$

$$\rho_2(x,\tau)=\frac{1}{2\pi i}\int_{C_2}A(\sigma',\tau)x^{-\sigma'}d\sigma', \tag{1.17}$$

where the contours C_1 and C_2 are shown on Fig. (1 .)

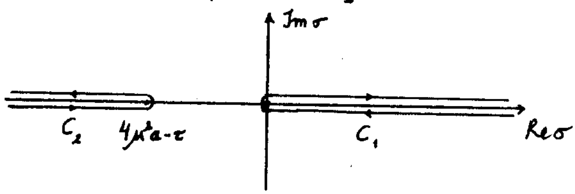

Fig. 1

Now we can deform C_1 and C_2 making them coincident and parallel to the imaginary axis. (As far as $A(\sigma,\tau)$ satisfies the dispersion relation without subtractions, the contributions of the integral over the infinite circle are equal to zero). Then by comparison of eq. (1.16) and (1.17) we get

$$\rho_1\left(\frac{1}{x}, \tau\right) = \rho_2(x, \tau). \tag{1.18}$$

This last equation allows us to transform eq. (1.14) into the form:

$$\mathcal{A}(\sigma, \tau) = \int_0^\infty \rho_1(x, \tau)\, x^{-\sigma-1}\, dx \tag{1.19}$$

$$4\mu^2 a - \tau < \operatorname{Re}\sigma < 0.$$

Instead of eq. (1.16) we get

$$\rho_1(x, \tau) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \mathcal{A}(\sigma, \tau)\, x^{\sigma}\, d\sigma \qquad 4\mu^2 a - \tau < c < 0. \tag{1.20}$$

Eqs. (1.19) and (1.20) show that the amplitude $\mathcal{A}(\sigma, \tau)$ is the Mellin transform of $\rho_1(x, \tau)$ and vice versa: $\rho_1(x, \tau)$ is an inverse Mellin transform of the amplitude.

Here we list some simple properties of $\rho(x, \tau)$ (in what follows we drop out the index "1"). The first one is a consequence of the $\sigma \leftrightarrow \omega$ crossing-symmetry of the amplitude:

$$\mathcal{A}(\sigma, \tau) = \mathcal{A}(\omega, \tau). \tag{1.21}$$

Using eqs. (1.19), (1.21) and (1.3) we get the following functional equation

$$\rho(x, \tau) = x^{4\mu^2 a - \tau}\, \rho\left(\frac{1}{x}, \tau\right). \tag{1.22}$$

It shows, that the behaviour of $\rho(x,\tau)$ at the point $x=0$ determines the behaviour for $x\to\infty$ and vice versa. But (see Eq. (1.19)) the behaviour of $\rho(x,\tau)$ for $x\to 0$ fixes the position and the kind of the right-hand singularity of $A(\sigma,\tau)$. If, for instance $\rho(x,\tau)\xrightarrow[x\to 0]{} x^{-\alpha} f(\tau)$ $(\alpha>0)$, then the r.h. branch point shifts to the left.
This fact contradicts our assumption, that $A(\sigma,\tau)$ is an analytical function on the interval $(4\mu^2 a-\tau, 0)$.
Therefore such a behaviour of $\rho(x,\tau)$ for $x\to 0$ is eliminated. On the other hand, if $\rho(x,\tau)\xrightarrow[x\to 0]{} x^{\alpha} f(\tau)$ ($\alpha>0$), the same singularity shifts to the right, thus extending the analyticity interval. A similar situation takes place in the narrow-resonance approximations, where the simple poles on the real axis are the only singularities of $A(\sigma,\tau)$.

Without loss of generality we assume, that for $x\to 0$ $\rho(x,\tau)$ has no power singularities and does not go to zero as a power of x. In the case of narrow-resonance approximation one has to replace in eqs. (1.19) and (1.20) $x^{-\sigma-1}$ and x^{σ} with $x^{-(\sigma-b)-1}$ and $x^{\sigma-b}$, respectively. The constant b fixes the displacement of the first pole with respect to the threshold $4\mu^2 a$. The new $\rho(x,\tau)$ which we obtain after such a replacement yields the same $A(\sigma,\tau)$ and has no zeros for $x=0$. In order to simplify the formulae further we use (1.19) and (1.20) and only in the final results, if necessary, the narrow-resonance approximation (i.e. the replacement $\sigma \to \sigma - \beta$) should be assumed.

Now from eq. (1.22) we see, that $\rho(x,\tau) \longrightarrow x^{4\mu^2 a-\tau}$ for $x \to \infty$.

Summing the results of this section we note, that the crossing-symmetrical scattering amplitude for two identical scalar particles, which satisfies the dispersion relation (1.1) in the region (1.9) can be represented in the form of Mellin integral (1.19), where $\rho(x,\tau)$ (eq. (1.20) satisfies the crossing-symmetry condition (1.22).

2. Duality and the group $SL(2,R)$.

The Mellin transformation is a very convenient tool for investigation of the duality. As a matter of fact, the integral representations for the simple dual amplitudes have the form of the Mellin transform. In addition, the pole structure of these amplitudes leads to simple expressions for the function $\rho(x,\tau)$.

In this section we consider duality from the point of view of irreducible representations of the group $SL(2,R)$. At first we define duality in a way, which is most convenient for our further discussion. The definition we are going to exploit is contained in a series of papers [8-13], but we shall discuss it in the form, as given in [14]. It has been shown in this last paper, that if the scattering amplitude for two identical scalar particles:

1. is a meromorphic function;

2. satisfies the crossing-symmetry condition;

3. possesses Regge asymptotic;

4. is such a function, that the finite energy sum rules [15-18] are saturated with poles only (i.e. the sum of pole terms in a channel equals the sum of asymptotic terms in the same channel, or so-called mathematical duality [19]). then the amplitude can be represented as a sum of Euler's B-functions. As is well known, the set of these four conditions determines the narrow-resonance approximation of the amplitude.

An attempt to take into aocount the presence of the kinematical cut has been done in the same paper. To do this one has to give up the first condition and to change the fourth one [20]. The asymptotical power series for the amplitude is determined by the pole terms too, but the existence of the cut leads to the emergence of some logarithmic therms in the asymptotic. We see, that in this case the finite energy sum rules lead to local connection of the asymptotic not only with the pole structure, but also with branch points in the same channel. Suoh a wider connection could be used as a natural generalization of the common mathematioal duality. On the basis of this more general definition the authors of the above-mentioned paper have been able to write down a Mellin-type integral representation for the amplitude. The essenoe of their results, expressed in our notation (Seotion 1), oonsists in the statement, that the function $\rho(x,\tau)$ has to be in the form:

$$\rho(x,\tau)=(1+x)^{4\mu^2 a-\tau} f\left(\frac{x}{1+x}\right), \tag{2.1}$$

where $f(y)$ is an arbitrary function. (The crossing-symmetry condition, which leads to some simple requirement for $f(y)$ is not under consideration here). The assumption, that $f(y)$ has a Taylor's expansion around the point $y=0$ leads to Veneziano-type amplitudes. If $f(y)$ has definite logarithmic behaviour at the same point, then the amplitude has a cut in the s-plane. Eq. (2.1) is a straight consequence of the above-mentioned generalized mathematical duality. That is why we accept eq. (2.1) as a definition of duality.

The connection between the dual amplitudes and the group of the projective-transformations has been considered in many papers [1,2,5,21-25]. A lot of the properties of the amplitudes have been described by means of these transformations and their representations. The operatorial factorization has revealed definite group structure of the amplitudes. We would like to note the papers [26], where, in our opinion, a group sense of the Veneziano type amplitudes has been given. In particular, the author turns one's attention to the following fact which is well known in the theory of the representations of the group $SL(2,R)$ [27]; there are representations of the triangular matrices subgroup, in which the Euler's B-function plays the role of the kernel in the integral operator of these representation. This result is respectively generalized for the case of the Veneziano N-point function.

Let us analyse in some details this fact for the case N==4. Denote by Δ_γ the subgroup of the triangular matrices in $SL(2,R)$ (we define it later). To the given representation of Δ_γ there corresponds a global representation of the whole group $SL(2,R)$. Then the kernel of the integral operator is in fact a matrix element of the global representations operator. Therefore the kernel (i.e. the Euler's B- function) could be considered as a function, given on the subgroup Δ_γ . Further we recall, that together with Δ_γ $SL(2,R)$ contains another, complementary subgroup S_β of triangular matrices. Each element of Δ_γ determines a coset with respect to S_β and can be chosen as representative of this coset (See Appendix A). Consequently every function, defined on Δ_γ can be considered as a function, defined on the set of cosets $S_\beta \backslash SL(2,R)$. Thus, one can state, that the dual elastic scattering amplitude is a function, defined on a certain homogeneous space $S_\beta \backslash SL(2,R)$ of the group $SL(2,R)$. It is known, that usual spherical functions are also functions, defined on a homogeneous space, as far as the sphere is a homogeneous space with respect to the rotation group. That is why we call the functions, defined on $S_\beta \backslash SL(2,R)$ spherical, too.

We shall construct such spherical functions and shall show that the functions (2.1) are spherical ones.

By definition, $SL(2,R)$ is the group of all real unimodulator 2×2 mattices. An arbitrary element $g\in SL(2,R)$ can be written in the form:

$$g=\begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix} \qquad \alpha\delta-\beta\gamma=1. \tag{2.2}$$

The above-mentioned subgroup S_β contains all triangular matrices of the type:

$$s_\beta=\begin{pmatrix}1 & \beta \\ 0 & 1\end{pmatrix}. \tag{2.3}$$

Obviously S_β is an abbelian subgroup of $SL(2,R)$. Let us consider the space of all cosets

$$s_\beta g \tag{2.4}$$

It is easy to see, that two elements

$$g_1=\begin{pmatrix}\alpha_1 & \beta_1 \\ \gamma_1 & \delta_1\end{pmatrix} \qquad g_2=\begin{pmatrix}\alpha_2 & \beta_2 \\ \gamma_2 & \delta_2\end{pmatrix}$$

belong to the same coset (2.4) if

$$\gamma_1 = \gamma_2 \qquad \delta_1 = \delta_2. \tag{2.5}$$

Therefore as representative of given coset (2.4) we can accept the matrix of the type:

$$\bar{g} = \begin{pmatrix} \delta^{-1} & 0 \\ \gamma & \delta \end{pmatrix}. \tag{2.6}$$

The matrices $\bar{g}$ form a subgroup of $SL(2,R)$ which we have denoted by Δ_γ.

Now, consider the space $\mathfrak{U}$ of all functions $f(x)$ defined on the real axis. To every $g \in SL(2,R)$ we put into correspondence an operator V_g^λ which acts on $f(x) \in \mathfrak{U}$ according to the definition

$$V_g^\lambda f(x) = |\beta x + \delta|^\lambda f\left(\frac{\alpha x + \gamma}{\beta x + \delta}\right), \tag{2.7}$$

where λ is an arbitrary number. It is easy to check, that V_g^λ give a representation of $SL(2,R)$ in $\mathfrak{U}$. (One should note, that as far as $\mathfrak{U}$ contains all functions on the real axis, this representation is in general reducible).

Let $\mathfrak{M}$ be the set of all functions $\varphi(g,f)$ defined on the group $SL(2,R)$ where $g \in SL(2,R)$, $f \in \mathfrak{U}$.

Obviously, every $\varphi(g,f)$ corresponds to certain $f(x)\in \mathfrak{U}$.
We assume $\varphi(g,f)$ to be a linear functional on $\mathfrak{U}$, which satisfies the following two conditions:

$$\varphi(g, V_h^\lambda f) = \varphi(gshs^{-1}, f) \qquad h\in SL(2,R) \tag{2.8}$$

$$\varphi(s_\beta, f) = \varphi(e,f), \tag{2.9}$$

where e is the unite element of $SL(2,R)$ and s is a matrix from s_β for which $\beta = 1$.

According to eq. (2.8) transforming f we can vary the argument g of $\varphi(g,f)$. As far as f is the "index" of the function $\varphi(g,f)$, eq. (2.8) shows, that if we know the value of $\varphi(g,f)$ in a given point (i.e. for fixed g), by means of the index transformation we can obtain its value in every other point. Obviously, this property is similar to the analogous one, which takes place for usual spherical functions.

Let us consider the functional $\varphi(s_\beta g, f)$. It follows from eqs. (2.8) and (2.9) that

$$\varphi(s_\beta g, f) = \varphi(g,f). \tag{2.10}$$

Consequently, as a function of g, $\varphi(g,f)$ is defined on the space of cosets (2.4), i.e. $\varphi(g,f)$ does not change its value when g varies inside a given coset. Hence eqs. (2.8) and (2.9) define $\varphi(g,f)$ as spherical functions of the homogeneous space, which is equivalent to the set of all the cosets (2.4).

One can define two representations of $SL(2,R)$ in $\mathfrak{M}$. The operators T_h of the first one are determined as follows:

$$T_h \varphi(g,f) = \varphi(gh, f) \tag{2.11}$$

and the operators R_h of the second one(the "index" transformation), respectively:

$$R_h \varphi(g,f) = \varphi(g, V^{\lambda}_{s^{-1}} V^{\lambda}_{h^{-1}} V^{\lambda}_{s} f). \tag{2.12}$$

Using eq. (2.8) we can find, that

$$R_h \varphi(g,f) = \varphi(gh^{-1}, f). \tag{2.13}$$

Consequently, if we know the value of a spherical function at one point g, by means of the transformation R_h we can get its value at every other point. In particular:

$$\varphi(g,f) = R_{g^{-1}}\varphi(e,f). \tag{2.14}$$

Using eqs. (2.7),(2.8),(2.9),(2.12) and (2.14) we get:

$$\varphi\left(e, f(x) - |\beta x+1|^{\lambda} f\left(\frac{x}{\beta x+1}\right)\right) = 0 \tag{2.15}$$

for arbitrary $f(x) \in \mathcal{U}$. As far as $\varphi(e,f)$ is not identically zero, from eq. (2.15) it follows, in particular,

$$\varphi(e,f) = \int_{-\infty}^{\infty} \delta(x) f(x) dx = f(0). \tag{2.16}$$

Then, in view of eqs. (2.14) and (2.12) we have

$$\varphi(g,f) = \left[V^{\lambda}_{s^{-1}} V^{\lambda}_{g} V^{\lambda}_{s} f(z) \right]_{z=0}. \tag{2.17}$$

Transforming the r.h.s. of eq. (2.17) with the help of eq. (2.7) we get:

$$\varphi(g,f) = |\gamma+\delta|^{\lambda} f\left(\frac{\gamma}{\gamma+\delta}\right). \tag{2.18}$$

Since $\varphi(g,f)$ depends on δ and γ only as a function of g it is oonstant on every one from the cosets (2.4). When a $SL(2,R)$ transformation is applied, both variables δ and γ change according to the definition (2.11). If we denote

$$\delta = y$$

and choose γ in the form

$$\gamma = xy$$

we get

$$\varphi(g,f) \equiv \varphi(x,y;f) = |y|^{\lambda} |1+x|^{\lambda} f\left(\frac{x}{1+x}\right), \tag{2.19}$$

where x changes in a linear fractional way by the operation (2.11). Using the homogeneity of the two dimensional (x,y) spaoe we fix $y=1$:

$$\varphi(x,1;f) \equiv \varphi(x,f) = |1+x|^{\lambda} f\left(\frac{x}{1+x}\right). \tag{2.20}$$

Then instead of eq. (2.11) we have

$$T_h \varphi(x,f) = |\beta x+\delta|^{\lambda} \varphi\left(\frac{\alpha x+\gamma}{\beta x+\delta}, f\right) \qquad h \in SL(2,R) \tag{2.21}$$

Obviously, for $\lambda = 4\mu^2 a - \tau$ the expression (2.20) coincides with (2.1) and this proves our statement. Eq. (2.21) shows that both the spherical functions and the functions f are transformed in the same way. Now we notice, that in the whole set of functions $f(x)$ and $\varphi(g, f)$ there exists a subset of functions, which asymptotic is x^{λ} for $x \to \infty$. On this subset both representations (2.7) and (2.21) are irreducible and equivalent. In what follows we assume, that they are irreducible and will not distinguish between them.

Thus, in Section 2 we have shown, that the inverse Mellin transform $\varrho(x,\tau)$ of the dual amplitude can be considered as a function, defined on the homogeneous space of the group $SL(2,R)$. This space is isomorphic to the cosets (2.4). By analogy with the usual sperical functions we call these functions spherical too.

3. A basis in the spherical function space

In this section we introduce a basis in the space of our spherical functions. This gives us the possibility of expressing an arbitrary inverse Mellin transform $\varrho(x,\tau)$ of the dual amplitude as a superposition of the basis elements.

For this purpose we first consider the $SL(2,R)$ algebra. Its generators are easily obtained through successive differentiation with respect to all three parameters of the group. Here we use a parametrization, which is similar to that from ref. [5] :

$$g(\alpha_1,\alpha_2,\alpha_3)=\begin{pmatrix} e^{\frac{\alpha_2}{2}} & \frac{1}{2}(\alpha_1+\alpha_3) \\ \frac{1}{2}(-\alpha_1+\alpha_3) & \left(1-\frac{\alpha_1^2-\alpha_3^2}{4}\right)e^{-\frac{\alpha_2}{2}} \end{pmatrix}. \qquad (3.1)$$

The generators of the representation (2.21) are defined as follows:

$$I_i=\left(\frac{\partial T_{g(\alpha_1,\alpha_2,\alpha_3)}}{\partial \alpha_i}\right)_{\alpha_1=\alpha_2=\alpha_3=0}. \qquad (3.2)$$

Then we have

$$\begin{aligned} I_1 &= \frac{\lambda}{2}x-\frac{1}{2}(1+x^2)\frac{\partial}{\partial x} \\ I_2 &= -\frac{\lambda}{2}+x\frac{\partial}{\partial x} \\ I_3 &= \frac{\lambda}{2}x+\frac{1}{2}(1-x^2)\frac{\partial}{\partial x} \end{aligned} \qquad (3.3)$$

and I_1, I_2, I_3 satisfy the commutation relations

$$[I_1,I_2]=-I_3, \quad [I_2,I_3]=I_1, \quad [I_3,I_1]=-I_2. \qquad (3.4)$$

The simplest spherical function of the type (2.20) can be obtained by putting

$$\varphi(x,const)=const\,|1+x|^{\lambda}. \qquad (3.5)$$

It is easy to see, that $\varphi(x,const)$ is an eigenfunction of I_3 :

$$I_3\varphi(x,const)=\frac{\lambda}{2}\varphi(x,const). \tag{3.6}$$

Acting on $\varphi(x,const)$ with I_1-I_2 and I_1+I_2 we get:

$$(I_1-I_2)\varphi(x,const)=0 \tag{3.7}$$

$$(I_1+I_2)\varphi(x,const)=const\,|1+x|^{\lambda}\left(\frac{1-x}{1+x}\right). \tag{3.8}$$

The function $(I_1+I_2)\varphi(x,const)$ is an eigenfunction of I_3 too, but its eigenvalue is $\frac{\lambda}{2}-1$. Thus $\varphi(x,const)$ plays the role of the highest vector in the space,in which the generators (3.3) act.Now we introduce the notation:

$$e_0^{\lambda}\equiv\varphi(x,const)=const\,|1+x|^{\lambda} \tag{3.9}$$

and apply the same prodedure to the vector (3.8), getting a new eigenvector of I_3 and so on. As a result we obtain the veotors:

$$e_k^{\lambda}(x)=N\,|1+x|^{\lambda}\left(\frac{1-x}{1+x}\right)^k \tag{3.10}$$

which satisfy the equations:

$$\begin{aligned} A_0 e_k^{\lambda} &= \left(\frac{\lambda}{2}-k\right)e_k^{\lambda} \\ A_+ e_k^{\lambda} &= (k-\lambda)e_{k+1}^{\lambda} \\ A_- e_k^{\lambda} &= k\,e_{k-1}^{\lambda}\quad, \end{aligned} \tag{3.11}$$

where we have introduced the notation

$$\mathcal{A}_+ = I_1 + I_2 \qquad \mathcal{A}_- = I_1 - I_2 \qquad \mathcal{A}_0 = I_3 \, . \tag{3.12}$$

The form of the functions (3.10) is similar to that of our spherical functions. All linear combinations of e_κ^λ form a space, which for κ integer is invariant with respect to the action of the operators (3.12). For κ even e_κ^λ satisfy the even crossing symmetry condition (1.22) and for odd κ — the odd crossing symmetry condition.

If it is possible to choose e_κ^λ as a basis in the spherical functions space, then an arbitrary $\rho(x,\tau)$ of the type (2.1) can be represented as a sum of e_κ^λ . When we go from $\rho(x,\tau)$ to $\mathcal{A}(\sigma,\tau)$ this sum will be transformed into the sum of Eiler's B-functions (as far as B-function is the Mellin transform of e_κ^λ):

$$\int_0^\infty x^{-\sigma-1}(1+x)^\lambda \left(\frac{1-x}{1+x}\right)^\kappa dx = \sum_{n_1+n_2=\kappa} \frac{\kappa!(-1)^{n_1}}{n_1!\,n_2!} B(-\sigma+n_1, -\lambda+\sigma+n_2) \tag{3.13}$$

$$\lambda < \sigma .$$

Now the question arises whether or not we can integrate the representation (3.11) of the $SL(2,R)$ algebra and get a global group representation. If this integration could be done, then e_κ^λ would form a basis not only for the algebra, but for the group representation too (the definition of the generators guarantees, that the last one coincides with (2.21)).

We have mentioned, that $\lambda = 4\mu^2 a - \tau$, but we did not specify what kind of irreducible representation we are going to exploit. As a function of τ λ changes continuously. Hence, we have to determine (2.21) in a way, which allows continuous variation of λ. The only unitary irreducible representations, which satisfy these conditions are those with

$$-2 < \lambda = 4\mu^2 a - \tau < 0. \tag{3.14}$$

They form the so-called supplementary series of representations of the group $SL(2,R)$ [28]. For all other unitary series either λ takes integer values, or only $\mathrm{Im}\,\lambda$ changes continuously and hence they are not suitable for our purpose.

Let $\varphi_1(x)$ and $\varphi_2(x)$ belong to the space of functions, which are transformed with the help of the representations from the supplementary series. By means of the formula [28]

$$(\varphi_1, \varphi_2) = \frac{1}{\Gamma(-\lambda-1)} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, |x-y|^{-\lambda-2} \varphi_1(x) \overline{\varphi_2(y)} \tag{3.15}$$

one can introduce in this space an invariant scalar product, which satisfies the condition $(\varphi,\varphi) > 0$. (Here $\overline{\varphi}$ means complex conjugation). This fact allows us to verify whether or not the vector (3.10) belong to the space of the global representation of the group. If we calculate $(e_\kappa^\lambda, e_\kappa^\lambda)$ we shall find, that the integral in the r.h.s. of (3.15) diverges. Consequently, for positive integer κ the vectors e_κ^λ do not belong to the space of these representations.

Therefore we can make the following conclusion. We have seen, that the meromorphic approximation and simple mathematical duality assumption lead to a series of Eiler's B-functions for the amplitude. This is a decomposition in the basis of the representation of the $SL(2,R)$ algebra. That is why the inverse Mellin transform $\rho(x,\tau)$ for such an amplitude is not a spherical function. The fact, that the eigenvalues of $I_3 = A_0$ are real, shows onoe more, that the algebra representation (3.11) cannot be integrated in the space of veotors (3.10). Actually, if the group representation (2.21) is unitary, it follows from our definition (3.2), that $\mathcal{A}_0$, $\mathcal{A}_+$ and $\mathcal{A}_-$ should be skew-Hermitian ($\mathcal{A}_0$, $\mathcal{A}_+$ and $\mathcal{A}_-$ are real linear combinations of I_1, I_2 and I_3). Then, if the representation (3.11) of the algebra could be integrated, $\mathcal{A}_0$ could not have real eigenvalues.

Therefore, the vectors (3.10) do not form a basis in the spherioal function space. However, the last remark concerning the eigenvalues of $\mathcal{A}_0$ shows the way, in which this basis could be obtained. Taking into account, that κ in eq. (3.11) can be arbitrary complex number, we assume

$$\frac{\lambda}{2} - \kappa = i\nu \tag{3.16}$$

Thus $\mathcal{A}_0$ becomes a skew-Hermitian operator. Nevertheless, this assumption leads to some complications. First of all, because of the branch-point $x = 1$, the functions (3.10) are not determined uniquely. Second, for arbitrary κ the veotors e^{λ}_{κ} do not have definite crossing symmetry properties. That is why we have to redefine the functions (3.10). (One encounters similar complications in Regge theory by analytical continuation of Legendre-polynomials from integer to arbitrary indices). If we substitute K from (3.16) into (3.10), we can choose such a branch of the multivalued function, which still keeps the crossing-symmetry properties of e_k^λ . As far as the new vectors depend on the index ν we denote them e_ν^λ and define:

$$e_\nu^{\lambda\pm} = \begin{cases} N(1-x^2)^{\frac{\lambda}{2}}\left(\frac{1-x}{1+x}\right)^{i\nu} & |x|<1 \\ \\ \pm N(x^2-1)^{\frac{\lambda}{2}}\left(\frac{x-1}{x+1}\right)^{i\nu} & |x|>1, \end{cases} \tag{3.17}$$

where (+) and (–) concern the cross-even and cross-odd functions, correspondingly. In what follows we omit this signs, having in mind the cross-even functions only. Every further result could be easily obtained for the cross-odd funotions, too (see Appendix B).

Let us consider the most important properties of the functions (3.17):

1. The orthonormality condition. Using the definition (3.15) of the scalar product we get

$$\left(e_{\nu_1}^\lambda, e_{\nu_2}^\lambda\right) = N(\lambda,\nu_1)\,\delta(\nu_1-\nu_2), \tag{3.18}$$

where

$$N(\lambda,\nu)=|N|^2 2^{\lambda+3}\sin\frac{\pi}{2}(\lambda+2)\left[ch\,\pi\nu+\cos\frac{\pi}{2}(\lambda+2)\right]\left|\Gamma\left(\frac{\lambda}{2}+i\nu+1\right)\right|^2.$$

Obviously, for different ν the vectors e_ν^λ are orthogonal. Using the inequality (3.14) we can get:

$$\mathcal{N}(\lambda,\nu_1)>0. \tag{3.19}$$

Taking into account, that

$$|\Gamma(x+iy)|^2 \underset{|y|\to\infty}{\simeq} 2\pi |y|^{2x-1} e^{-\pi|y|} \tag{3.20}$$

we find out the asymptotic behaviour of $\mathcal{N}(\lambda,\nu_1)$ for large $|\nu_1|$:

$$\mathcal{N}(\lambda,\nu_1) \underset{|\nu_1|\to\infty}{\simeq} \pi|N|^2 2^{\lambda+3} |\nu_1|^{\lambda+1} \sin\frac{\pi}{2}(\lambda+2). \tag{3.21}$$

2. Completeness. Consider the function

$$\varphi(x)=\int_{-\infty}^{\infty} d\nu\, c(\nu)\, e_\nu^\lambda(x). \tag{3.22}$$

The integral from the r.h.s. of eq. (3.22) converges at the ends of the integration interval if the integral

$$\int_{-\infty}^{\infty} d\nu\, |c(\nu)| \tag{3.23}$$

converges. ($|e_\nu^\lambda(x)|$ does not depend on ν). Thus, to every modulo-integrable function $c(\nu)$ there corresponds a function $\varphi(x)$, defined by eq. (3.22). We shall show, that

$\varphi(x)$ is spherical function. First of all we substitute $e^{\lambda}_{\nu}(x)$ from (3.1) into (3.22) and get

$$\varphi(x)=\int_{-\infty}^{\infty}d\nu\, c(\nu)\left|1-x^2\right|^{\frac{\lambda}{2}}\left|\frac{1-x}{1+x}\right|^{i\nu}, \tag{3.24}$$

i.e. indeed $\varphi(x)$ is a function of the type (2.20). Second we have to show, that (φ,φ) exists. Substituting (3.24) into (3.13) and taking into account the orthonormality condition we obtain

$$(\varphi,\varphi)=\int_{-\infty}^{\infty}d\nu\,|c(\nu)|^2\mathcal{N}(\lambda,\nu). \tag{3.25}$$

The convergence of this integral is guaranteed by (3.21) and (3.23). Because of the inequality (3.19) (φ,φ) is positive. In this way we have proved, that the functions $\varphi(x)$ are spherical.

The orthogonality of $e^{\lambda}_{\nu}(x)$ allows us to express $c(\nu)$ through $\varphi(x)$. To do this we form the scalar product $(\varphi(x), e^{\lambda}_{\nu}(x))$ and get:

$$c(\nu)=\frac{1}{\mathcal{N}(\lambda,\nu)}(\varphi,e^{\lambda}_{\nu}) \tag{3.26}$$

It is easy to see, that for arbitrary spherical function $\varphi(x)$ the integral in the r.h.s. of (3.26) converges.

Thus the vectors $e^{\lambda}_{\nu}(x)$ form a complete set in the space of the spherical functions, which are transformed according to the unitary irreducible representation (2.21) of $SL(2,R)$. We note, however, that the functions $e^{\lambda}_{\nu}(x)$ themselves do not belong to this space.

This result shows, that by means of the integral (3.22) the inverse Mellin transform $\rho(x,\tau)$ of the dual amplitude can be represented as a superposition of the vectors $e_{\nu}^{\lambda}(x)$.

3. The reality condition. Eq. (3.22) defines $\varphi(x)$ as a complex function of the real argument x. On the other hand, the inverse Mellin transform $\rho(x,\tau)$ of the physical amplitude is a real function. Consequently we have to impose some conditions on $C(\nu)$ which provide the reality of $\varphi(x)$. Using the identity:

$$e_{-\nu}^{\lambda} = \overline{e_{\nu}^{\lambda}} \tag{3.27}$$

we can write of (3.22) into the form

$$\varphi(x) = \int_0^\infty d\nu \left[c(\nu) e_{\nu}^{\lambda}(x) + c(-\nu) \overline{e_{\nu}^{\lambda}(x)} \right]. \tag{3.28}$$

Thus, the condition, which leads to real $\varphi(x)$ is:

$$C(-\nu) = \overline{C(\nu)}. \tag{3.29}$$

In particular, for real $c(\nu)$ we have

$$C(-\nu) = C(\nu)$$

and

$$\varphi(x) = \varphi(-x) = \int_0^\infty d\nu \, c(\nu) \left[e_{\nu}^{\lambda}(x) + \overline{e_{\nu}^{\lambda}(x)} \right]. \tag{3.30}$$

4. Let us briefly consider the "index" transformation R_h (2.12), applied to the basis vectors $e_\nu^\lambda(x)$. The operators R_h are integral operators with kernel $R(\nu,\nu';h)$. According to eq. (2.13) we have

$$\int_{-\infty}^{\infty} d\nu' \begin{pmatrix} R_{++}^\lambda(\nu,\nu';h) & R_{+-}^\lambda(\nu,\nu';h) \\ R_{-+}^\lambda(\nu,\nu';h) & R_{--}^\lambda(\nu,\nu';h) \end{pmatrix} \begin{pmatrix} e_{\nu'}^\lambda(x)_+ \\ e_{\nu'}^\lambda(x)_- \end{pmatrix} = \tag{3.31}$$

$$= |-\beta x+\alpha|^\lambda \begin{pmatrix} e_\nu^\lambda\left(\frac{\delta x-\gamma}{-\beta x+\alpha}\right)_+ \\ e_\nu^\lambda\left(\frac{\delta x-\gamma}{-\beta x+\alpha}\right)_- \end{pmatrix},$$

where

$$e_\nu^\lambda(x)_+ = e_\nu^\lambda(x) \qquad |x|<1$$

$$e_\nu^\lambda(x)_- = e_\nu^\lambda(x) \qquad |x|>1$$

and

$$h=\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \qquad h^{-1} = \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix}.$$

The form of the kernel R and the proof of eq. (3.11) are given in Appendix C.

Let us sum the results of this Section. We have seen, that the simple mathematical duality, which leads to Veneziano type [x] models, is a consequence of the assumption, that $\rho(x,\tau)$ is a function from the space of the nonintegrable representations of $SL(2,R)$ algebra. The spherical functions, which we have obtained starting from the more general definition (2.1) of mathematical duality, can be treated as elements of a space, in which acts a unitary representation of $SL(2,R)$. Therefore the definition (2.1) and the statement, that $\rho(x,\tau)$ belong to this space are equivalent. This allows us to state, that the inverse Mellin transform $\rho(x,\tau)$ of the dual amplitude can be represented in the form (3.28) or (3.30).

4. Mellin transform of the basis vectors

In Section 3 for the inverse Mellin transform $\rho(x,\tau)$ of the scattering amplitude we have found the decomposition

$$\rho(x,\tau)=\int_{-\infty}^{\infty} d\nu\, c(\nu)\, e_{\nu}^{\lambda}(x), \tag{4.1}$$

where $\lambda = 4\mu^2 a - \tau$ and $c(\nu)$ [xx] satisfy the condition:

$$c(-\nu)=\overline{c(\nu)}.$$

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[x] i.e. the meromorphic approximation for the scattering amplitude when the last one is represented as an arbitrary finite sum of B-funotions (finite number of satellites).

[xx] As far as λ (i.e. τ) is an invariant of the representation of $SL(2,R)$, $c(\nu)$ oould depend on λ too.

If we go from $\rho(x,\tau)$ to the amplitude by means of eq. (1.19) we get

$$\mathcal{A}(\sigma,\tau)=\int_{-\infty}^{\infty} d\nu\, c(\nu)\, T_\nu(\sigma,\tau), \tag{4.2}$$

where

$$T_\nu(\sigma,\tau)=\int_0^{\infty} dx\, e_\nu^{4\mu^2 a-\tau}(x)\, x^{-\sigma-1} . \tag{4.3}$$

Both equations (4.3) and the reverse one

$$e_\nu^{4\mu^2 a-\tau}(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} T_\nu(\sigma,\tau)\, x^{\sigma}\, d\sigma \tag{4.4}$$

give the Mellin transformation for the basis veotors (3.17). Eq. (4.2) is an expansion of the amplitude with respect to the functions (4.3). Here we formulate some statements concerning $T_\nu(\sigma,\tau)$ and $\mathcal{A}(\sigma,\tau)$.

I. In the space of the veotors $T_\nu(\sigma,\tau)$ acts a unitary irreducible representation of $SL(2,R)$ which is equivalent to (2.21). It is realized by means of integral operators, which are considered in details in [27] (See Appendix C).

2. Eq. (4.2) shows, that $\mathcal{A}(\sigma,\tau)$ is an element from the space of the vectors $T_\nu(\sigma,\tau)$. Consequently we have a unitary irreducible representation of $SL(2,R)$ which transforms $\mathcal{A}(\sigma,\tau)$ and $4\mu^2 a-\tau$ is an invariant of this representation.

3. Using eq. (3.15) we can define an invariant scalar product in the space of the vectors (4.3) too. To do this we substitute in eq. (3.15) the Mellin transform of $\varphi_1(x)$ and $\varphi_2(x)$

$$\varphi_i^+(s) = \int_0^\infty \varphi_i(x)\, x^{-s-1}\, dx$$
$$\varphi_i^-(s) = \int_0^\infty \varphi_i(-x)\, x^{-s-1}\, dx. \tag{4.5}$$

From here we get

$$\varphi_i(x) = \begin{cases} \dfrac{1}{2\pi i}\displaystyle\int_{c-i\infty}^{c+i\infty} \varphi_i^+(s)\, x^s\, ds & x>0 \\[2ex] \dfrac{1}{2\pi i}\displaystyle\int_{c-i\infty}^{c+i\infty} \varphi_i^-(s)\, (-x)^s\, ds & x<0. \end{cases} \tag{4.6}$$

After some simple calculations we get from (3.15):

$$(\varphi_1, \varphi_2) = \int_{\frac{\lambda}{2}-i\infty}^{\frac{\lambda}{2}+i\infty} ds\, \left(\varphi_1^-(s), \varphi_1^+(s)\right) \begin{pmatrix} K_{--}(\lambda; s, \bar{s}) & K_{-+}(\lambda; s, \bar{s}) \\ K_{-+}(\lambda; s, \bar{s}) & K_{--}(\lambda; s, \bar{s}) \end{pmatrix} \begin{pmatrix} \bar{\varphi}_2^-(\bar{s}) \\ \bar{\varphi}_2^+(\bar{s}) \end{pmatrix}, \tag{4.7}$$

where $\bar{\varphi}_i^\pm(s)$ denotes complex conjugation of the form of function, and the kernels K_{--}, K_{-+} are as follows:

$$K_{-+}(\lambda; s, \bar{s}) = \frac{-1}{2\pi\Gamma(-\lambda-1)}\, B(s+1, \bar{s}+1)$$

$$K_{--}(\lambda; s, \bar{s}) = \frac{-1}{2\pi\Gamma(-\lambda-1)}\left[B(s+1, -\lambda-1) + B(\bar{s}+1, -\lambda-1)\right]. \tag{4.8}$$

The r.h.s. of eq. (4.7) expresses the scalar product of two elements from the space of the vectors $T_\nu(\sigma,\tau)$. One should note, that

$$T^+_\nu(\sigma,\tau) \equiv T_\nu(\sigma,\tau) \tag{4.9}$$

$$T^-_\nu(\sigma,\tau) \equiv \overline{T}_\nu(\sigma,\tau).$$

Substituting the last ones into eq. (4.7) we get the orthonormality condition for $T_\nu(\sigma,\tau)$:

$$(T_{\nu_1}, T_{\nu_2}) = \int\limits_{\frac{\lambda}{2}-i\infty}^{\frac{\lambda}{2}+i\infty} d\sigma \left(\overline{T}_{\nu_1}, T_{\nu_1}\right) \begin{pmatrix} K_{--} & K_{-+} \\ K_{-+} & K_{--} \end{pmatrix} \begin{pmatrix} T_{\nu_2} \\ \overline{T}_{\nu_2} \end{pmatrix} = \mathcal{N}(\lambda,\nu_1)\,\delta(\nu_1-\nu_2), \tag{4.10}$$

where $\mathcal{N}(\lambda,\nu)$ is given by eq. (3.18).

4. For the physical amplitude we choose $c(\nu)$ to be real. This assumption essentially simplifies our considerations without restriction of their generality. Indeed, in Section 1 we have seen, that the physical amplitude defines $\rho(x,\tau)$ for $x>0$ only. The simplest assumption for $x<0$ is:

$$\rho(x,\tau) = \frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty} A(\sigma,\tau)(-x)^\sigma d\sigma \qquad x<0, \tag{4.11}$$

i.e. we postulate $\rho(x,\tau)$ to be even function of x and from here it follows, that $c(\nu)$ are real and even too. Then, instead of eqs. (4.1) and (4.2) we get:

$$\rho(x,\tau)=\int_0^\infty d\nu\, c(\nu)(e_\nu^\lambda+e_{-\nu}^\lambda) \tag{4.12}$$

and

$$\mathcal{A}(\sigma,\tau)=\int_0^\infty d\nu\, c(\nu)\left[T_\nu(\sigma,\tau)+T_{-\nu}(\sigma,\tau)\right]. \tag{4.13}$$

As far as

$$e_{-\nu}^\lambda = \bar{e}_\nu^\lambda \tag{4.14}$$

$$T_{-\nu}(\sigma,\tau)=\overline{T}_\nu(\sigma,\tau)$$

the quantities $\rho(x,\tau)$ and $\mathcal{A}(\sigma,\tau)$, given by eqs. (4.12) and (4.13) are real. They are the final expressions for the expansion of the physical amplitude and its inverse Mellin transform with respect to the spherical functions of the group $SL(2,R)$.

5. Consider the function $T_\nu(\sigma,\tau)+T_{-\nu}(\sigma,\tau)$. Substituting $e_\nu^{4\mu^2 a-\tau}$ from eq. (3.17) into eq. (4.3) we get:

$$T_\nu(\sigma,\tau)=\int_0^1 dx\, x^{-\frac{\sigma}{2}-1}(1-x)^{\frac{4\mu^2 a-\tau}{2}+i\nu}(1+\sqrt{x})^{-2i\nu}+(\sigma\to\omega). \tag{4.15}$$

Then

$$T_\nu(\sigma,\tau)+T_{-\nu}(\sigma,\tau)=\int_0^1 dx\, x^{-\frac{\sigma}{2}-1}(1-x)^{\frac{4\mu^2 a-\tau}{2}}\left[\left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)^{i\nu}+\left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)^{-i\nu}\right]+$$
$$+(\sigma\to\omega). \tag{4.16}$$

It is easy to see, that $\left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)^{i\nu}+\left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)^{-i\nu}$ is an even function of $\sqrt{x}$ and has a Taylor's expansion around the point $x=0$, thus leading to an infinite series of Euler's B-functions for $T_\nu+T_{-\nu}$. This series shows, that in the case of the meromorphic approximation we have to choose $a=2\alpha'$, where α' is the slope of the Regge-trajectories.

The main analytical properties of $T_\nu+T_{-\nu}$ can be obtained by means of the integral representation (4.16). Obviously, they are meromorphic functions of σ and τ (their poles are analogous to the poles of the Veneziano-type amplitudes). The asymptotic of the functions (4.16) depends on ν. That is why a definite asymptotical behaviour of $A(\sigma,\tau)$ can be reached appropriately choosing $c(\nu)$. In particular, if the integral (4.13) is uniformly convergent and $c(\nu)$ contains a term of the type:

$$\frac{1}{\nu^2+c^2(\tau_0-\tau)}$$

in the asymptotic of the amplitude we get the term:

$$\sim \sigma^{\frac{1}{2}\left(4\mu^2a-\tau+c\sqrt{\tau_0-\tau}\right)}.$$

Thus the appropriate choice of $c(\nu)$ provides an imaginary correction to the linear Regge-trajectory.

5. The point $\sigma = 0$ is a branch-point of the physical amplitude, while for $\sigma = 0$ T_ν has only a simple pole. This fact originates some dificulties when we have to choose $C(\nu)$. It has been shown [14] that the behatiour of $\mathcal{A}(\sigma,\tau)$ near the point $\sigma = 0$ and the behaviour of $\rho(x,\tau)$ near $x=0$ are connected. In particular, if

$$\mathcal{A}(\sigma,\tau) \sim (-\sigma)^{\gamma} \tag{4.17}$$

for $\sigma \to -0$, then

$$\rho(x,\tau) \sim (-\ln x)^{-\gamma-1} \tag{4.18}$$

for $x \to +0$. As far as the integral (3.15) converges for functions, the behaviour of which around the point $x=0$ is (4.18), we can find $C(\nu)$ giving the right behaviour of the physical amplitude near its branch point.

Conclusion

Let us summarize our main results.

1. The analytical properties, expressed in the dispersion relations (1.1) allow us to write down the scattering amplitude $\mathcal{A}(\sigma,\tau)$ as the Mellin transform (1.19) of the function $\rho(x,\tau)$

2. The duality of $\rho(x,\tau)$ can be considered as a consequence of some group-symmetry of the scattering amplitude. Here this symmetry is given by the group $SL(2,R)$. It turns

out, that the dual $\varphi(x,\tau)$ are spherical functions in the sense of definitions (2.8) and (2.9). In the space of these functions the unitary irreducible representation of $SL(2,R)$ is realized (2.21).

3. Taking into account the crossing symmetry condition we have introduced a basis in the spherical functions space and this leads us to the integral representation (4.2) for the scattering amplitude.

4. Except the vector $e_0^{\lambda}(x)$, the meromorphic approximations of the dual amplitude, which are expressed through finite number of Euler's B- functions, are not elements of the unitary space. Instead of the whole $SL(2,R)$ symmetry in this case we have symmetry with respect to the non-unitary representation of the $SL(2,R)$ algebra.

On the basis of this results and expecially from the integral representation (4.2) the hope arises, that following this way we could combine the duality and unitarity conditions for the scattering amplitude.

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APPENDIX A

1. Let h_1 and h_2 be 2x2 matrices and $h_1, h_2 \in SL(2,R)$:

$$h_1 = \begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{pmatrix} \qquad h_2 = \begin{pmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & \delta_2 \end{pmatrix} \qquad \begin{aligned} &\alpha_1\delta_1 - \beta_1\gamma_1 = 1 \\ &\alpha_2\delta_2 - \beta_2\gamma_2 = 1. \end{aligned} \tag{A.1}$$

If S_β is the subgroup, containing all the matrices of the type

$$\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \tag{A.2}$$

then h_1 and h_2 are in the same coset $S_\beta \backslash SL(2,R)$ when

$$h_1 h_2^{-1} \in S_\beta . \tag{A.3}$$

This last condition gives the following equations:

$$\begin{aligned} &\gamma_1\delta_2 - \gamma_2\delta_1 = 0 , \\ &\alpha_2\delta_1 - \gamma_1\beta_2 = 1 . \end{aligned} \tag{A.4}$$

Using the connection $\alpha_2\delta_2 - \beta_2\gamma_2 = 1$ to eliminate β_2 from (A.4) we get:

$$\delta_1 = \delta_2 , \tag{A.5}$$

$$\gamma_1 = \gamma_2 .$$

(A.6)

So, all the matrices with equal elements of the second line belong to the same coset. Let $\bar{g}$ be a matrix from a given coset. Every other matrix g from this coset can be represented in the form:

$$g = s\bar{g}. \tag{A.7}$$

Obviously $\bar{g}$ is a two-parametric manifold. We shall show, that for the cosets under consideration we can choose $\bar{g}$ in the form

$$\bar{g} = \begin{pmatrix} \delta^{-1} & 0 \\ \gamma & \delta \end{pmatrix} \qquad \delta \neq 0. \tag{A.8}$$

Eqs. (A.5) and (A.6) show, that every coset contains only one matrix of this type. Multiplying $\bar{g}$ with arbitrary $s \in S_\beta$ we get:

$$g = \begin{pmatrix} \delta^{-1} + \beta\gamma & \beta\delta \\ \gamma & \delta \end{pmatrix}. \tag{A.9}$$

Thus we see, that for fixed γ and δ all the matrices (A.9) belong to the same coset. Moreover, it is easy to check, that every matrix from $SL(2,R)$ can be represented in the form (A.9). This completes the proof of our statement.

II. The unitary representations of the group $SL(2,R)$ are realized in the space of the functions $f(x)$ of one real variable. The operators T_h of these representations are defined as follows:

$$T_h f(x) = \mathrm{sign}^{\varepsilon}(\beta x+\delta)\,|\beta x+\delta|^{s-1} f\left(\frac{\alpha x+\gamma}{\beta x+\delta}\right), \tag{A.10}$$

where ε = 0,1. For our purposes we use the case $\varepsilon = 0$. Obviously, the connection between the invariant of the representation s and λ from eq. (2.7) is $\lambda = s-1$.

There exist several series of unitary representations of the group $SL(2,R)$;

1. The principal series. In this case s is arbitrary imaginary number. The invariant scalar product has the form:

$$(f_1, f_2) = \int_{-\infty}^{\infty} f_1(x)\,\overline{f_2(x)}\,dx. \tag{A.11}$$

Using this scalar product one can introduce the norm of the vectors, thus transforming the space into a Hilbert space.

2. The supplementary series. In this case s is real and

$$-1 < s < 1 \qquad s \neq 0. \tag{A.12}$$

The invariant scalar product is defined as follows:

$$(f_1, f_2) = \frac{1}{\Gamma(-s)} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, |x-y|^{-s-1} f_1(x)\,\overline{f_2(y)}. \tag{A.13}$$

It is just this series that we have applied in the present paper.

A more detailed information about the unitary representations of $SL(2,R)$ is available in [28].

Appendix B

In order to find out the scalar product of two cross-even vectors $e^{\lambda+}_{\nu_1}$ and $e^{\lambda+}_{\nu_2}$ one has to evaluate the integral, which is obtained when substituting $e^{\lambda+}_{\nu_1}$ and $e^{\lambda+}_{\nu_2}$ from (3.17) into (3.15):

$$\left(e^{\lambda+}_{\nu_1}, e^{\lambda+}_{\nu_2}\right) = \frac{|N|^2}{\Gamma(-\lambda-1)} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \, |x-y|^{-\lambda-2} |1-x^2|^{\frac{\lambda}{2}} \left|\frac{1-x}{1+x}\right|^{i\nu_1} |1-y^2|^{\frac{\lambda}{2}} \left|\frac{1-y}{1+y}\right|^{-i\nu_1} . \tag{B.1}$$

The domain of integration is decomposed into 12 subdomains, in which every one from the quantities $x-y$, $1-x^2$, $\frac{1-x}{1+x}$, $1-y^2$, $\frac{1-y}{1+y}$ do not change its sign. Consequently, $\left(e^{\lambda+}_{\nu_1}, e^{\lambda+}_{\nu_2}\right)$ appears as a sum of 12 integrals. The first of them:

$$J_1(\nu_1,\nu_2) = \frac{|N|^2}{\Gamma(-\lambda-1)} \int_{-1}^{1} dx \int_{-1}^{x} dy \, (x-y)^{-\lambda-2} (1-x^2)^{\frac{\lambda}{2}} \left(\frac{1-x}{1+x}\right)^{i\nu_1} (1-y^2)^{\frac{\lambda}{2}} \left(\frac{1-y}{1+y}\right)^{-i\nu_2} \tag{B.2}$$

is easily evaluated through the substitution

$$x = \frac{1-u}{1+u} \qquad y = \frac{v-u}{v+u} . \tag{B.3}$$

after which it takes the form

$$\mathcal{J}_1(\nu_1,\nu_2)=\frac{|N|^2 2^\lambda}{\Gamma(-\lambda-1)}\int_0^\infty du\, u^{i(\nu_1-\nu_2)-1}\int_0^1 dv\, v^{\frac{\lambda}{2}+i\nu_2}(1-v)^{-\lambda-2}.$$

As far as

$$\int_0^\infty du\, u^{i(\nu_1-\nu_2)-1}=2\pi\delta(\nu_1-\nu_2) \tag{B.4}$$

and

$$\int_0^1 dv\, v^{\frac{\lambda}{2}+i\nu_2}(1-v)^{-\lambda-2}=B\left(\frac{\lambda}{2}+i\nu_2+1,-\lambda-1\right) \tag{B.5}$$

we have

$$\mathcal{J}_1(\nu_1,\nu_2)=\frac{2^{\lambda+1}\pi|N|^2}{\Gamma(-\lambda-1)}B\left(\frac{\lambda}{2}+i\nu_2+1,-\lambda-1\right)\delta(\nu_1-\nu_2). \tag{B.6}$$

The second integral can be expressed through $\mathcal{J}_1$:

$$\mathcal{J}_2(\nu_1,\nu_2)=\frac{|N|^2}{\Gamma(-\lambda-1)}\int_{-1}^1 dy\int_{-1}^y dx\,(y-x)^{-\lambda-2}(1-x^2)^{\frac{\lambda}{2}}\left(\frac{1-x}{1+x}\right)^{i\nu_1}(1-y^2)^{\frac{\lambda}{2}}\left(\frac{1-y}{1+y}\right)^{-i\nu_2}=\mathcal{J}_1(-\nu_2,-\nu_1). \tag{B.7}$$

Making the substitution

$$x=\frac{1+u}{1-u}\qquad y=\frac{1+uv}{1-uv} \tag{B.8}$$

into the integral

$$\mathcal{J}_3(\nu_1,\nu_2)=\frac{|N|^2}{\Gamma(-\lambda-1)}\int_1^\infty dx\int_1^x dy\,(x-y)^{-\lambda-2}(x^2-1)^{\frac{\lambda}{2}}\left(\frac{x-1}{x+1}\right)^{i\nu_1}(y^2-1)^{\frac{\lambda}{2}}\left(\frac{y-1}{y+1}\right)^{-i\nu_2} \tag{B.9}$$

we transform it into the form

$$\mathcal{J}_3(\nu_1,\nu_2)=\frac{2^{\lambda}|N|^2}{\Gamma(-\lambda-1)}\int_0^1 du\, u^{i(\nu_1-\nu_2)-1}\int_0^1 dv\, v^{\frac{\lambda}{2}-i\nu_2}(1-v)^{-\lambda-2} . \tag{B.10}$$

As far as

$$\int_0^1 du\, u^{i(\nu_1-\nu_2)-1} = \pi\delta(\nu_1-\nu_2)-i\mathcal{P}\frac{1}{\nu_1-\nu_2} = 2\pi\delta_-(\nu_1-\nu_2), \tag{B.11}$$

for $\mathcal{J}_3$ we get

$$\mathcal{J}_3(\nu_1,\nu_2)=\frac{2^{\lambda+1}\pi|N|^2}{\Gamma(-\lambda-1)}B\left(-\lambda-1,\frac{\lambda}{2}-i\nu_2+1\right)\delta_-(\nu_1-\nu_2). \tag{B.12}$$

As before

$$\mathcal{J}_4(\nu_1,\nu_2)=\frac{|N|^2}{\Gamma(-\lambda-1)}\int_1^{\infty}dy\int_1^{y}dx\,(y-x)^{-\lambda-2}(x^2-1)^{\frac{\lambda}{2}}\left(\frac{x-1}{x+1}\right)^{i\nu_1}(y^2-1)^{\frac{\lambda}{2}}\left(\frac{y-1}{y+1}\right)^{-i\nu_2}=\mathcal{J}_3(-\nu_2,-\nu_1). \tag{B.13}$$

It is easy to see, that changing $x\rightarrow x$ and $y\rightarrow -y$ the next two integrals are reduced to $\mathcal{J}_3$ and $\mathcal{J}_4$:

$$\mathcal{J}_5(\nu_1,\nu_2)=\frac{|N|^2}{\Gamma(-\lambda-1)}\int_{-\infty}^{-1}dx\int_{x}^{-1}dy\,(y-x)^{-\lambda-2}(x^2-1)^{\frac{\lambda}{2}}\left(\frac{x-1}{x+1}\right)^{i\nu_1}(y^2-1)^{\frac{\lambda}{2}}\left(\frac{y-1}{y+1}\right)^{-i\nu_2}=\mathcal{J}_3(-\nu_1,-\nu_2) \tag{B.14}$$

$$\mathcal{J}_6(\nu_1,\nu_2)=\frac{|N|^2}{\Gamma(-\lambda-1)}\int_{-\infty}^{-1}dy\int_{y}^{-1}dx\,(x-y)^{-\lambda-2}(x^2-1)^{\frac{\lambda}{2}}\left(\frac{x-1}{x+1}\right)^{i\nu_1}(y^2-1)^{\frac{\lambda}{2}}\left(\frac{y-1}{y+1}\right)^{-i\nu_2}=\mathcal{J}_4(-\nu_1,-\nu_2). \tag{B.15}$$

The integral

$$\mathcal{J}_7(\nu_1,\nu_2)=\frac{|N|^2}{\Gamma(-\lambda-1)}\int_1^\infty dx\int_{-1}^1 dy\,(x-y)^{-\lambda-2}(x^2-1)^{\frac{\lambda}{2}}\left(\frac{x-1}{x+1}\right)^{i\nu_1}(1-y^2)^{\frac{\lambda}{2}}\left(\frac{1-y}{1+y}\right)^{-i\nu_2} \tag{B.16}$$

after the substitution

$$x=\frac{1+u}{1-u} \qquad y=\frac{1-uv}{1+uv} \tag{B.17}$$

takes the form

$$\mathcal{J}_7(\nu_1,\nu_2)=\frac{2^\lambda|N|^2}{\Gamma(-\lambda-1)}\int_0^1 du\,u^{i(\nu_1-\nu_2)-1}\int_0^\infty dv\,v^{\frac{\lambda}{2}-i\nu_2}(1+v)^{-\lambda-2}. \tag{B.18}$$

Using the equality

$$\int_0^\infty dv\,v^{\frac{\lambda}{2}-i\nu_2}(1+v)^{-\lambda-2}=B\left(\frac{\lambda}{2}-i\nu_2+1,\frac{\lambda}{2}+i\nu_2+1\right) \tag{B.19}$$

and taking into account eq. (B.11) we get

$$\mathcal{J}_7(\nu_1,\nu_2)=\frac{2^{\lambda+1}\pi|N|^2}{\Gamma(-\lambda-1)}B\left(\frac{\lambda}{2}-i\nu_2+1,\frac{\lambda}{2}+i\nu_2+1\right)\delta_-(\nu_1-\nu_2). \tag{B.20}$$

In an analogous way one obtains

$$\mathcal{J}_8(\nu_1,\nu_2)=\frac{|N|^2}{\Gamma(-\lambda-1)}\int_{-1}^1 dx\int_1^\infty dy\,(y-x)^{-\lambda-2}(1-x^2)^{\frac{\lambda}{2}}\left(\frac{1-x}{1+x}\right)^{i\nu_1}(y^2-1)^{\frac{\lambda}{2}}\left(\frac{y-1}{y+1}\right)^{-i\nu_2}=\mathcal{J}_7(-\nu_2,-\nu_1) \tag{B.21}$$

$$\mathcal{J}_9(\nu_1,\nu_2)=\frac{|N|^2}{\Gamma(-\lambda-1)}\int_{-1}^1 dx\int_{-\infty}^{-1} dy\,(x-y)^{-\lambda-2}(1-x^2)^{\frac{\lambda}{2}}\left(\frac{1-x}{1+x}\right)^{i\nu_1}(y^2-1)^{\frac{\lambda}{2}}\left(\frac{y-1}{y+1}\right)^{-i\nu_2}=\mathcal{J}_8(-\nu_1,-\nu_2) \tag{B.22}$$

$$\mathcal{J}_{10}(\nu_1,\nu_2)=\frac{|N|^2}{\Gamma(-\lambda-1)}\int_{-\infty}^{-1} dx\int_{-1}^1 dy\,(y-x)^{-\lambda-2}(x^2-1)^{\frac{\lambda}{2}}\left(\frac{x-1}{x+1}\right)^{i\nu_1}(1-y^2)^{\frac{\lambda}{2}}\left(\frac{1-y}{1+y}\right)^{-i\nu_2}=\mathcal{J}_7(-\nu_1,-\nu_2). \tag{B.23}$$

The integral

$$J_{11}(\nu_1,\nu_2)=\frac{|N|^2}{\Gamma(-\lambda-1)}\int_{-\infty}^{-1}dy\int_1^{\infty}dx\,(x-y)^{-\lambda-2}(x^2-1)^{\frac{\lambda}{2}}\left(\frac{x-1}{x+1}\right)^{i\nu_1}(y^2-1)^{\frac{\lambda}{2}}\left(\frac{y-1}{y+1}\right)^{-i\nu_2} \tag{B.24}$$

after the substitution

$$x=\frac{1+u}{1-u} \qquad y=\frac{v+u}{v-u} \tag{B.25}$$

takes the form

$$J_{11}(\nu_1,\nu_2)=\frac{2^{\lambda}|N|^2}{\Gamma(-\lambda-1)}\int_0^1 dv\,v^{\frac{\lambda}{2}+i\nu_2}(1-v)^{-\lambda-2}\int_v^1 du\,u^{i(\nu_1-\nu_2)-1}=$$

$$=\frac{i2^{\lambda}|N|^2}{\Gamma(-\lambda-1)}\left[B\left(-\lambda-1,\frac{\lambda}{2}+i\nu_1+1\right)-B\left(-\lambda-1,\frac{\lambda}{2}+i\nu_2+1\right)\right]\mathcal{P}\frac{1}{\nu_1-\nu_2}. \tag{B.26}$$

Finally

$$J_{12}=\frac{|N|^2}{\Gamma(-\lambda-1)}\int_{-\infty}^{-1}dx\int_1^{\infty}dy\,(y-x)^{-\lambda-2}(x^2-1)^{\frac{\lambda}{2}}\left(\frac{x-1}{x+1}\right)^{i\nu_1}(y^2-1)^{\frac{\lambda}{2}}\left(\frac{y-1}{y+1}\right)^{-i\nu_2}=J_{11}(-\nu_2,-\nu_1). \tag{B.27}$$

Summing up all the expressions $J_1, J_2, \ldots J_{12}$ we get:

$$(e^{\lambda+}_{\nu_1},e^{\lambda+}_{\nu_2})=\frac{2^{\lambda+2}\pi|N|^2}{\Gamma(-\lambda-1)}\Big[B\left(-\lambda-1,\frac{\lambda}{2}+i\nu_1+1\right)+B\left(-\lambda-1,\frac{\lambda}{2}-i\nu_1+1\right)+$$
$$+B\left(\frac{\lambda}{2}+i\nu_1+1,\frac{\lambda}{2}-i\nu_1+1\right)\Big]\delta(\nu_1-\nu_2). \tag{B.28}$$

After some simple calculations this result takes the form:

$$(e^{\lambda+}_{\nu_1},e^{\lambda+}_{\nu_2})=\mathcal{N}(\lambda,\nu_1)\delta(\nu_1-\nu_2) \tag{B.29}$$

where

$$\mathcal{N}(\lambda,\nu_1)=|N|^2 2^{\lambda+3}\left|\Gamma\left(\tfrac{\lambda}{2}+i\nu_1+1\right)\right|^2\left[ch\,\pi\nu_1+\cos\tfrac{\pi}{2}(\lambda+2)\right]\sin\tfrac{\pi}{2}(\lambda+2). \tag{B.30}$$

In an analogous way one can show, that

$$\left(e^{\lambda-}_{\nu_1}, e^{\lambda-}_{\nu_2}\right)=\mathcal{N}'(\lambda,\nu_1)\,\delta(\nu_1-\nu_2), \tag{B.31}$$

where

$$\mathcal{N}'(\lambda,\nu_1)=|N|^2 2^{\lambda+3}\left|\Gamma\left(\tfrac{\lambda}{2}+i\nu_1+1\right)\right|^2\left[ch\,\pi\nu_1-\cos\tfrac{\pi}{2}(\lambda+2)\right]\sin\tfrac{\pi}{2}(\lambda+2) \tag{B.32}$$

and

$$\left(e^{\lambda+}_{\nu_1}, e^{\lambda-}_{\nu_2}\right)=0. \tag{B.33}$$

APPENDIX C

1. Let $f(x)$ transform according to some representation (A.10). Besides, let we have the Mellin transform of $f(x)$:

$$\begin{aligned}\varphi_+(\mu)&=\int_0^\infty f(x)\,x^{\mu-1}\,dx\equiv\int_{-\infty}^{\infty} f(x)\,x_+^{\mu-1}\,dx\\ \varphi_-(\mu)&=\int_0^\infty f(-x)\,x^{\mu-1}\,dx\equiv\int_{-\infty}^{\infty} f(x)\,x_-^{\mu-1}\,dx\,,\end{aligned} \tag{C.1}$$

where the function $f(x)$ can be expressed through φ_+ and φ_-:

$$f(x)=\begin{cases}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\varphi_+(\mu)\,x^{-\mu}\,d\mu & x>0\\ \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\varphi_-(\mu)\,(-x)^{-\mu}\,d\mu & x<0\,.\end{cases} \tag{C.2}$$

Here we have introduced the notations:

$$x_+^{\mu}=\theta(x)\,x^{\mu}$$
$$x_-^{\mu}=\theta(-x)(-x)^{\mu}. \tag{C.3}$$

Let us consider the operation K_h^{s-1} defined as follows:

$$K_h^{s-1}\varphi_\pm(\mu)=\int_{-\infty}^{\infty}T_h^{s-1}f(x)\,x_\pm^{\mu-1}\,dx, \tag{C.4}$$

where T_h^{s-1} is given by eq. (A.1o), i.e.

$$K_h^{s-1}\varphi_\pm(\rho)=\int_{-\infty}^{\infty}|\beta x+\delta|^{s-1}f\left(\frac{\alpha x+\gamma}{\beta x+\delta}\right)x_\pm^{\rho-1}\,dx. \tag{C.5}$$

Obviously the correspondence $h\rightarrow K_h^{s-1}$ defines a representation of $SL(2,R)$. One can unite both equations (C.2) into the form:

$$f(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\varphi_+(\mu)\,x_+^{-\mu}\,d\mu+\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\varphi_-(\mu)\,x_-^{-\mu}\,d\mu. \tag{C.6}$$

Expressing $f\left(\frac{\alpha x+\gamma}{\beta x+\delta}\right)$ through (C.6) substituting it into eq. (C.5) we get

$$K_h^{s-1}\varphi_\pm(\rho)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}d\mu\,\varphi_+(\mu)\int_{-\infty}^{\infty}dx\,x_\pm^{\rho-1}\left(\frac{\alpha x+\gamma}{\beta x+\delta}\right)_+^{-\mu}|\beta x+\delta|^{s+1}+$$

$$+\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}d\mu\,\varphi_-(\mu)\int_{-\infty}^{\infty}dx\,x_\pm^{\rho-1}\left(\frac{\alpha x+\gamma}{\beta x+\delta}\right)_-^{-\mu}|\beta x+\delta|^{s-1} \tag{C.7}$$

Thus we see, that

$$K_h^{s-1}\varphi_+(\rho)=\int_{c-i\infty}^{c+i\infty}d\mu\,K_{++}^{s-1}(\rho,\mu;h)\varphi_+(\mu)+$$

$$+\int_{c-i\infty}^{c+i\infty}d\mu\,K_{+-}^{s-1}(\rho,\mu;h)\varphi_-(\mu), \tag{C.8}$$

where

$$K_{++}^{s-1}(\rho,\mu;h)=\frac{1}{2\pi i}\int_0^{\infty}x^{\rho-1}\left(\frac{\alpha x+\gamma}{\beta x+\delta}\right)_+^{-\mu}|\beta x+\delta|^{s-1}dx \tag{C.9}$$

$$K_{+-}^{s-1}(\rho,\mu;h)=\frac{1}{2\pi i}\int_0^{\infty}x^{\rho-1}\left(\frac{\alpha x+\gamma}{\beta x+\delta}\right)_-^{-\mu}|\beta x+\delta|^{s-1}dx.$$

In an analogous way for $K_h^{s-1}\varphi_-(\rho)$ we get:

$$K_h^{s-1}\varphi_-(\rho)=\int_{c-i\infty}^{c+i\infty}d\mu\,K_{-+}^{s-1}(\rho,\mu;h)\varphi_+(\mu)+$$

$$+\int_{c-i\infty}^{c+i\infty}d\mu\,K_{--}^{s-1}(\rho,\mu;h)\varphi_-(\mu), \tag{C.10}$$

where

$$K_{-+}^{s-1}(\rho,\mu;h)=\frac{1}{2\pi i}\int_0^\infty x^{\rho-1}\left(\frac{-\alpha x+\gamma}{-\beta x+\delta}\right)_+^{-\mu}|-\beta x+\delta|^{s-1}dx \tag{C.11}$$

$$K_{--}^{s-1}(\rho,\mu;h)=\frac{1}{2\pi i}\int_0^\infty x^{\rho-1}\left(\frac{-\alpha x+\gamma}{-\beta x+\delta}\right)_-^{-\mu}|-\beta x+\delta|^{s-1}dx.$$

More information about the convergency conditions of the integrals and about the validity of all operations, which lead to eqs. (C.8)-(C.11) is available in [27].

Eqs. (C.8) and (C.10) determine the way in which the Mellin transform of $f(x)$ is transformed. Essentially eqs. (C.9) and (C.11) determine the matrix elements of this transformation.

II. Let us find out the index-transformations (2.12) and(213) of the basis vectors $e_\nu^\lambda(x)$ (3.17). For this purpose we write down e_ν^λ in the form:

$$e_\nu^\lambda(x)_+=|1-x^2|^{\frac{\lambda}{2}}\left(\frac{1-x}{1+x}\right)_+^{i\nu} \tag{C.12}$$

$$e_\nu^\lambda(x)_-=|1-x^2|^{\frac{\lambda}{2}}\left(\frac{1-x}{1+x}\right)_-^{i\nu},$$

where for definitness consider $x>0$. (The signs + and - have the same meaning as in eq. (C.3)). To the matrix

$$h=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$$

we put into oorrespondenoe the matrix

$$\tilde{h} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \alpha-\beta-\gamma+\delta & -\alpha-\beta+\gamma+\delta \\ -\alpha+\beta-\gamma+\delta & \alpha+\beta+\gamma+\delta \end{pmatrix} \tag{C.13}$$

It is easy to check, that

$$\tilde{h} = \ell^{+} h \ell, \tag{C.14}$$

where ℓ is a unitary matrix:

$$\ell = \frac{i}{\sqrt{2}}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \qquad \ell^{+}\ell = 1. \tag{C.15}$$

Let us evaluate the expression

$$R_h e_\nu^\lambda(x)_+ = i\int_{-\infty}^{\infty} d\nu' K_{++}^\lambda\left(-\frac{\lambda}{2}-i\nu', -\frac{\lambda}{2}-i\nu; \tilde{h}^{-1}\right) e_{\nu'}^\lambda(x)_+ + \tag{C.16}$$

$$+ i\int_{-\infty}^{\infty} d\nu' K_{-+}^\lambda\left(-\frac{\lambda}{2}-i\nu', -\frac{\lambda}{2}-i\nu; \tilde{h}^{-1}\right) e_{\nu'}^\lambda(x)_- .$$

It follows from (C.9) and (C.11) that

$$K_{++}^\lambda\left(-\frac{\lambda}{2}-i\nu', -\frac{\lambda}{2}-i\nu; \tilde{h}^{-1}\right) = \frac{1}{2\pi i}\int_0^\infty y^{-\frac{\lambda}{2}-1-i\nu'}\left(\frac{Dy-C}{-By+A}\right)_+^{\frac{\lambda}{2}+i\nu} |-By+A|^\lambda dy \tag{C.17}$$

$$K_{-+}^\lambda\left(-\frac{\lambda}{2}-i\nu', -\frac{\lambda}{2}-i\nu; \tilde{h}^{-1}\right) = \frac{1}{2\pi i}\int_0^\infty y^{-\frac{\lambda}{2}-1-i\nu'}\left(\frac{-Dy-C}{By+A}\right)_+^{\frac{\lambda}{2}+i\nu} |By+A|^\lambda dy .$$

Substituting eqs. (C.12) and (C.17) into eq. (C.16) we get:

$$R_h e_\nu^\lambda(x)_+ = |1-x^2|^{\frac{\lambda}{2}} \int_0^\infty dy\, \delta\left(y - \frac{1-x}{1+x}\right)\left(\frac{1-x}{1+x}\right) y^{-\frac{\lambda}{2}-1} \left(\frac{Dy-C}{-By+A}\right)_+^{\frac{\lambda}{2}+i\nu} |-By+A|^\lambda +$$

$$+ |1-x^2|^{\frac{\lambda}{2}} \int_0^\infty dy\, \delta\left(y - \frac{x-1}{x+1}\right)\left(\frac{x-1}{x+1}\right) y^{-\frac{\lambda}{2}-1} \left(\frac{-Dy-C}{By+A}\right)_+^{\frac{\lambda}{2}+i\nu} |By+A|^\lambda =$$

$$= \theta\left(\frac{1-x}{1+x}\right)\left(\frac{D\frac{1-x}{1+x} - C}{-B\frac{1-x}{1+x} + A}\right)_+^{\frac{\lambda}{2}+i\nu} |-B(1-x)+A(1+x)|^\lambda +$$

$$+ \theta\left(\frac{x-1}{x+1}\right)\left(\frac{D\frac{1-x}{1+x} - C}{-B\frac{1-x}{1+x} + A}\right)_+^{\frac{\lambda}{2}+i\nu} |-B(1-x)+A(1+x)|^\lambda =$$

$$= |-B(1-x)+A(1+x)|^\lambda \left(\frac{D\frac{1-x}{1+x} - C}{-B\frac{1-x}{1+x} + A}\right)_+^{\frac{\lambda}{2}+i\nu} .$$

Using the explicit expressions for A,B,C,D from eq. (C.13) we have:

$$R_h e_\nu^\lambda(x)_+ = |-\beta x+\alpha|^\lambda \left|1 + \frac{\delta x-\gamma}{-\beta x+\alpha}\right|^\lambda \left(\frac{1 - \frac{\delta x-\gamma}{-\beta x+\alpha}}{1 + \frac{\delta x-\gamma}{-\beta x+\alpha}}\right)_+^{\frac{\lambda}{2}+i\nu} =$$

$$= |-\beta x+\alpha|^\lambda \left|1 - \left(\frac{\delta x-\gamma}{-\beta x+\alpha}\right)^2\right|^{\frac{\lambda}{2}} \left(\frac{1 - \frac{\delta x-\gamma}{-\beta x+\alpha}}{1 + \frac{\delta x-\gamma}{-\beta x+\alpha}}\right)_+^{i\nu} .$$

The comparison with eq. (C.12) shows, that

$$R_h e_\nu^\lambda(x)_+ = |-\beta x+\alpha|^\lambda e_\nu^\lambda\left(\frac{\delta x-\gamma}{-\beta x+\alpha}\right)_+ . \tag{C.18}$$

Consequently the operation R_h defined with (C.16), coincides with the index-transformation (2.12) and (2.13).

In an analogous way for the operation

$$R_h e_\nu^\lambda(x)_- = i\int_{-\infty}^{\infty} d\nu' K_{+-}^\lambda\left(-\tfrac{\lambda}{2}-i\nu', -\tfrac{\lambda}{2}-i\nu; \tilde{h}^{-1}\right) e_{\nu'}^\lambda(x)_+ + \tag{C.19}$$

$$+ i\int_{-\infty}^{\infty} d\nu' K_{--}^\lambda\left(-\tfrac{\lambda}{2}-i\nu', -\tfrac{\lambda}{2}-i\nu; \tilde{h}^{-1}\right) e_{\nu'}^\lambda(x)_-$$

one can find the equality

$$R_h e_\nu^\lambda(x)_- = |-\beta x+\alpha|^\lambda e_\nu^\lambda\left(\frac{\delta x-\gamma}{-\beta x+\alpha}\right)_- . \tag{C.20}$$

The equations (C.16), (C.18), (C.19), and (C.20) prove the formula (3.31) and

$$R_{++}^\lambda(\nu,\nu';h) = K_{++}^\lambda\left(-\tfrac{\lambda}{2}-i\nu', -\tfrac{\lambda}{2}-i\nu; \tilde{h}^{-1}\right)$$

$$R_{+-}^\lambda(\nu,\nu';h) = K_{-+}^\lambda\left(-\tfrac{\lambda}{2}-i\nu', -\tfrac{\lambda}{2}-i\nu; \tilde{h}^{-1}\right)$$

$$R_{-+}^\lambda(\nu,\nu';h) = K_{+-}^\lambda\left(-\tfrac{\lambda}{2}-i\nu', -\tfrac{\lambda}{2}-i\nu; \tilde{h}^{-1}\right) \tag{C.21}$$

$$R_{--}^\lambda(\nu,\nu';h) = K_{--}^\lambda\left(-\tfrac{\lambda}{2}-i\nu', -\tfrac{\lambda}{2}-i\nu; \tilde{h}^{-1}\right).$$

It is seen, that the index-transformation can be obtained from the transformation of the argument (C.8) and (C.11) through the replacement $h \to \tilde{h}^{-1}$ and transposition fixing the constant c in eqs. (C.8) and (C.11) equal to $\frac{\lambda}{2}$.

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