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E.A.Tagirov

ON THE LOWER LIMIT OF BOSON MASSES IN A CLASS OF COSMOLOGICAL MODELS

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Тагиров Э.А.

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О нижней границе масс бозонов в одном классе космологических моделей

На основе строгого вычисления и изучения свойств сингулярных функций квантованного скалярного поля в мире Де Ситтера ($S_{1,3}$) выдвигается возражение против имеющегося в литературе утверждения о том, что в $S_{1,3}$ (и, как следствие, в асимптотически совпадающих с $S_{1,3}$ космологических моделях) массы бозонов не могут быть меньше $\frac{h}{cr}$, r – радиус $S_{1,3}$.

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Tagirov E.A.

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On the Lower Limit of Boson Masses in a Class of Cosmological Models

As has been concluded by some authors, there cannot exist bosons, for which $m^2 < m_0^2 \sim \frac{h}{cr}$, m^2 being the boson rest mass, in the De Sitter worls of the radius r and consequently in a certain class of cosmological models. The significance of this bound is discussed and an objection against it is given on the ground of a strict calculation and study of the properties of two-point functions of the quantum scalar field with $m^2 < m_0^2$ and $m^2 > m_0^2$.

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1. Introduction

In the papers by Nachtmann[1] and Börner and Dürr[2] on quantum field theory (QFT) in the De Sitter world ($S_{1,3}$) there is the assertion, that the possible values of the squared rest mass of the scalar particle in $S_{1,3}$ have a lower positive limit:

$$m^2 \geqslant m_0^2 = 9/4r^2, \tag{1}$$

where r is the radius of $S_{1,3}$ and the units for which $c = \hbar = 1$ are used. There is an analogous limit for vector fields, too[2].

On the other hand, quantization of the free scalar field had been performed by Chernikov and the author[3] under the naive supposition that in $S_{1,3}$ $m^2 \geqslant 0$ as well as in $E_{1,3}$, and this supposition gives no rise to difficulties. In addition, the results of the paper[3] are particularly simple for m =0. At the same time arguments leading to the bound (1) (see Sec.3) are of purely mathematical nature and physical consequences of breaking of this bound are not apparent. Therefore in the present paper the commutator and the propagator of the free scalar field in $S_{1,3}$ will be calculated in a strict manner with the aim to find physical arguments for rejecting masses for which $0 \leqslant m^2 < m_0^2$. But the result will be that there is no sufficient reason for this.

The problem of the lower bound of m^2 (0 or m_0^2) may seen as nonessential for the following two reasons.

Firstly, the substitution of the radius of the Universe $\sim 10^{10}$ light years into (1) instead of r gives $m_o \sim 10^{-56} g$ i.e. m_o^2 is negligible. Secondly, $S'_{1,3}$ is not a realistic cosmological model and the problem of existence of massless particles in $S_{1,3}$ may seem abstract.

Against this one can give the following objections. Firstly, there is an example of the physical system for which any difference of the mass from zero is essential. Namely, the quantum theory of the massless Yang-Mills field is not a limiting case of the massive field theory /4/. Possibly some interactions have an analogous property under any small breaking of conformal symmetry. Secondly, it will be clear further that the bound of the type (1) arises in a rather wide class of cosmological models. At last, we are interested in the possibility of principle of observable qualitative manifestations of vanishing a curvature in QFT. The $S_{1,3}$ is an appropriate model to look for these manifestations, because its group of motions $SO(1,4)$ is ten-parametric as the Poincare group, but essentially differs from the latter by the structure. This reflects the topological nonequivalence of $S_{1,3}$ and the Minkowsky space-time $(E_{1,3})$.

Already the canonical quantization of the free scalar field in $S_{1,3}$ results in consequences which, in a certain sense, do not depend on the numerical value of r /3/. They are directly related to the problem of bound (1) and therefore we will briefly consider them in the following section. We shall use the same notations as in the paper /3/ with seldom exception and we shall not explain them in evident cases.

2. The Conformal Covariant Scalar Field Equation

In the paper[3] it was shown that the corpuscular interpretation of the free quantum scalar field in $S_{1,3}$ can be achieved only if the field equation in an arbitrary Riemannian space-time ($V_{1,3}$) will be *)

$$(\Box + R/6 + m^2)\varphi = 0, \tag{2}$$

where

$$\Box \equiv \frac{1}{\sqrt{-g}}\partial_\alpha\left(\sqrt{-g}\, g^{\alpha\beta}\partial_\beta\right), \quad \partial_\alpha \equiv \frac{\partial}{\partial x^\alpha}, \quad \alpha, \beta, \gamma \ldots = 0,1,2,3.$$

instead of the conventional one

$$(\Box + m^2)\varphi = 0. \tag{3}$$

This conclusion is based on that in the theory[5] with Eq.(2) the one-particle states correspond in the quasiclassic situation to the motion along a geodesic line in $S_{1,3}$ contrary to the theory with Eq. (3).

An important distinction between Eqs. (2) and (3) consists in conformal covariance for $m=0$ of the former[5], i.e. if the given $V_{1,3}$ and $\tilde{V}_{1,3}$ with metric tensors $g_{\alpha\beta}$ and $\tilde{g}_{\alpha\beta}$ are mutually conformal

$$\tilde{g}_{\alpha\beta}(x) = \Omega^2(x)\, g_{\alpha\beta}(x), \tag{4}$$

$\Omega(x)$ being some function, and if φ is a solution of

$$(\Box + R/6)\varphi = 0 \tag{5}$$

in $V_{1,3}$, then $\tilde{\varphi} = \Omega^{-1}\varphi$ is a solution of the same equation in $\tilde{V}_{1,3}$. From general and conformal covariance of Eq. (5) it follows its conformal invariance, i.e. if $V_{1,3}$ admits

*) More exactly Eq. (2) is the simplest equation satisfying the requirements of the paper[3].

conformal coordinate transformations:

$$'x^{\alpha} = f^{\alpha}(x): \ 'g_{\alpha\beta}(x) = \frac{\partial f^{\gamma}}{\partial x^{\alpha}} \frac{\partial f^{\delta}}{\partial x^{\beta}} g_{\gamma\delta}(x) = \omega^2(x) g_{\alpha\beta}('x) , \tag{6}$$

where $\omega(x)$ is an appropriate function, and if $\varphi(x)$ is a solution of (5) then $'\varphi(x) = \omega(x)\,\varphi[f(x)]$ is a solution of the same equation.

The following two properties of Eq. (2) are due to its conformal covariance:

1. The commutator of the quantum field satisfying Eq. (2) is not zero only on the light cone in an arbitrary conformally flat $V_{1,3}$ when $m = 0$ (See also Sec. 4). This means the validity of Huygens' principle. There is no such property in the theory with Eq. (3).

2. Variation of the Lagrangian, corresponding to Eq. (2) with respect to $g_{\alpha\beta}$ gives the symmetric energy-momentum tensor:

$$t_{\alpha\beta} = T_{\alpha\beta} + \frac{1}{6}\left(R_{\alpha\beta} + \nabla_{\alpha}\nabla_{\beta} - g_{\alpha\beta}\Box\right)\varphi^2, \tag{7}$$

where ∇_{α} means covariant differentiation, $T_{\alpha\beta}$ is the canonical energy-momentum tensor, i.e.

$$T_{\alpha\beta} = \partial_{\alpha}\varphi\,\partial_{\beta}\varphi - \frac{1}{2} g_{\alpha\beta}\left(g^{\mu\nu}\partial_{\mu}\varphi\,\partial_{\nu}\varphi - m^2 - R/6\right)\varphi^2.$$

Owing to Eq. (2)

$$\nabla^{\alpha} t_{\alpha\beta} = 0 , \quad \nabla^{\alpha} T_{\alpha\beta} = 0 , \tag{8}$$

$$t^{\alpha}_{\alpha} = \frac{1}{2} m^2 \varphi^2. \tag{9}$$

Let us assume that there is a group of conformal transformations of $V_{1,3}$. Equation

$$\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha = 2 f g_{\alpha\beta} \tag{10}$$

for vector field ξ_α and a function $f(x)$ defines generators of the group $i\xi^\alpha \partial_\alpha$. Then for $m = 0$, as a consequences of Eqs. (8), (9), a conserved (independent of a choice of space-like hypersurface Σ) quantity

$$M = \int_\Sigma d\sigma^\alpha \xi^\beta T_{\alpha\beta} \tag{11}$$

corresponds to each solution of Eq. (10). If $m \neq 0$, M is conserved only when ξ^α satisfies Eq. (10) with $f = 0$, i.e. when ξ^α defines an infinitesimal motion. Then the value of M is not changed by substitution $T_{\alpha\beta}$ instead of $t_{\alpha\beta}$. So conserved quantities corresponding to both motions and conformal transformations are uniformly defined by the same formula (11).

Though the terms additional to $T_{\alpha\beta}$ in (7) have as an origin the term $R\varphi/6$ in Eq. (2), nevertheless $t_{\alpha\beta} \neq T_{\alpha\beta}$ even in $E_{1,3}$ where $R \equiv 0$ and Eq. (2) and (3) coincide (1). The tensor $t_{\alpha\beta}$ in $E_{1,3}$ is nothing but the "new improved energy-momentum tensor" of Callan, Coleman and Jackiw[6]. Having been added the term $(\partial_\mu \partial_\nu - \eta_{\mu\nu} \Box)\varphi^2$ to $T_{\alpha\beta}$ they show that the tensor thus obtained has finite matrix elements in any perturbation order, if the interaction is renormalizable *). Substituting $T_{\alpha\beta}$ into the Einstein equations as a source of gravitation they come to Eq. (2) for the scalar field in $V_{1,3}$. Then $t_{\alpha\beta}$ was essentially

*) In paper[10] the energy-momentum tensor of several interacting fields of spin 0,1/2, 1 is considered but it is really "new improved" only in terms related to the scalar fields.

used for investigation of broken conformal symmetry [7].

It is important to emphasize once more that QFT in $S_{1,3}$ results in the necessity of passing from (3) to (2) [3] in an arbitrary $V_{1,3}$. Though the difference is very small in any observable gravitational field it is of principal interest by itself, and at the same time its consequences play a certain role in current problems of elementary particle dynamics even without immediate account of gravitation.

3. The Wave Function of the Scalar Particle in $S_{1,3}$ and Its Properties

In coordinates τ, ξ^i used in paper [3/*)] the interval of $S_{1,3}$ has the form

$$ds^2 = \frac{1}{\cos^2\tau}\left(d\tau^2 - \omega_{ij}\, d\xi^i d\xi^j\right), \quad i, j = 1, 2, 3, \qquad -\pi/2 < \tau < \pi/2, \tag{12}$$

where $\omega_{ij}(\xi)$ is the metric tensor of 3-dimensional sphere (S_3) of unit radius, ξ^i are curvilinear coordinates on S_3. If one introduces homogeneous coordinates $k^a(\xi)$ on S_3, i.e. $a = 1,2,3,4$ and $\sum_a (k_a(\xi))^2 = 1$, then $\omega_{ij} = \sum_a \partial_i k_a \partial_j k_a$. Remote "future" and "past" of $S_{1,3}$ corresponds to $\tau = \pm\pi/2$.

One obtains the following system of linear-independent solutions of Eq. (2) in $S_{1,3}$ by separation of variables [3]

$$\varphi^{\pm}_{s\sigma}[\tau, k(\xi)] = \cos\tau\, u^{\pm}_{s-1}(\tau)\, \mathcal{J}_{s\sigma}[k(\xi)] \qquad s = 0, 1, 2, \ldots\ ; \quad \sigma = 1, \ldots, (s+1)^2. \tag{13}$$

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*) The time-like coordinate τ was denoted there by θ.

$\mathcal{P}_{s\sigma}$ are basis orthonormal harmonic polynomials in k of degree s. They are labelled by index σ. The $u^{\pm}_{s-1}$ are expressed through a hypergeometric function

$$u^{\pm}_{s-1}(\tau) = \frac{1}{s!}\sqrt{\Gamma(s+\mu)\Gamma(s-\mu+1)}\, e^{\pm is\tau} F\left(\mu, 1-\mu; s+1; \frac{1\pm i\,tg\tau}{2}\right), \qquad (14)$$

where $\mu = \frac{1}{2}\left(1-\sqrt{1-4\mathfrak{m}^2}\right)$, $\mathfrak{m} = m r$, $u^{-}_{s} = (u^{+}_{s})^{*}$.

The $u^{\pm}_{s}$ can be expressed in terms of Legendre functions on the cut P^{λ}_{ν} and Q^{λ}_{ν}:

$$u^{\pm}_{s-1}(\tau) = \left[\frac{\cos\tau}{2\pi}\frac{\Gamma(s+\mu+1)}{\Gamma(s-\mu+2)}\right]^{1/2} e^{\pm i\pi(s-\mu+1)/2}\left[P^{1/2-\mu}_{s+1/2}(\sin\tau) \pm \frac{2i}{\pi} Q^{1/2-\mu}_{s+1/2}(\sin\tau)\right]$$

Expressing $P^{\lambda}_{\nu}(x)$, $Q^{\lambda}_{\nu}(x)$ through $Q^{\lambda}_{\nu}(x+i\epsilon)$ and using Wipple's formula /8/ one has

$$u^{\pm}_{s-1} = \frac{\Gamma(1-\mu-s)}{\Gamma(1-\mu+s)}\sqrt{\Gamma(s+\mu)\Gamma(s-\mu+1)}\left[\theta(\tau)e^{\mp is\pi/2} + \theta(-\tau)e^{\pm is\pi/2}\right]\lim_{\epsilon\to 0} P_{-\mu}(\epsilon \pm i\,tg\tau), \qquad (15)$$

where

$$\theta(\tau) = \begin{cases} 1 & \tau > 0 \\ 0 & \tau < 0 \,. \end{cases}$$

As was shown in /3/ $\varphi^{+}_{s\sigma}$ ($\varphi^{-}_{s\sigma}$) are analogues of positive (negative)- frequency exponentials in $E_{1,3}$ and coefficients of field operator expansions in basis (13) are the creation and annihilation operators of particles with quantum numbers s, σ in $S_{1,3}$.

Let us turn to representations of the connected component of the De Sitter group containing unity. It is often denoted as $SO_0(1,4)$. Irreducible representations of $SO_0(1,4)$ are generally determined by eigenvalues of the two Casimir operators $\mathcal{J}_1$

and $\mathfrak{J}_2$. For the representations under consideration in the space of scalar functions $\mathfrak{J}_1 = \Box$, $\mathfrak{J}_2 = 0$ (degenerate representations).

Consequently using the set of solutions (12) of Eq. (2) for fixed m^2 one may construct a linear manifold in which an irreducible representation of $SO_0(1,4)$ is realized. Particularly such a manifold is spanned by $\varphi^+_{s\sigma}$ for all s,σ.

The unitary irreducible representations of $SO_0(1,4)$ are known to form three series: principal, supplementary, discrete (see e.g. [9,10]). In the degenerate case under consideration $(\mathfrak{J}_2 = 0)$ we have :

principal series, if $m^2 \geqslant 1/4$, i.e. $\mu = 1/2 - i\Lambda, \Lambda \geqslant 0$,

Supplementary series, if $0 < m^2 < 1/4$, i.e. $1/2 > \mu > 0$,

discrete series, if $m^2 = -n(n+1)$, $n = 0, 1, \ldots$

An essential distinction between the principal and supplementary series consists in the asymptotic behaviour of $u^{\pm}_s(\tau)$ for $|\tau| \to \pi/2$:

1) for $m^2 > 1/4$, i.e. $\mu = 1/2 - i\Lambda, \Lambda > 0$

$$u^+_{s-1}(\tau) \approx const \cdot \cos^{1/2}\tau \, \cos(\Lambda \ln \cos\tau - \alpha(\mu) \pm i\pi\Lambda/2)$$

$$\alpha(\mu) = \arg[\Gamma(s-\mu+1)\Gamma(i\Lambda)] + \ln 2$$

2) for $m^2 < 1/4$, i.e. $0 < \mu < 1/2$

$$u^+_{s-1}(\tau) = const \cdot \cos^{\mu}\tau .$$

Using these expressions and well known relations for the Legendre functions one may show that

$$\int d^4x \sqrt{-g}\, \varphi^-_{s\sigma}(x; m_1^2)\, \varphi^+_{s\sigma}(x; m_2^2) = \begin{cases} \delta(m_1^2 - m_2^2), & m_1^2, m_2^2 > 1/4 \\ \infty & 0 \leq m_1 < 1/4, \end{cases} \quad (16)$$

where the integration is performed over the whole $S_{1,3}$.

Special consideration is necessary for the case $m^2 = 0$

Then

$$\varphi^{\pm}_{s\sigma}(x;0) = (s+1)^{-1/2}\cos\tau \exp\{\pm i(s+1)\tau\} \mathcal{P}_{s\sigma}. \tag{17}$$

However the basis of the irreducible representation from the discrete series corresponding to $n = 0$ is formed by the functions

$$\varphi_{ns\sigma}\Big|_{n=0} = \begin{cases} (s+1)^{-1/2}\cos\tau \cos(s+1)\tau\, \mathcal{P}_{s\sigma}, & if\ s = 2n \\ (s+1)^{-1/2}\cos\tau \sin(s+1)\tau\, \mathcal{P}_{s\sigma}, & if\ s = 2n+1. \end{cases}$$

All $\varphi_{ns\sigma}$ are square-integrable. The representations of $SO_0(1,4)$ which are realized in the linear spaces spanned by functions $\varphi^{+}_{s\sigma}(x;0)$ or $\varphi^{-}_{s\sigma}(x;0)$ belong neither to discrete nor to supplementary series and should be considered as a limiting case of the latter.

An arbitrary square integrable over $S_{1,3}$ function $f(x)$ may be represented as

$$f(x) = \sum_{n=0}^{\infty} \sum_{s,\sigma} f_{ns\sigma}\, \varphi_{ns\sigma}(x) + \int d\rho(m^2) \sum_{s,\sigma} \Big\{ f^{+}_{s\sigma}(m^2)\varphi^{+}_{s\sigma}(x;m^2) + $$

$$+ f^{-}_{s\sigma}(m^2)\varphi^{-}_{s\sigma}(x;m^2)\Big\}, \tag{18}$$

where $f_{ns\sigma}$, $f^{\pm}_{s\sigma}(m^2)$ are constants of expansion and the measure $\rho(m^2)$ is such that $d\rho(m^2) = 0$, if $m^2 < 1/4$. This expansion is strictly associated with the asymptotic properties of $\varphi^{\pm}_{s\sigma}(x;m^2)$ and square integrability of $\varphi_{ns\sigma}$ and is a consequence of the well-known assertion that the decomposition of the quasiregular representation of $SO_0(1,4)$ into the irreducible ones contains only the discrete and principal series (see, e.g. [10]).

The difference between $\varphi^{\pm}(x; m^2)$ for $m^2 > 1/4$ and $m^2 < 1/4$,expressed by Eqs. (16),(20) or, more exactly, the non-unitarity of representations with $m^2 < 1/4$ with respect to the scalar product defined as an invariant-integral over $S_{1,3}$ was a reason[1,2] to conclude that in $S_{1,3}$ the squared masses of scalar particles cannot be less, than m_0^2, $m_0^2 = 1/4r^2$, if Eq. (2) is used and $m_0^2 = 9/4r^2$, i.e. the bound (1) takes place, if one chooses Eq. (3). However, in the aspect under consideration of importance is only the fact that $m_0^2 > 0$. It means that the massless particles are forbidden in $S_{1,3}$ (and some other cosmological models) taking into account global space-time properties in QFT.

Now,let is try to look for physical manifestations of invalidity, of the mass for which $m^2 < m_0^2$. It is rather hard to explain what the author means by "physical manifestations" since in his opinion they do not exist when $m^2 \geqslant 0$. They could be,for example, an acausal behaviour of the field commutator more strong (compared to the conventional ones) singularities of Green functions. We pass to the comparison of properties of fields with $m^2 < m_0^2$ and $m^2 > m_0^2$ from this viewpoint.

4. Two-Point Functions of the Scalar Field in $S_{1,3}$.

In the papers [3,11] closed expressions for the scalar field commutator[3] and Green functions[11] were given, but the latter were obtained by somewhat formal calculation procedure. Then again, in these papers it was considered that $m^2 \geqslant 0$

in $S_{1,3}$ as well as in $E_{1,3}$. Here we caloulate these two-point functions and analyse their properties in a more striot and detailed manner. It is sufficient to find

$$\mathcal{D}^{-}(x_1,x_2) = i\langle 0|\varphi(x_1),\varphi(x_2)|0\rangle = -\mathcal{D}^{+}(x_2,x_1) \tag{19}$$

Then the field commutator,

$$\mathcal{D}(x_1,x_2) = i\left[\varphi(x_1),\varphi(x_2)\right] = \mathcal{D}^{-}(x_1,x_2) + \mathcal{D}^{+}(x_1,x_2) \tag{20}$$

and Green functions, for example, the causal one

$$\mathcal{D}^{c}(x_1,x_2) = i\langle 0|T\{\varphi(x_1)\varphi(x_2)\}|0\rangle = \theta(\tau_1-\tau_2)\mathcal{D}^{-}(x_1,x_2) - \theta(\tau_2-\tau_1)\mathcal{D}^{+}(x_1,x_2). \tag{21}$$

can easily be obtained.

After insertion of expression (14) $\varphi^{\pm}_{s\sigma}(x)$ and use of the summation theorem for harmonic polynomials (see e.g. /13/)

$$\sum_{\sigma=1}^{(s+1)^2} \mathcal{P}_{s\sigma}[k(\xi_1)]\,\mathcal{P}_{s\sigma}[k(\xi_2)] = \frac{s+1}{2\pi^2}\,\frac{\sin(s+1)\gamma}{\sin\gamma},$$

$\gamma = \gamma(\xi_1,\xi_2)$ being defined by relation $\cos\gamma = \sum_{\alpha=1}^{4} k_\alpha(\xi_1)k_\alpha(\xi_2)$, one gets

$$\mathcal{D}^{-}(x_1,x_2) = i\frac{\cos\tau_1\cos\tau_2}{2\pi^2\sin\gamma}\sum_{s=0}^{\infty}(s+1)\,u^{-}_{s}(\tau_1)u^{+}_{s}(\tau_2)\sin(s+1)\gamma. \tag{22}$$

This series is divergent everywhere, but if the series

$$\Delta^{-} = \sum_{s=0}^{\infty} u^{-}_{s}(\tau_1)u^{+}_{s}(\tau_2)\cos(s+1)\gamma \tag{23}$$

then the series (22) has a generalized sum in the sense of Poisson-Abel, whioh is equal to $-\frac{\cos\tau_1\cos\tau_2}{2\pi^2\sin\gamma}\frac{\partial}{\partial\gamma}\Delta^{-}$.

To prove the oonvergence of (23) we use the expression (15) for $u^{\pm}_{s}(\tau)$ and the asymptotic formulae

$$F\left(\mu,1-\mu;s+1;\frac{1\pm i\,tg\,\tau}{2}\right) = 1+O\left(\frac{1}{s^2}\right),\quad \frac{\Gamma(s+\mu)\Gamma(s-\mu+1)}{s!} = \frac{1}{s}+O\left(\frac{1}{s^2}\right).$$

Then we get $$\Delta^-(x_1,x_2)=\sum_{s=1}^{\infty}\left\{\frac{e^{-is(\tau_1-\tau_2)}}{s}+O\left(\frac{1}{s^2}\right)\right\},$$
i.e. the series converges if $\tau_1-\tau_2\pm\gamma\neq 0$

The geodesic distance between events (τ_1,ξ_1) and (τ_2,ξ_2) in $S_{1,3}$ is equal to *) $\Gamma(x_1,x_2)=\tau\, Ar\,Ch\,G$, $G=\frac{\cos\gamma-\sin\tau_1\sin\tau_2}{\cos\tau_1\cos\tau_2}$

Therefore the equation $\tau_1-\tau_2\pm\gamma=0$ determines the light cone. We emphasize that there were no limitations on m^2 up to here.

The series (23) can be summed. To this end expression (15) for $\mathcal{U}_s^{\pm}$ and the summation theorem of adjoint Legendre functions should be used. The interchange of the transition to the limit $\epsilon=0$ and the summation is possible in view of the convergence of (27) proved above. Finally we have

$$\Delta^-(x_1,x_2)=\frac{\pi}{\sin\mu\pi}\lim_{\epsilon\to 0}P_{-\mu}\left[-G(x_1,x_2)+i(\tau_1-\tau_2)\epsilon\right]$$

where $G(x_1,x_2)=Ch\,\frac{\Gamma(x_1,x_2)}{\tau}$. It is easy to make oneself sure that $\frac{\partial\Delta^-}{\partial\gamma}=-\frac{\sin\gamma}{\cos\tau_1\cos\tau_2}\frac{\partial\Delta^-}{\partial G}$ and consequently

$$\mathcal{D}^-(x_1,x_2)=\frac{i}{8\pi\tau^2\sin\mu\pi}\frac{\partial}{\partial G}\lim_{\epsilon\to+0}P_{-\mu}\left[-G+i(\tau_1-\tau_2)\epsilon\right],$$

Taking into account relations between $P_{-\mu}(\pm z)$ and that $P_{-\mu}(z)$ and $Q_{-\mu}(z)$ are defined on the complex plane with outs along real axis correspondingly from $-\infty$ to -1 and from $-\infty$ to 1, one may pass to the limit $\epsilon=0$:

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*) Thus the function $\Gamma(x_1,x_2)$ is imaginary when $-1<G<1$ i.e. when x_1 and x_2 are space-like situated. In $S_{1,3}$ there are events for which $G<-1$ and there exists no $\Gamma(x_1,x_2)$ because they cannot be connected by any geodesic line.

$$\mathcal{D}^{-}(x_1,x_2)=\frac{\varepsilon(\tau_1-\tau_2)}{8\pi\tau^2}\frac{d}{dG}\left[\theta(G-1)P_{-\mu}(G)\right]+$$

$$+\frac{i}{8\pi\tau^2\sin\mu\pi}\frac{d}{dG}\left\{\theta(1-G)P_{-\mu}(-G)+\theta(G-1)\left[P_{-\mu}(G)+\frac{2}{\pi}\sin\mu\pi\, Q_{-\mu}(G)\right]\right\},$$

where $\varepsilon(\tau)=\pm 1$ for $\tau \gtrless 0$. According to (20), (21) we have

$$\mathcal{D}(x_1,x_2)=\frac{\varepsilon(\tau_1-\tau_2)}{4\pi\tau^2}\frac{d}{dG}\left[\theta(G-1)P_{-\mu}(G)\right], \tag{24}$$

$$\mathcal{D}^{c}(x_1,x_2)=\frac{i}{8\pi\tau^2\sin\mu\pi}\frac{d}{dG}\lim_{\epsilon\to+0}P_{-\mu}(-G+i\epsilon)= \tag{25}$$

$$=\frac{i}{8\pi\tau^2\sin\mu\pi}\frac{d}{dG}\left\{\theta(1-G)P_{-\mu}(-G)+\theta(G-1)\left[e^{-i\mu\pi}P_{-\mu}(G)+\frac{2}{\pi}\sin\mu\pi\, Q_{-\mu}(G)\right]\right\}.$$

Transition to the limit $\mu=0$ $(m=0)$ in Eqs. (24), (25) gives the corresponding functions for massless field:

$$\mathcal{D}_0(x_1,x_2)=\frac{\varepsilon(\tau_1-\tau_2)}{4\pi\tau^2}\delta(G-1), \tag{26}$$

$$\mathcal{D}^{c}_0(x_1,x_2)=\frac{1}{4\pi\tau^2}\left\{\delta(G-1)+\frac{i}{2\pi}(G-1)^{-1}\right\} \tag{27}$$

Now consider properties of $\mathcal{D}$ and $\mathcal{D}^{c}$. Since conditions $G=1$ and $G<1$ define the light cone and its exterior respectively, $\mathcal{D}=0$ outside the light cone for any m^2. According to Eq. (26) the massless field commutator is not zero only on the light cone. This property is a consequence of the conformal covariance of Eq. (2) and occurs in any conformal flat space-time too. This is in contradiction with the assertion /2/ that there exist no conformal covariant free field having a causal field commutator in $S_{1,3}$. This contradiction originates from the difference of starting points: conformal covariant solutions (17) are rejected in the paper /2/ because representations of $SO_0(1,4)$ realized by these functions are non-unitary. However, as is obvious from the paper /13/ and the present section the massless field operator constructed with use of (17) results in $\mathcal{D}_o$ and $\mathcal{D}_o^c$ which coincide with $\mathcal{D}_o$ and $\mathcal{D}_o^c$ in $E_{1,3}$ up to the substitution $G-1 \longrightarrow \lambda^2(x_1, x_2)$, λ being the distance between $x_1, x_2 \in E_{1,3}$.

It is easy to be convinced that the leading singularities of $\mathcal{D}, \mathcal{D}^c$ on the light cone are represented by formulae (26), (27). There is no distinction between the regions $m^2 > 1/4$ and $m^2 < 1/4$ in this respect.

At the exterior of the light cone $G < 1$ and

$$\mathcal{D}^c = \frac{-i\, P^1_{-\mu}(-G)}{8\pi\tau^2 \sin\mu\pi \sqrt{1-G^2}} = -\frac{i m^2 F(\mu, 1-\mu; 2; \frac{1+G}{2})}{4\pi\tau^2 \sin\mu\pi\,(1-G)} . \tag{28}$$

For $G < 1$ and $m^2 > 0$, $\mathcal{D}^c(x_1, x_2)$ diminishes with increasing $|G|$ more rapidly, than $\mathcal{D}_o^c$, Eq. (27). This is seen from the positivness of each term of the hypergeometric series in (28) for $-1 < G < 1$ and from the asymptotic expression for $P^1_{-\mu}(-G)$ when $G \gg 1$.

At last we give an asymptotic form of $\mathcal{D}$ for $G \longrightarrow \infty$ (large distances between time-like situated x_1 and x_2):

if $\mu = \frac{1}{2} - i\Lambda$, $\Lambda > 0$ $(m^2 > \frac{1}{4})$

$$\mathcal{D}(x_1, x_2) = \frac{-m^2\, \varepsilon(\tau_1 - \tau_2)}{(2\pi G)^{3/2}\, \tau^2 \sin\mu\pi} \left| \frac{\Gamma(i\Lambda)}{\Gamma(\frac{3}{2} + i\Lambda)} \right| \left\{ \cos\left[\Lambda \ln 2G + \psi(\Lambda) + i\Lambda\pi\right] + O(G^{-5/2}) \right\} \tag{29}$$

if $\mu < 1/2$ $(m^2 < 1/4)$

$$\mathcal{D}(x_1, x_2) = \frac{\varepsilon(\tau_1 - \tau_2)\, \Gamma(\frac{1}{2} - \mu)}{4\pi^{3/2}\, \tau^2\, 2^\mu\, \Gamma(1-\mu)}\, G^{-1-\mu} + O(G^{-2+\mu}) \tag{30}$$

The asymptotic behaviour of $\mathcal{D}^c$ is of course the same up to an evident difference in coefficients.

It is seen from (29),(31) that for $m^2 < 1/4$ and are decreasing aperiodically and more slowly than for $m^2 > 1/4$. One may assert the same about the decrease of $\mathcal{D}^c$ out the cone. But this difference can hardly be a reason for the fields for which $0 < m^2 < 1/4$ to be rejected.

An example of a situation which is really unacceptable is the case $m^2 < 0$. As is well known, $\mathcal{D}(x_1, x_2)$ determines the solution of the Cauchy problem for Eq. (2) with arbitrary initial data $\varphi(x)\big|_\Sigma$, $\partial_\alpha \varphi(x)\big|_\Sigma$ an arbitrary space-like hypersurface Σ:

$$\varphi(x_1) = \int_\Sigma d\sigma^\alpha(x) \left\{ \mathcal{D}(x_1, x) \partial_\alpha \varphi(x)\big|_\Sigma - \partial_\alpha \mathcal{D}(x_1, x) \varphi(x)\big|_\Sigma \right\}.$$

According to Eq. (30) $\dfrac{\partial \mathcal{D}(x_1, x_2)}{\partial \tau_1} \xrightarrow[\tau_1 \to \pi/2]{} f(r, \tau_2) \cos^\mu \tau$

and consequently $\varphi(x_1)$ is increasing unbounded in time, if $\mu < 0$ ($m^2 < 0$). In classical terms this means an increase of a signal as it propagates. In the spirit of the end of Sec. 3 this may be called a physical manifestation of unacceptability of $m^2 < 0$. This also means that the conclusion about the lack of massless scalar particles in $S_{1,3}$ follows from the theory with Eq. (3) even if the arguments giving rise to the bound (1) (Sec.3) are abandoned.

5. A Generalization

Whether the integral (16) is divergent or convergent in the generalized sense this depends on only asymptotic properties of $u_s^\pm(\tau)$ for $|\tau| \to \pi/2$. Therefore both the question about the validity of the bound (1) .(with $m_0^2 = 1/4r^2$ for Eq. (2) and $m_0^2 = 9/4r^2$ for Eq. 3) and some or other answer to it arise in any cosmological model coinciding asymptotically in time with $S_{1,3}$. It is known that the asymptotic space-time isotropization is a property of a rather wide class of models /12/. For example, it is sufficient for this that the dust-filled Universe be spherically symmetric and the cosmological constant be positive.

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