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# CALCULATION OF THE $\pi e$ - SCATTERING CROSS SECTION INVOLVING THE RADIATIVE CORRECTIONS AND REALISTIC EXPERIMENTAL CONDITIONS

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# CALCULATION OF THE $\pi$ e - SCATTERING CROSS SECTION INVOLVING THE RADIATIVE CORRECTIONS AND REALISTIC EXPERIMENTAL CONDITIONS

1. Problems, need to be solved for comparing theoretical calculations of the $\pi e$ -scattering with experimental data, have much in common with those for the $ep$ -scattering.

In works [1-3] the corresponding formulae have been found for the $ep$ -scattering [x]. In principle, just following the authors of the above papers, it is possible to perform the necessary calculations for the $\pi e$ -scattering, too. Up to now only the first steps have been taken along these lines. In works [4-5] dealing with the $\pi e$ -scattering, some calculations have been made. However these must be considered only as estimatious; for the experimental situation is reproduced there nonrealistically.

The basic difficulties, one encounters in such a work, are of the computative nature. The principal questions as those connected with accounting strong interactions, applicability of perturbation theory and so on, of course, are present as well. Nevertheless, experiments on the $\pi e$ - and $ep$ -scattering are preferable for studying form factors just because the contributions of roughly considered diagrams are usually small, and they can be neglected or fairly taken into account by means of some model (for instance, the $\rho$ -dominance model).

At the same time the calculations prove to be very bulky, and we think that it is quite unreasonable to do them analytically. The analytic methods suffer mainly because of nonuniversality in the sense that if some new factor has to be involved, the calculations should be made practically anew. In the suggested numerical approach using the Monte Carlo method (MCM), any changes are easily introduced, since

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[x] A more detailed list of papers can be found in [3].

only the integrands should be changed, and not the calculation method of integrals.

It takes pleasure, of course, to possess the closed analytic formulae - results. However, they practically prove to be so cumbersome that need to be analyzed again numerically, by means of a computer.

In the present work we formulate the calculation method for the quantity $\sigma_{ex}$ -the experimentally observed $\pi e$-scattering cross section, to order $\alpha^3$, and the corresponding differential cross section over the experimentally registered value of the momentum transfer $d\sigma / dt^e_{ex}$ or, which is the same, $d\sigma / d\epsilon'_{ex}$ ($t^e_{ex} = 2m(\epsilon'_{ex} - m)$, where $\epsilon'_{ex}$ is the experimentally observed value of energy of the final electrons, $m$ - the electron mass). The factors involved into the calculation are the following: the change of electron energy due to the radiation within a target, spread of the primary pion beam, imprecisions in the measurements of momenta and angles of electrons and pions caused by the finite resolution of registering apparatus and, finally, experimental selection criteria (ESC) of elastic events.

The radiation losses were taken into account following the work /3/ by using the formulae (B.2 - B.6). The primary beam spectrum was given by the histogram obtained in the Serpukhov $\pi e$ - scattering experiment. The distributions of experimentally measured values of energies and angles for the electrons and pions with respect to their true values were given via the Gaussian functions of the probability with the experimental widths. The selection criteria, imposed on the experimental values of momenta and restricting the integration region over these variables, included cuts on deviation from the energy conservation law, on deviation from zero transverse component of the total momentum and on coplanarity of the primary pion momentum with respect to the final particle momenta. Calculations were carried out for the mean energy of primary beam $E$ = 50 GeV.

In Section 2 the matrix element for the process is briefly considered; in Sect. 3 we present the calculation technique; in Sect. 4 a comparison with work /4/ is given, and in Sect. 5 the results are discussed.

2. The matrix element of elastic $\pi e$ -scattering, to order $\alpha$ was considered in detail in works /4-5/ . Our calculations have employed the formulae from ref. /4/ , modified by inserting the pion form factor. A possibility of such a phemomenological insertion of the piom form factor into the $\pi e$ -scattering diagrams is now briefly discussed. Let us write the $\pi e$ -scattering cross section to order $\alpha^3$ in the form

$$\sigma = \sigma^{el} + \sigma^{el\,soft}(\bar{\omega}) + \sigma^{hard}(\bar{\omega}) . \qquad (I)$$

Here the quantity $\sigma^{el}$ is of order $\alpha^2$ , and the only diagram

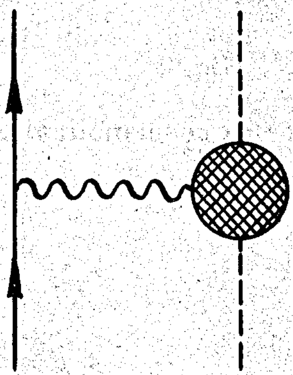

Fig. I

contributes to that. In the matrix element corresponding to the diagram of Fig. I, the pion form factor is contained as a factor and depends on the only variable $t^e_k$ , $t^e_k = 2m(\epsilon'_k - m)$ is the square of lab.momentum transferred to electron, and $\epsilon'_k$ the final electron energy x/ . Two other terms in (I) are of order $\alpha^3$ and to these the following diagrams contribute:

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x/ In what follows we will name kinematical and denote by index "$k$" those values of particle energies, momenta and angles, which enter the matrix element of $\pi e$ -scattering and satisfy the conservation laws, against the experimental values, i.e. those which result from the measurement and will be denoted by index "$ex$".

1) due to electron vertex renormalization

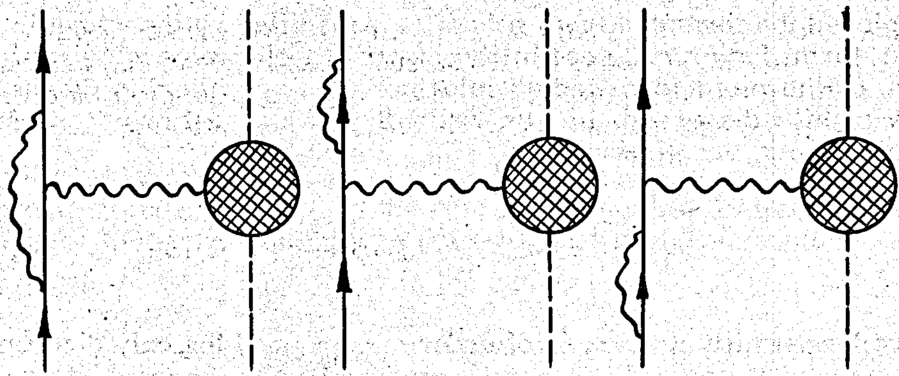

Fig. 2

2) due to meson vertex renormalization

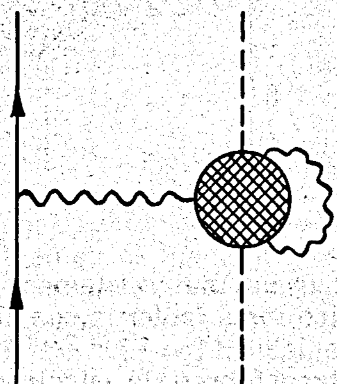

Fig. 3

3) two-photon exchange

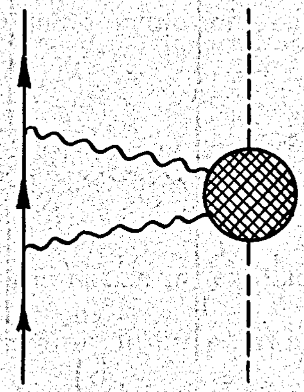

Fig. 4

4) vacuum polarization

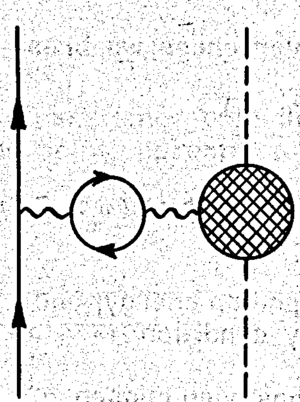

Fig. 5

5) nonobservable photon radiation

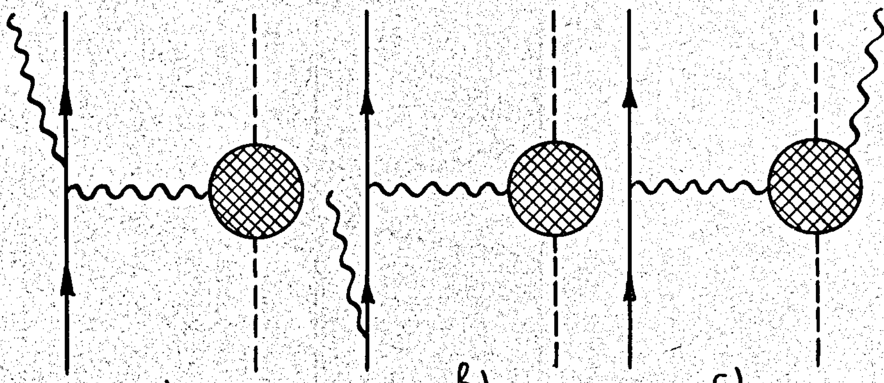


Fig. 6

A contribution to the cross section of the interference of the diagrams in Figs. 2-5 with that in Fig. I is denoted via $\sigma^{int}$. A contribution to (I) of the diagrams of Fig. 6 is denoted by $\sigma^{b}$. To cancel the infrared divergences we break $\sigma^{b}$ into two parts, as usually (see e.g. ref. /4/ )

$$\sigma^{b} = \sigma^{soft}(\omega < \bar{\omega}) + \sigma^{hard}(\omega > \bar{\omega}), \qquad (2)$$

where $\omega$ is the lab.photon energy, and $\bar{\omega}$ is an arbitrary quantity satisfying the only condition $\bar{\omega} \ll m$. In the concrete calculations $\omega = m/100$ has been chosen x/ .

In computing $\sigma^{soft}$ in the bremsstrahlung matrix element described by the diagrams of Fig. 6, only the term $\sim \frac{1}{\omega}$ is considered and the process kinematics is taken to be elastic /4/ . In such

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x/ In order to check whether our results are stable with respect to $\bar{\omega}$, we performed some calculations also for $\bar{\omega} = m/300$. Within the statistical accuracy results are the same. The value of $\bar{\omega} = m/10$ somewhat changes the result for electron energies close to maximum.

an approximation the pion form factor enters as a factor into the matrix element for soft photon radiation, and depends, like in elastic case, on the only variable $t_k^e$.

In calculating the interference of the diagrams of Figs. 2 and 5 with that of Fig. I it is not difficult to take account of the pion form factor. This again is present in the diagrams as a factor depending on $t_k^e$. However, this is not so in the diagrams of Figs. 3 and 4. A contribution of these diagrams, in the point approximation, has been calculated in work /4/. As the numerical calculation shows, at the primary pion energy ~ 50 GeV, their contribution amounts less than I% of the elastic cross section. Corrections to the point approximation due to strong interactions, can be estimated, for example, in the $\rho$-dominance model. In work /5/ these corrections have been found to be of order $t_k^e / m_\rho^2$ thus they can be neglected in the energy region under consideration. Hence only the contribution of these diagrams computed in the point approximation, is here considered. The quantity

$$\sigma^{elsoft} = \sigma^{int} + \sigma^{soft} \tag{3}$$

is free of the infrared divergences and represents thus the total contribution to the cross section of terms $\sim \alpha^3$ with the elastic kinematics.

The quantity $\sigma^{hard} (\sim \alpha^3)$ is a contribution of hard nonobservable photons and in its calculations it is already impossible to consider only the term $\sim \frac{1}{\omega}$ and the kinematics cannot be regarded as the elastic one. In diagrams 6a and 6b the form factor can be allowed for trivially, for it again is contained there as a factor but depending on the other variable $t_k^\pi = -(q-q')^2$ which does not coincide with $t_k^e$ for inelastic kinematics.

Diagram 6c cannot be calculated exactly, however by means of the standard technique we can exactly take account of two principal terms in an expansion of 6c over the photon energy $\omega$ ( the terms $\sim\frac{1}{\omega}$ and $\sim 1$ ). In these two terms the pion form factor is contained as a factor depending on $t_k^e$. Since the diagrams of emission by pion contribute to the cross section much less than those of emission by electron (the square of the pion emission diagrams equals ~ 5% from the total contribution $\sim\alpha^3$ ), then it seems quite sufficient to consider only the above two terms.

Thus we see the phenomenological insertion of the pion form factor is actually possible into all the $\pi e$ -scattering diagrams except for those of Figs. 3 and 4, giving small contribution.

At the pion energy not above 50 GeV, $\sqrt{t_{max}} \lesssim$ 190 MeV, and in this case one can expand the form factor $F(t)$ in the Taylor series over the variable $t$ and consider only a few first terms

$$F(t)=1+\frac{\partial F}{\partial t}\Big|_{t=0} t+\frac{1}{2}\frac{\partial^2 F}{\partial t^2}\Big|_{t=0} t^2+\ldots=1-\frac{1}{6}<r_\pi^2>t+\frac{1}{6\cdot 20}<r_\pi^4>t^2+\ldots^{x/} \quad (4)$$

At such energies the second term in (4) does not exceed 0.07 for $\sqrt{<r_\pi^2>} < 0.7\ f$. Inserting (4) into (I) we can factorize unknown parameters $<r_\pi^2>$ and $<r_\pi^4>$ and represent the cross section in the form

$$\sigma = \sigma_1^{el} + <r_\pi^2>\sigma_2^{el} + (\frac{5}{3}<r_\pi^2>^2 + <r_\pi^4>)\,\sigma_3^{el} + $$
$$+\sigma_1^{elsoft} + <r_\pi^2>\sigma_2^{elsoft} + \sigma_1^{hard} + <r_\pi^2>\sigma_2^{hard}. \quad (5)$$

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x/ Here and in the following the units used for the parameters $<r_\pi^2>$ and $<r_\pi^4>$ are (fermi)$^2$ and (fermi)$^4$, respectively.

If the parameter $<r_\pi^4>$ is not anomalously large, it is doubtful whether the relevant terms in (5) can be measured at the primary beam energy $E = 50$ GeV. It is no sence, obviously, to keep the terms proportional to $<r_\pi^4>$ and $<r_\pi^2>^2$ in the contributions $\sim a^3$.

In what follows we will calculate the quantities $\sigma_{1ex}^{el}, \ldots, \sigma_{2ex}^{hard}$, the corresponding differential cross sections with respect to the quantity $t_{ex}^e$ as well as a radiative correction by which we mean here the following ratio (in %)

$$\delta(<r_\pi^2>, t_{ex}^e) = \frac{\frac{d\sigma_1^{elsoft}}{dt_{ex}^e} + \frac{d\sigma_1^{hard}}{dt_{ex}^e} + <r_\pi^2>\left(\frac{d\sigma_2^{soft}}{dt_{ex}^e} + \frac{d\sigma_2^{hard}}{dt_{ex}^e}\right)}{\frac{d\sigma_1^{el}}{dt_{ex}^e} + <r_\pi^2>\frac{d\sigma_2^{el}}{dt_{ex}^e}} \cdot 100. \tag{6}$$

3. Consider $\sigma_p^s$ any of quantities $\sigma_1^{el}, \ldots, \sigma_2^{hard}$, introduced in the previous section. With the experimental conditions involved the following expression holds for the quantity $(\sigma_p^s)_{ex}$

$$(\sigma_p^s)_{ex} = \int_{-\infty}^{\infty} dE_k f(E_k) \int_m^{\epsilon_k^{max}(E_k)} d\epsilon_k' \int_{\Omega_k} \prod_{l=1}^{n-1} d\xi_k^l M_p^s(E_k, \epsilon_k', \{\xi_k^l\}) \times$$

$$\times \frac{1}{\ell_{max}} \int_0^{\ell_{max}} d\ell \int_m^{\epsilon_k'} d\epsilon_\ell' N(\epsilon_k', \ell) L(\ell, v) \int_{-\infty}^{\infty} d\epsilon_{ex}' G(\epsilon_\ell' - \epsilon_{ex}') \times \tag{7}$$

$$\times \int_{-\infty}^{\infty} \prod_{l=1}^{m-1} G(\xi_{ex}^l - \xi_k^l)\, d\xi_{ex}^l\, C(E_{ex}, \epsilon_{ex}', \{\xi_{ex}^l\}).$$

Here: $E$ - is the energy of the pion primary beam, $E_{ex}$ - its mean value, and $f(E_k)$ the energy distribution density, $\int_{-\infty}^{\infty} f(E_k)\,dE_k = 1$ ; $\epsilon'_k$, $\{\xi'_k\}$ - $n$ essential kinematical parameters, which characterize the considered process and satisfy the conservation laws ( $\epsilon'_k$ - the electron energy, and $\{\xi'_k\}$ - the remaining parameters); $\epsilon'^{max}_k(E_k)$ - the maximum kinematical electron energy for given $E_k$ ; $M^s_p(E_k, \epsilon'_k, \{\xi'_k\})$ - the total contribution of all the diagrams to the corresponding cross section multiplied by some degree of momentum transfer arising from the form factor expansion. The quantities $M^s_p$ depend on the essential kinematical variables: $E_k$ and $\epsilon'_k$ for terms with the elastic kinematics and on $E_k$ and four variables, as well, for terms with the inelastic kinematics;

$\Omega_k$ - is the phase space of $\{\xi'_k\}$ integration, $\ell_{max}$ - the target length, $\ell$ - the path of the final electron within a target;

$\epsilon'_\ell$ - the value of the final electron energy after the external bremsstrahlung within the target, $N(\epsilon'_k, \ell)\,L(\ell, v)$ - the distribution density of the quantity $\epsilon'_\ell$ ,

$$\int_m^{\epsilon'_k} N(\epsilon'_k, \ell)\,L(\ell, v)\,d\epsilon'_\ell = 1 , \qquad (8)$$

and

$$v = \frac{\epsilon'_k - \epsilon'_\ell}{\epsilon'_k} ,$$

for $L(\ell, v)$ we employ the conventional expression /3/

$$L(\ell, v) = v^{b\ell - 1}\left(1 - v + \frac{3}{4} v^2\right) \qquad (9)$$

(for definition of the quantity $b$ see also ref. /3/ ).

The normalization condition (8) for $N(\epsilon'_k, \ell)L(\ell, v)$ provides us with the following expression

$$N(\epsilon'_k, \ell) = \frac{1}{\epsilon'_k} c(a),$$

$$c(a) = \left(\frac{1}{a(a+1)} + \frac{3}{4}\frac{1}{a+2}\right)^{-1}, \qquad a = b\ell. \tag{10}$$

$G(x - x_1) = N \cdot exp\left(-\frac{(x-x_1)^2}{D}\right)$ - are the normalized Gaussian functions in such a way that $\int_{-\infty}^{\infty} G(x - x_1)dx = 1$, these functions involve the errors of measuring apparatus; $\epsilon'_{ex}$, $\{\xi^l_{ex}\}$ - the electron energy and (m-l) other parameters measured in experiment; $C(E_{ex}, \epsilon'_{ex}, \{\xi^l_{ex}\})$ -a function, equal to unity or zero depending on that whether ESC have been satisfied or not for an event.

We note that under used normalization conditions for the functions $f(E_k)$, $N(\epsilon'_k, \ell) L(\ell, v)$ and Gaussian functions, the quantity

$$\sigma = \sum_{p,s} \int_m^{\epsilon'^{max}_k(E_k)} d\epsilon'_k \int_{\Omega_k} \left(\prod_{l=1}^{n-1} d\xi^l_k\right) M^s_p(E_k, \epsilon'_k, \{\xi^l_k\})$$

is the total cross section for the process not accounting the experimental restrictions on the kinematics, unlike the quantity

$$\sigma_{ex} = \sum_{p,s} (\sigma^s_p)_{ex},$$

representing the total experimental cross section for the process.

In principle, interesting for us cross section $d\sigma / d\epsilon'_{ex}$ may be looked for by means of MCM using directly the expression (7). In such an approach the following calculating procedure would take place:

1) An initial pion energy $E_k$ is randomly chosen with the density $f(E_k)$ ;

2) An interaction point in a target is randomly chosen (the path $\ell$ is looked for);

3) A random star is generated, i.e. the random values of

$\{\epsilon'_k, \xi'_k\}$ -kinematical parameters are chosen;

4) At $\epsilon'_k$ fixed the loss $v$ within the target is chosen with density $N(\epsilon'_k, \ell)L(\ell, v)$ and from $v$ the quantity $\epsilon'_\ell$ is obtained;

5) With the density of Gaussian functions near $\epsilon'_\ell$ and $\xi'_k$ the values of experimental parameters $\epsilon'_{ex}$ and $\xi'_{ex}$ are chosen;

6) The function $C(E_{ex}, \epsilon'_{ex}, \{\xi'_{ex}\})$ is calculated;

7) The quantity $M^s_p(E_k, \epsilon'_k, \{\xi'_k\})$ is calculated;

8) The quantities $d\sigma^s_p / d\epsilon'_{ex}$ are obtained by histogramming with respect to $\epsilon'_{ex}$ .

We see that in such an approach the calculations must be performed over the whole range $\epsilon'_{ex}$ while to obtain a good accuracy the quantity, $d\sigma^s_p / d\epsilon'_{ex}$ should be calculated only at the fixed points interesting for us. We also note, that using this technique with histogramming, we lose accuracy because of a fluctuation of event number in the histogram cells. It is well-known, as well that the calculation method by means of histogramming works unwell on the intervals of rapid change of a function. As will be seen below, the radiation correction changes rapidly near $\epsilon'^{max}_k(E_k)$ . Thus, even if the calculation would be necessary to perform over the whole interval $\epsilon'_{ex}$ the calculation method at the fixed points $\epsilon'_{ex}$ would possess the indisputable advantage over the histogramming.

To be able to calculate $d\sigma_p^s / d\epsilon'_{ex}$ at $\epsilon'_{ex}$ fixed it is necessary to transform exp. (7). It is clear that

$$\frac{d\sigma_p^s}{d\epsilon'_{ex}} = \int_{-\infty}^{\infty} f(E_k)\, dE_k \int_m^{\epsilon_k'^{max}(E_k)} d\epsilon'_k \frac{1}{\ell_{max}} \int_0^{\ell_{max}} d\ell \int_m^{\epsilon'_k} d\epsilon'_\ell N(\epsilon'_k, \ell) \times$$

$$\times L(\ell, v)\, G(\epsilon'_\ell - \epsilon'_{ex})\, \Phi_p^s(\epsilon'_k, \epsilon'_{ex}), \qquad (12)$$

where via $\Phi_p^s(\epsilon'_k, \epsilon'_{ex})$ we denote the following integral

$$\Phi_p^s(\epsilon'_k, \epsilon'_{ex}) = \int_{\Omega_k} \prod_{i=1}^{n-1} d\xi_k^i M_p^s \int_{-\infty}^{\infty} \prod_{j=1}^{m-1} G(\xi_{ex}^j - \xi_k^j)\, d\xi_{ex}^j\, C(E_{ex}, \epsilon'_{ex}, \{\xi_{ex}^j\}). \qquad (13)$$

Let us insert the $\theta$ function into the integral over $d\epsilon'_\ell$, in order to make constant the upper limit of this integral. At the same time we change the integration order in (12). Then we get

$$\frac{d\sigma_p^s}{d\epsilon'_{ex}} = \int_{-\infty}^{\infty} f(E_k)\, dE_k \frac{1}{\ell_{max}} \int_0^{\ell_{max}} d\ell \int_m^{\epsilon_k'^{max}(E_k)} d\epsilon'_\ell\, G(\epsilon'_\ell - \epsilon'_{ex}) \times$$

$$\times \int_m^{\epsilon_k'^{max}(E_k)} \theta(\epsilon'_k - \epsilon'_\ell)\, d\epsilon'_k\, N(\epsilon'_k, \ell)\, L(\ell, v)\, \Phi_p^s(\epsilon'_k, \epsilon'_{ex})$$

Now let us use the $\theta$-function in the integral over $d\epsilon'_k$. Finally we arrived at

$$\frac{d\sigma_p^s}{d\epsilon'_{ex}} = \int_{-\infty}^{\infty} f(E_k)\, dE_k \frac{1}{\ell_{max}} \int_0^{\ell_{max}} d\ell \int_m^{\epsilon_k'^{max}(E_k)} d\epsilon'_\ell\, G(\epsilon'_\ell - \epsilon'_{ex}) \times$$

$$\times \int_{\epsilon'_\ell}^{\epsilon_k'^{max}(E_k)} d\epsilon'_k\, N(\epsilon'_k, \ell)\, L(\ell, v)\, \Phi_p^s(\epsilon'_k, \epsilon'_{ex}). \qquad (14)$$

To the expression (14) there corresponds the following calculation procedure:

1) With the density $f(E_k)$ an initial pion energy is randomly chosen;

2) An interaction point within a target is chosen ( $\ell$ is looked for);

3) Near the fixed value of $\epsilon'_{ex}$ with the density of Gaussian function the quantity $\epsilon'_\ell$ is randomly chosen, the choosing being done in the interval $m < \epsilon'_\ell < \epsilon'^{max}_k(E_k)$ ;

4) Near the quantity $\epsilon'_\ell$ obtained with the density proportional to $N(\epsilon'_k, \ell) L(\ell, v) d\epsilon'_k$ the quantity $v$ is chosen, and from it the quantity $\epsilon'_k$ x/ is found;

5) For the obtained $\epsilon'_k$ a random star is generated, i.e., values of remaining essential kinematical parameters $\xi^!_k$ are found;

6) With the density of Gaussian functions near the obtained kinematical values of $\xi^!_k$ the random experimental values of $\xi^!_{ex}$ are chosen;

7) The function $C(E_{ex}, \epsilon'_{ex}, \{\xi^!_{ex}\})$ is calculated;

8) For the events satisfying ESC the quantity $M^s_p$ is calculated.

As is seen from the above procedure,the simulation of an electron "fate" does not correspond to the real development of events in time: At first, errors and loss are taken into account, and only after this the interaction is involved. However, this must not be surprising,

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x/ The interchange of the integration order gives rise to that a distribution of loss $v$ is described by completely different function $N(\epsilon'_k, \ell) L(\ell, v) d\epsilon'_k$ and not by $L(\ell, v) d\epsilon'_\ell$ as in the first case.

for the calculation technique corresponds to the integral (14) derived from the starting expression (7) by means of identical transformations.

Now, we would like to describe briefly the calculation technique for the function $\Phi_p^s(\epsilon_k', \epsilon_{ex}')$. Before choosing random values of variables entering the integrand of $\Phi_p^s(\epsilon_k', \epsilon_{ex}')$ the quantity $\epsilon_k'$ has been found. If to $\Phi_p^s$ just the diagrams with elastic kinematics contribute, then the two quantities $E_k$ and $\epsilon_k'$ completely define $M_p^s$ and the function $\Phi_p^s$ is calculated immediately in straightforward way. Otherwise, if to $\Phi_p^s$ the diargams with inelastic kinematics contribute, the corresponding $M_p^s$ are sharply peaked functions. The basic contribution to the inelastic cross section proceeds from diagrams 6a and 6b describing the emission by electron. Direct calculations reveal the main contribution proportional to $1/pk \cdot p'k$ (where $p, p', k$ are four-momenta of the initial, final electrons and photon, respectively). Thus, the corresponding $M_p^s$ possess a sharp peak at small $x = -2p \cdot k$ and $y = -2p'k$, and to arrive at the high accuracy it is necessary to go to the variables $x$ and $y$, and choose them with the density $1/xy\, dx\, dy$.

On completing such a procedure in the matrix element of inelastic scattering three out of four essential variables have already been chosen. As the remaining variable there is taken the angle between vectors $\vec{q} \times \vec{q}'$ and $\vec{p}' \times \vec{k}$ which varies within the most simple limits:

$$0 \leq \phi \leq 2\pi .$$

For new variables $x, y$ and $\phi$ the integral over phase space of the hard photons $\Omega_k$ looks as follows

$$\int\limits_{\Omega_k} \prod_{l=1}^{n-1} d\xi^l_k M^s_p = \frac{\pi}{8\sqrt{\lambda}} \int\limits_{x_{min}}^{x_{max}} dx \int\limits_{y_{min}}^{y_{max}} dy \int\limits_0^{2\pi} d\phi \frac{M^s_p}{\sqrt{(x-t^e_k)^2-4m^2(t^e_k+y-x)}} . \qquad (15)$$

The $x$ and $y$ integration region is given by the inequalities:
if

$$\bar{x} \leq x \leq x_1 ,$$

then

$$\frac{x}{2m^2}(t^e_k + 2m^2 - z) \leq y \leq \frac{x}{2m^2}(t^e_k + 2m^2 + z) ;$$

if

$$x_1 \leq x \leq x_2$$

then

$$\frac{x}{2m^2}(t^e_k + 2m^2 - z) \leq y \leq \frac{1}{2m^2}[ (t^e_k + x)(S + m^2 - \mu^2) -$$

$$- \lambda + \sqrt{\lambda} \sqrt{(S - t^e_k - x - m^2 - \mu^2)^2 - 4m^2\mu^2} .$$

Here $\bar{x} = 2m\bar{\omega}$

$$x_{\substack{2\\1}} = \frac{t^e_k(\mu^2 - m^2 - S) + \sqrt{\lambda}\, z}{S - m^2 - \mu^2 - t^e_k \mp (\sqrt{\lambda} - z)}$$

$\mu$ - is the pion mass,

$$S = m^2 + \mu^2 + 2mE_k$$

$$\lambda = \lambda(S, m^2, \mu^2) = (S - m^2 - \mu^2)^2 - 4m^2\mu^2,$$

$$z = \sqrt{t_k^e(t_k^e + 4m^2)} .$$

In the actual calculation the choosing of random values of variables was carried out with the density proportional to $v^{b\ell-1} \cdot 1/x \cdot 1/y$ and Gaussian functions. All other factors in (14) were accounted as the event weight. The calculation has shown that this weight is now rather slowly varying function, which ensures the rapid convergence of the calculation results by MCM.

4. To check up the validity of the program realizing the method described in the previous section, numerous test calculations were made. In particular, to test the program of generation of random $\pi e \gamma$ -events with density $1/x \cdot y$ we have reproduced the result of the Kahane's work /4/. In this work the radiation corrections have been calculated without the radiation loss within a target, beam spread and resolution of devices, under the only condition $\omega_k < \Lambda$ or, the same $|E_k - \epsilon_k' - E_k'| < \Delta$ (Here $E_k'$ is the final pion energy). We have used the formulae of work /4/ for $E_k = 50$ GeV and $\Lambda = \epsilon_k' \times (0.07, 0.04, 0.01)$ and compared with our calculation results under the same conditions. At energies $\epsilon_k' < 22$ GeV an agreement is achieved within a relative accuracy better than 0.5%. At energies $\epsilon_k' > 22$ GeV there begins to display a discrepancy increasing with $\epsilon_k'$. That Kahane /4/ has used the approximations in deriving the formulae accounts for this discrepancy. Thus,

e.g. there have been neglected terms of the type $\Lambda \cdot m^2 \cdot E_k$ / $(4m^2 E_k E_k' - t_k^e \mu^2)$ very growing as $\epsilon_k' \rightarrow \epsilon_k'^{max}(E_k)$ . Because of this the Kahane's formulae, at energies equal, for instance, to $\epsilon_k'$ = 32 GeV, cannot pretend to an accuracy of radiative correction calculations better than 1% of the elastic scattering; and at $\epsilon_k' \geq$ 34 GeV they give rise to absolutely noncorrect result. In the region 22 GeV $< \epsilon_k' <$ 32 GeV agreement holds at the values $\Delta \lesssim$ 0.04 GeV only, unacceptable in the experimental treating.

5. The method described was realized in two programs written in FORTRAN, the so-called programs of "elastic" and "inelastic" kinematics. The calculation results are listed in Tables 1 and 2. In the first Table the results of calculations for elastic kinematics are presented. As is seen from the upper part of Table, of the calculation results without any cuts, an account of the radiation loss and beam spread influences rather strongly $d\sigma_p^s / dt_{ex}^e$ as compared to $d\sigma_p^s / dt_k^e$ computed analytically. Except for the point $\epsilon_{ex}'$= 36 GeV, close to highest possible kinematical energy, the above difference for all points is caused mainly by the radiation loss; but at the point $\epsilon_{ex}'$ = 36 GeV both the factors are essential. The experimental cuts change the result weakly enough, for these have been chosen to be fairly soft - their values equal four standard deviations in the measurement of an appropriate quantity.

Analogous conclusions hold for the terms $d\sigma_{1,2}^{hard} / dt_{ex}^e$ with the only difference that due to inelastic kinematics the other part of events does not satisfy ESC (see Table 2).

From Table 2 it is seen however that the radiative correction depends rather weakly on ESC. This is due to the fact that the soft photons contribute mainly to the bremsstrahlung, and effectively the kinematics of inelastic process differs slightly from that of elastic process, and ESC influence almost similarly the terms with elastic and inelastic kinematics. An effect of ESC increases with decreasing energy, since the less is the electron energy, the larger the photon phase space and the stronger difference from the elastic kinematics.

To investigate an influence of the involved experimental factors, there has been made the calculation of the radiative correction not involving radiation loss, beam spread and measurement errors.

The summary effect of all these factors proves to change the radiative correction by a value equal almost to 2 % of elastic cross section. Thus, an account of all the factors seems to be necessary in the terms $\sim \alpha^3$, as well. (This statement for terms $\sim \alpha^2$ was stressed earlier).

An important feature of the obtained results also is that the radiative correction (6) within the calculation accuracy does not depend on the value of $\langle r_\pi^2 \rangle$ which we took equal to 0 and 0.7 $f$. Thus, within this accuracy the pion form factor is factorized from terms $\sim \alpha^3$ in the same way as from terms $\sim \alpha^2$, and their ratio does not depend on $\langle r_\pi^2 \rangle$. This property results from the fact, that the contribution of diagrams calculated in the point approximation is small, and also that the difference of $t^\pi$ from $t^e$ is nonessential for the inelastic kinematics because of the overwhelming contribution of soft photons. This property is very favourable for a model independent determination of $\langle r_\pi^2 \rangle$, and in fact it means that for the considered energy $E$, the radiation correction can be calculated rather accurately in the point approximation. Summarizing one can say that a comparison with experiment is possible by means of the following simple formula:

$$\frac{d\sigma}{dt^e_{ex}} = \frac{d\sigma^{el}}{dt^e_{ex}} \left(1 + \delta\left(t^e_{ex}\right)\right), \qquad (16)$$

where $d\sigma^{el}/dt^e_{ex}$ contains the pion form factor and depends on $\langle r^2_\pi \rangle$, and $\delta$ does not depend on $\langle r^2_\pi \rangle$.

The test calculation at $E = 100$ GeV has already revealed a weak dependence of the radiation correction on $\langle r^2_\pi \rangle$, however, our method remains valid also in this region, with only difference that in comparing with experiment one should use not the simple formula (16) but more complicated representation analogous with (5).

Now let us discuss the accuracy which we like to get in calculating the radiative correction. As it was said in Sect. 2, an error due to the account of diagrams 3 and 4 in the point approximation, is of order $t^e_k / m^2_\rho \lesssim 0.06$. Since these diagrams make a contribution ~1% of the elastic scattering, such an approximation brings to the cross section an absolute inaccuracy less than 0.06%. Other error is because of neglecting the terms $\sim\alpha^4$ which are known to be roughly estimated by means of the exponential representation. In such a case, the absolute inaccuracy in the calculation of $\delta$ is equal roughly to $\delta^2/2$. Since $\delta$ does not exceed 0.13, the conclusion is possible that the absolute inaccuracy of this approximation is not worse than 0.8% at the point $\epsilon'_{ex} = 36$ GeV, is much less than 0.8% at other points and is the main error of calculations.

There exists, of course, also an error due to inaccurate account of the experimental situation. However, this can be estimated by the Monte-Carlo method only. In the present work such estimations have not been performed.

In Tables the calculation statistical error is given. To get such an accuracy, 1.5 h. of computing time at the computer BESM-6 has been required. The statistical error, of course, can be cut down.

In conclusion we would like to present a few remarks on a possible application of our calculation procedure to the region $E$ = 100-300 GeV. In the part concerning an account of the experimental situation, selection criteria and generation of random stars, there do not evidently happen any principal difficulties, except for the extreme ultrarelativism which may require a calculation with double precision. However, it is no longer possible to use the expansion of the pion form factor in the Taylor series, because the second term of expansion becomes of the same order as the first one, and some different parametrization for the form factor is necessary. The account of two-photon exchange diagrams and meson vertex renormalization in the point approximation becomes unsatisfactory, as well, since the terms

$\sim t / m_\rho^2$ which are small at $E \lesssim$ 50 GeV, now become ~ 1.

Fortunately, the contribution of these diagrams (denote it as $\sigma^p$ ), computed in the point approximation, remains small also at energies 100-300 GeV. Therefore the following calculation method seems to us to be reasonable. Everywhere the pion form factor is factorized, its some new parametrization is put. For example, a deviation from the $\rho$ -dominance model is investigated in the form

$$F(t) = \frac{1}{1 + \frac{t}{m_\rho^2}} (a + bt + \dots) \, , \qquad (17)$$

or in some other, and the contribution $\sigma^p$ as being small is calculated in a simple model, e.g. in the $\rho$ -dominance one. Thus, it is sufficient to introduce into our programs an expression of the type (17) everywhere the pion form factor is factorized, and to replace expressions for $\sigma^p$ by formulae calculated in some model. These formulae will be presented elsewhere.

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Table 1

MONTE-CARLO CALCULATION OF RADIATIVE CORRECTIONS TO PI-E SCATTERING (SERPUKHOV ACCELERATOR-71)

AVERAGE INITIAL ENERGY OF PIONS= 50.00GEV
ACCURACY IN ANGULAR MEASUREMENT DT= 0.20MRAD
ACCURACY IN MOMENTUM MEASUREMENT DP=DT*(P**2)/ 1.625

ENERGY BALANCE CUT 50.00GEV
PERPENDICULAR MOMENTUM CUT= 50.0+003MEV
COPLANARITY CUT= 10.00-001

| ELECTRON ENERGY(GEV) | TRANSFER MOMENTUM(MEV) | ELASTIC CROSS SECTION ANALYTIC | EXPERIMENTAL | MONTE-CARLO ERROR | ALPHA-ORDER ELASTIC AND SOFT PHOTONS CROSS SECTION ANALYTIC | EXPERIMENTAL | MONTE-CARLO ERROR |
|---|---|---|---|---|---|---|---|
| | | ( I N | U N I T S | | 1.E - 34 C M ** 2 / M E V ** 2 | ) | |
| | | T E R M S | I N D E P E N D E N T | O N | R A D I U S | | |
| 18.0 | 135.6 | 3,871+000 | 3,690+000 | 4.932-003 | -1.960+000 | -1,870+000 | 2,499-003 |
| 27.0 | 166.1 | 8,695-001 | 8,146-001 | 1.124-003 | -4.384-001 | -4,109-001 | 5,765-004 |
| 30.0 | 175.1 | 4,745-001 | 4,430-001 | 6.201-004 | -2.359-001 | -2.201-001 | 3,149-004 |
| 33.0 | 183.6 | 2,023-001 | 1,888-001 | 3.383-004 | -9.796-002 | -9,160-002 | 1,682-004 |
| 36.0 | 191.8 | 1,044-002 | 1,966-002 | 1.627-004 | -4.876-003 | -9,272-003 | 7,722-005 |
| | | T E R M S | P R O P O R T I O N A L | T O | /RPI**2//FERMI**2 | | |
| 18.0 | 135.6 | -6,242-001 | -6,013-001 | 6.726-004 | 3.117-001 | 3.005-001 | 3,366-004 |
| 27.0 | 166.1 | -2,103-001 | -1,981-001 | 2.528-004 | 1.046-001 | 9,857-002 | 1,283-004 |
| 30.0 | 175.1 | -1,275-001 | -1,195-001 | 1.504-004 | 6.247-002 | 5.858-002 | 8,001-005 |
| 33.0 | 183.6 | -5,981-002 | -5,590-002 | 9.835-005 | 2.860-002 | 2.679-002 | 4,831-005 |
| 36.0 | 191.8 | -3,368-003 | -6,335-003 | 5.237-005 | 1.555-003 | 2.955-003 | 2,458-005 |
| | | T E R M S | P R O P O R T I O N A L | T O | (5/3*/RPI**2/**2+/RPI**4/)/FERMI**2 | | |
| 18.0 | 135.6 | 1,510-002 | 1,474-002 | 1.248-005 | -7,540-003 | -7,364-003 | 6,277-006 |
| 27.0 | 166.1 | 7,631-003 | 7,231-003 | 8.412-006 | -3,795-003 | -3,597-003 | 4,287-006 |
| 30.0 | 175.1 | 5,142-003 | 4,835-003 | 6.117-006 | -2.518-003 | -2.370-003 | 3,079-006 |
| 33.0 | 183.6 | 2,652-003 | 2,483-003 | 4.289-006 | -1.268-003 | -1.190-003 | 2,108-006 |
| 36.0 | 191.8 | 1,630-004 | 3,062-004 | 2.529-006 | -7.524-005 | -1,428-004 | 1,187-006 |

Table 1 (cont.)

ENERGY BALANCE CUT 4,00GEV
PERPENDICULAR MOMENTUM CUT= 56.0+000MEV
COPLANARITY CUT= 10,00-001

| ELECTRON ENERGY(GEV) | TRANSFER MOMENTUM(MEV) | ELASTIC CROSS SECTION ANALYTIC | ELASTIC CROSS SECTION EXPERIMENTAL | ELASTIC CROSS SECTION MONTE-CARLO ERROR | ALPHA-ORDER ELASTIC AND SOFT PHOTONS CROSS SECTION ANALYTIC | ALPHA-ORDER ELASTIC AND SOFT PHOTONS CROSS SECTION EXPERIMENTAL | ALPHA-ORDER ELASTIC AND SOFT PHOTONS CROSS SECTION MONTE-CARLO ERROR |
|---|---|---|---|---|---|---|---|
| | | ( I N U N I T S | 1.E - 34 C M ** 2 / M E V ** 2 | ) | | | |
| | | TERMS INDEPENDENT ON RADIUS | | | | | |
| 18.0 | 135.6 | 3,871+000 | 3,645+000 | 6.474-003 | -1.960+000 | -1.848+000 | 3,279-003 |
| 27.0 | 166.1 | 8,695-001 | 8,091-001 | 1.329-003 | -4,384-001 | -4.083-001 | 6,756-004 |
| 30.0 | 175.1 | 4,745-001 | 4,421-001 | 6.604-004 | -2.355-001 | -2,196-001 | 3,338-004 |
| 33.0 | 183.6 | 2,023-001 | 1,888-001 | 3.383-004 | -9.796-002 | -9.160-002 | 1,682-004 |
| 36.0 | 191.8 | 1,044-002 | 1,966-002 | 1.627-004 | -4.876-003 | -9,272-003 | 7,722-005 |
| | | TERMS PROPORTIONAL TO /RPI**2//FERMI**2 | | | | | |
| 18.0 | 135.6 | -6,242-001 | -5.914-001 | 1.023-003 | 3.117-001 | 2.956-001 | 5,111-004 |
| 27.0 | 166.1 | -2,103-001 | -1,965-001 | 3.138-004 | 1.046-001 | 9.780-002 | 1,574-004 |
| 30.0 | 175.1 | -1,275-001 | -1,192-001 | 1.723-004 | 6.247-002 | 5.844-002 | 8,596-005 |
| 33.0 | 183.6 | -5,981-002 | -5,590-002 | 9.835-005 | 2.860-002 | 2.679-002 | 4,831-005 |
| 36.0 | 191.8 | -3,368-003 | -6,335-003 | 5.237-005 | 1.555-003 | 2.955-003 | 2,458-005 |
| | | TERMS PROPORTIONAL TO (5/3*/RPI**2/**2+/RPI**4/)/FERMI**2 | | | | | |
| 18.0 | 135.6 | 1,510-002 | 1.440-002 | 2.441-005 | -7.540-003 | -7,196-003 | 1,220-005 |
| 27.0 | 166.1 | 7,631-003 | 7.161-003 | 1.113-005 | -3.795-003 | -3,564-003 | 5,584-006 |
| 30.0 | 175.1 | 5,142-003 | 4,821-003 | 6.736-006 | -2.518-003 | -2,364-003 | 3,364-006 |
| 33.0 | 183.6 | 2,652-003 | 2,483-003 | 4.289-006 | -1.268-003 | -1.190-003 | 2,108-006 |
| 36.0 | 191.8 | 1,630-004 | 3,062-004 | 2.529-006 | -7,524-005 | -1.428-004 | 1,187-006 |

Table 2

MONTE-CARLO CALCULATION OF RADIATIVE CORRECTIONS TO PI-E SCATTERING (SERPUKHOV ACCELERATOR-71)

AVERAGE INITIAL ENERGY OF PIONS= 50.00GEV
ANGULAR MEASUREMENT ACCURACY DT= 0.20MRAD
MOMENTUM MEASUREMENT ACCURACY DP=DT*(P**2)/ 1.625

ENERGY BALANCE CUT 50.00GEV
PERPENDICULAR MOMENTUM CUT= 50.0*003MEV
COPLANARITY CUT= 10.00-001

A L P H A - O R D E R   T E R M S

| ELECTRON ENERGY (GEV) | TRANSFER MOMENTUM (MEV) | ELASTIC CROSS SECTION | ELASTIC AND SOFT PHOTONS CROSS SECTION | HARD PHOTONS CONTRIBUTION | MONTE-CARLO ERROR | RADIATIVE CORRECTION WITH ITS MONTE-CARLO ERROR ( PER CENT ) (FOR RPI=0 AND 0.7F) |
|---|---|---|---|---|---|---|
| | | ( I N   U N I T S   1.E - 34   C M ** 2 / M E V ** 2 ) | | | | |
| | | T E R M S   I N D E P E N D E N T   O N   R A D I U S | | | | |
| 18.0 | 135.6 | 3.698+000 | -1.870+000 | 1.020+000 | 6.012-003 | -1.35 0.18 |
| 27.0 | 166.1 | 9.146-001 | -4.109-001 | 3.788-001 | 1.284-003 | -3.95 0.17 |
| 30.0 | 175.1 | 4.430-001 | -2.201-001 | 1.962-001 | 6.806-004 | -5.34 0.17 |
| 33.0 | 183.6 | 1.880-001 | -9.160-002 | 7.761-002 | 2.894-004 | -7.41 0.10 |
| 36.0 | 191.9 | 1.966-002 | -9.272-003 | 6.599-003 | 5.739-005 | -13.60 0.50 |
| | | T E R M S   P R O P O R T I O N A L   T O   /RPI**2//FERMI**2 | | | | |
| 18.0 | 135.6 | -6.013-001 | 3.005-001 | -2.914-001 | 9.521-004 | -1.34 0.19 |
| 27.0 | 166.1 | -1.981-001 | 9.857-002 | -9.115-002 | 3.065-004 | -3.98 0.20 |
| 30.0 | 175.1 | -1.195-001 | 5.850-002 | -5.245-002 | 1.022-004 | -5.43 0.20 |
| 33.0 | 183.6 | -5.590-002 | 2.679-002 | -2.207-002 | 0.329-005 | -7.48 0.21 |
| 36.0 | 191.9 | -6.335-003 | 2.955-003 | -2.123-003 | 1.044-005 | -13.69 0.60 |

ENERGY BALANCE CUT 4.00GEV
PERPENDICULAR MOMENTUM CUT= 50.0*000MEV
COPLANARITY CUT= 10.00-001

A L P H A - O R D E R   T E R M S

| ELECTRON ENERGY (GEV) | TRANSFER MOMENTUM (MEV) | ELASTIC CROSS SECTION | ELASTIC AND SOFT PHOTONS CROSS SECTION | HARD PHOTONS CONTRIBUTION | MONTE-CARLO ERROR | RADIATIVE CORRECTION WITH ITS MONTE-CARLO ERROR ( PER CENT ) (FOR RPI=0 AND 0.7F) |
|---|---|---|---|---|---|---|
| | | ( I N   U N I T S   1.E - 34   C M ** 2 / M E V ** 2 ) | | | | |
| | | T E R M S   I N D E P E N D E N T   O N   R A D I U S | | | | |
| 18.0 | 135.6 | 3.645+000 | -1.840+000 | 1.776+000 | 6.436-003 | -1.97 0.20 |
| 27.0 | 166.1 | 8.091-001 | -4.003-001 | 3.742-001 | 1.329-003 | -4.21 0.18 |
| 30.0 | 175.1 | 4.421-001 | -2.106-001 | 1.954-001 | 6.894-004 | -5.49 0.17 |
| 33.0 | 183.6 | 1.878-001 | -9.160-002 | 7.741-002 | 2.894-004 | -7.41 0.18 |
| 36.0 | 191.9 | 1.966-002 | -9.272-003 | 6.599-003 | 5.739-005 | -13.68 0.50 |
| | | T E R M S   P R O P O R T I O N A L   T O   /RPI**2//FERMI**2 | | | | |
| 18.0 | 135.6 | -5.914-001 | 2.956-001 | -2.817-001 | 1.038-003 | -1.94 0.22 |
| 27.0 | 166.1 | -1.965-001 | 9.780-002 | -8.902-002 | 3.214-004 | -4.23 0.21 |
| 30.0 | 175.1 | -1.192-001 | 5.844-002 | -5.219-002 | 1.849-004 | -5.53 0.20 |
| 33.0 | 183.6 | -5.390-002 | 2.679-002 | -2.287-002 | 0.529-005 | -7.48 0.21 |
| 36.0 | 191.9 | -6.335-003 | 2.955-003 | -2.123-003 | 1.844-005 | -13.69 0.60 |