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GROUP INTERPRETATION OF THE DUAL TRANSFORMATIONS IN THE COHERENT STATE SPACE

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Introduction

In the papers[1,2] a factorization of the dual N-point amplitude with the aid of coherent states of a final number 5-dimensional oscillators has been proposed. Here we find out a representation of the group $SL(2,R)$, that performs the dual transformations in the space of these coherent states. It allows us to factorize the dual N-point amplitude, corresponding to a semimultiperipherical configuration of the external particles.

We note, that in the Koba-Nielsen approach[3-5] (see also the review article[6]) the dual transformations are essentially equivalent to some projective (or linear-fractional) transformations of the integration variables which also belong to the group $SL(2,R)$.

Another connection between the dual amplitudes and the group $SL(2,R)$ has been noted in[7-9]. In particular, in these papers it has been shown, that the four-point amplitude appears as a kernel of a certain representation of the triangular matrix group which is contained in $SL(2,R)$.

The advantage of the factorization of the dual amplitudes with the aid of coherent states is due to the fact, that these states turn out to be a suitable basis for the given representation of the group $SL(2, R)$.

I.

Let us consider the 3-parameter group $SL\ (2\ ,\ R)$ of all two-dimensional real unimodular matrices. One can introduce the following parametrization for the elements of the group:

$$g_\varepsilon(\alpha,\beta,\gamma)=\begin{pmatrix} \varepsilon e^{\frac{\alpha}{2}} & \beta \\ \gamma & \varepsilon(1+\beta\gamma)e^{-\frac{\alpha}{2}} \end{pmatrix} \qquad \varepsilon=\pm 1 . \tag{1.1}$$

The parameters α , β , γ are arbitrary real numbers. The group multiplication law

$$g_\varepsilon(\alpha,\beta,\gamma)=g_{\varepsilon_1}(\alpha_1,\beta_1,\gamma_1)\,g_{\varepsilon_2}(\alpha_2,\beta_2,\gamma_2) \tag{1.2}$$

is given by

$$\begin{aligned} \alpha &= 2\ln\left(\varepsilon_1\varepsilon_2\beta_1\gamma_2+e^{\frac{\alpha_1+\alpha_2}{2}}\right) \\ \beta &= \varepsilon_1\beta_2 e^{\frac{\alpha_1}{2}}+\varepsilon_2\beta_1(1+\beta_2\gamma_2)e^{-\frac{\alpha_2}{2}} \\ \gamma &= \varepsilon_2\gamma_1 e^{\frac{\alpha_2}{2}}+\varepsilon_1\gamma_2(1+\beta_1\gamma_1)\,e^{-\frac{\alpha_1}{2}} \qquad \varepsilon=\varepsilon_1\varepsilon_2 . \end{aligned} \tag{1.3}$$

It follows from this, that the elements $g(\alpha,0,0)$, $g(0,\beta,0)$ and $g(0,0,\gamma)$ form three abelian subgroups of $SL\ (2\ ,\ R)$.

The parametrization (1.1) defines the local group $SL(2,R)$ in the neighbourhood of the identity element. In our consideration we shall use also matrices of the type

$$g(\beta)=\begin{pmatrix} 0 & \beta \\ \frac{-1}{\beta} & 0 \end{pmatrix} \tag{1.4}$$

which do not allow such a parametrization. In this case we shall use the decomposition

$$g(\beta)=g(0,\beta,0)\,g(0,0,\tfrac{-1}{\beta})\,g(0,\beta,0) \tag{1.5}$$

that follows from eqs (1.2)-(1.3). (All matrices in the right-hand side of eq. (1.5) are well defined).

Let us find out a representation of the $SL(2,R)$ group in the space of the coherent states of one 5-dimensional oscillator [1,2]. The creation and annihilation operators of this oscillator satisfy the commutation relations:

$$[a_i, a_k^+] = G_{ik}\,, \quad [a_i, a_k] = [a_i^+, a_k^+] = 0 \tag{1.6}$$
$$i = 0,1,2,3,4$$

and the elements of the metrix tensor are

$$-G_{00} = G_{11} = G_{22} = G_{33} = G_{44} = 1$$
$$G_{ik} = 0 \quad i \neq k\,. \tag{1.7}$$

If we consider a_i (i = 0,1,2,3,4) as the components of one 5-dimensional vector a , the scalar product of such two vectors is giwen by

$$za = \sum_{i,k=0}^{4} G^{ik} z_i a_k = \sum_{i=0}^{4} z^i a_i\,. \tag{1.8}$$

The coherent states of our oscillator are characterized by five complex numbers z_i and are defined as follows:

$$|z\rangle \equiv \exp\{za^+\}|0\rangle = \exp\Big\{\sum_{i=0}^{4} z^i a_i^+\Big\}|0\rangle . \qquad (1.9)$$

Now we define the representation of the group $SL(2,R)$ in the space of coherent states (1.9) as follows: to every element $g_\varepsilon(\alpha,\beta,\gamma) \in SL(2,R)$ we put into correspondence an operator $T(\alpha,\beta,\gamma)$*

$$T(\alpha,\beta,\gamma)\exp\{za^+\}|0\rangle = \exp\Big\{\sum_{\mu=0}^{3} z^\mu a_\mu^+ + \frac{e^{\frac{\alpha}{2}} z_4 + \beta}{\gamma z_4 + (1+\beta\gamma)e^{-\frac{\alpha}{2}}} a_4^+\Big\}|0\rangle . \qquad (1.10)$$

Thus the operator $T(\alpha,\beta,\gamma)$ performs a projective transformation of the coefficient of the operator a_4^+, the matrix of the transformation being $g_\varepsilon(\alpha,\beta,\gamma)$. It is obvious, that both $g_\varepsilon(\alpha,\beta,\gamma)$ from (1.1) and

$$g(\alpha,\beta,\gamma) = \varepsilon g_\varepsilon(\alpha,\beta,\gamma) \in SL(2,R) \qquad (1.11)$$

give the same projective transformation. Therefore we can restrict ourselves to considering the matrices of the type (1.11) only. (As far as β and γ can take both positive and negative values, we shall not write down ε explicitly).

In order to build up the operator $T(\alpha,\beta,\gamma)$ we write down for it the expression for the cases when two of the parameters are equal to zero:

---

*The Lorentz scalar product, which involves first four components of the 5-dimensional vectors is defined as usual:

$$\sum_{\mu=0}^{3} z^\mu z'_\mu = \sum_{\mu,\nu=0}^{3} g^{\mu\nu} z_\mu z'_\nu \qquad g_{00} = -g_{11} = -g_{22} = -g_{33} = 1$$
$$g_{\mu\nu} = 0 \quad \mu \neq \nu$$

$$T(\alpha,0,0)=\exp\{\alpha a_4^+ a_4\} \quad , \tag{1.12a}$$

$$T(0,\beta,0)=\exp\{\beta a_4^+\} \quad , \tag{1.12b}$$

$$T(0,0,\gamma)=\exp\{-\gamma a_4^+ a_4^2\} \quad . \tag{1.12c}$$

Eq. (1.12a) follows from the identity/1/:

$$x^H f(a_i^+)=f(x a_i^+)\, x^{-H}, \tag{1.13}$$

where

$$H=a^+a=\sum_{i,k=0}^{4} G^{ik} a_i^+ a_k \tag{1.14}$$

is the Hamiltonian of the 5 -dimensional oscillator.

Eq. (1.12b) does not need any proof.

The accuracy of eq. (1.12c) can be shown by means of the equality:

$$\exp(\gamma a_4^+ a_4^2)a_4 \exp(-\gamma a_4^+ a_4^2)=\frac{a_4}{1+\gamma a_4} \quad . \tag{1.15}$$

Multiplying from the left by $e^{-\gamma a_4^+ a_4^2}$ and acting on a coherent state we get

$$a_4 \exp\{-\gamma a_4^+ a_4^2\}\exp\{za^+\}|0\rangle=\frac{z_4}{1+\gamma z_4}\exp\{-\gamma a_4^+ a_4^2\}\exp\{za^+\}|0\rangle . \tag{1.16}$$

Here the well known identity [2]

$$f(a_1)\exp\{za^+\} = \exp\{za^+\} f(a_i + z_i) \tag{1.17}$$

has been applied. One can see from (1.16) that $e^{-\gamma a_4^+ a_4^2} e^{za^+}|0\rangle$ is an eigenstate of the annihilation operator a_4 with eigenvalue $\frac{z_4}{1+\gamma z_4}$. Therefore it can be written down in the following way:

$$\exp\{-\gamma a_4^+ a_4^2\}\exp\{za^+\}|0\rangle = c(z_4,\gamma)\exp\left\{-\sum_{\mu=1}^{3} z^\mu a_\mu^+ + \frac{z_4 a_4^+}{1+\gamma z_4}\right\}|0\rangle. \tag{1.18}$$

In order to get the coefficient $c(z_4,\gamma)$ we can make use of the formula

$$\exp(-\gamma a_4^+ a_4^2)\, a_4^+ \exp(\gamma a_4^+ a_4^2) = a_4^+(1-\gamma a_4)^2. \tag{1.19}$$

Comparing the matrix elements of the operators, that stand on both sides of eq. (1.19), evaluated between arbitrary coherent states we get $c(z_4,\gamma) \equiv 1$. Thus eq. (1.12c) has been proved.

If we make use of eq. (1.3) and of the explicit expressions (1.12) for the representations of the three abelian subgroups we get:

$$\begin{aligned} T(\alpha,\beta,\gamma) &= T(\alpha - 2\ln(1+\beta\gamma), 0, 0)\,. \\ &. T(0, \beta(1+\beta\gamma)e^{-\frac{\alpha}{2}}, 0)\, T\left(0, 0, \frac{\gamma e^{\frac{\alpha}{2}}}{1+\beta\gamma}\right). \end{aligned} \tag{1.20}$$

Finally we define the operators $T(g(\beta))$, that correspond to the matrices (1.4). Taking into account (1.5), (1.12a), (1.12b) and (1.12c) we define them by the equality:

$$T(g(\beta)) = exp(\beta a_4^+) exp(\tfrac{1}{\beta} a_4^+ a_4^2) exp(\beta a_4^+). \qquad (1.21)$$

One can show, that

$$T(g(\beta))\, exp(za^+)|0\rangle = exp\left\{-\sum_{\mu=0}^{3} z^\mu a_\mu^+ - \frac{\beta^2}{z_4} a_4^+\right\}|0\rangle. \qquad (1.22)$$

We can do in the same fashion in other cases, too, when the equalities (1.3) are no longer meaningful. It is seen from (1.21) that

$$T^{-1}(g(\beta)) = T(g(-\beta)). \qquad (1.23)$$

On the other hand, it follows from eq. (1.22) that

$$T(g(\beta)) = T(g(-\beta)) \qquad (1.24)$$

and therefore

$$T^{-1}(g(\beta)) = T(g(\beta)). \qquad (1.25)$$

If we denote

$$T_I \equiv T(g(1)) \qquad (1.26)$$

owing to eqs (1.22), (1.21) and (1.25) we can write

$$T_I \; exp\{za^+\}|0\rangle = exp\left\{-\sum_{\mu=0}^{3} z^\mu a_\mu^+ - \frac{1}{z_4} a_4^+\right\}, \qquad (1.27)$$

$$T_I = exp(a_4^+)\, exp(a_4^+ a_4^2)\, exp(a_4^+) =$$

$$= exp(-a_4^+)\, exp(-a_4^+ a_4^2)\, exp(-a_4^+) \qquad (1.28)$$

$$T_I^2 = 1.$$

In this manner, with the aid of the operators (1.10) and (1.21), we get a representation of the group $SL(2,R)$ in the space of the coherent states of the operator a_4^+ only. To get a representation of this group in the whole space of coherent states of the 5-dimensional oscillator we first introduce the operator

$$L \equiv :exp\left\{\sum_{\mu=0}^{3} a_\mu^+ a^\mu (1-a_4)\right\}: =$$
$$= \sum_{n=0}^{\infty} (1-a_4)^n \sum_{\sum_{\mu=0}^{3} n_\mu = n} \prod_{\mu=0}^{3} \frac{(a_\mu^+)^{n_\mu}}{\sqrt{n_\mu!}} \prod_{\mu=0}^{3} \frac{(a^\mu)^{n_\mu}}{\sqrt{n_\mu!}}. \qquad (1.29)$$

Using the formula (1.17) we can show, that

$$L\, exp\{za^+\}|0\rangle = exp\left\{-z_4 \sum_{\mu=0}^{3} z^\mu a_\mu^+ + z_4 a_4^+\right\}|0\rangle. \qquad (1.30)$$

Then in view of eq. (1.27) we have

$$T_I L T_I\, exp\{za^+\}|0\rangle = exp\left\{z_4 \sum_{\mu=0}^{3} z^\mu a_\mu^+ + z_4 a_4^+\right\}|0\rangle. \qquad (1.31).$$

and hence

$$\mathcal{L}^{-1} = T_I : exp\{\sum_{\mu=0}^{3} a_\mu^+ a^\mu (1+a_4)\} : T_I \cdot \qquad (1.32)$$

Here one has to take into account, that for $z_4 = 0$ the operator $\mathcal{L}^{-1}$ is undetermined.

The representation we want to define is given by the operator

$$\Theta(\alpha,\beta,\gamma) \equiv \mathcal{L}\, T(\alpha,\beta,\gamma)\, \mathcal{L}^{-1}. \qquad (1.33)$$

Using eqs (1.26),(1.19),(1.30) and (1.32) one can verify, that the action of this operator on a coherent state (with $z_4 \neq 0$) is expressed by the formula

$$\Theta(\alpha,\beta,\gamma)\, exp\{za^+\}|0\rangle = exp\{\frac{1}{z_4}\cdot\frac{e^{\frac{\alpha}{2}}z_4 + \beta}{\gamma z_4 + (1+\beta\gamma)e^{-\frac{\alpha}{2}}} za^+\}|0\rangle . \qquad (1.34)$$

The comparison of eqs (1.34) and (1.10) shows, that for $z_\mu = 0$ (μ = 0,1,2,3) both representations coincide. Therefore the representation (1.34) is not a direct product of five representations of the type (1.10).

It is obvious, that the representation (1.10) does not conserve the vacuum state. On the other hand, eq. (1.33) does not define the action of the operator $\Theta(\alpha,\beta,\gamma)$ on the vacuum state. This fact permits to impose an additional condition

$$\Theta(\alpha,\beta,\gamma)|0\rangle = T(\alpha,\beta,\gamma)|0\rangle . \tag{1.35}$$

One can prove the following equations:

$$\begin{aligned} \langle 0| T(\alpha,\beta,\gamma) &= \langle 0| \\ \langle 0| \Theta(\alpha,\beta,\gamma) &= \langle 0| \\ \langle 0| L &= \langle 0| . \end{aligned} \tag{1.36}$$

Obviously, in the space of states $\langle 0| e^{\bar{z}a}$ the hermitian conjugate representation $\Theta^{+}(\alpha,\beta,\gamma)$ holds for which the equality:

$$\Theta^{+}(\alpha,\beta,\gamma)|0\rangle = |0\rangle \tag{1.37}$$

is satisfied.

Finaly we note an important peculiarity of the representation (1.33). Although the operator $\Theta(\alpha,\beta,\gamma)$ transforms z_μ and z_4 , it does not depend on the numbers. To what extent the indicated representation is characteristic of the coherent states can been seen from the following.

Instead of eq. (1.10) we could define the representation of the group $SL(2,R)$ with the aid of the equality:

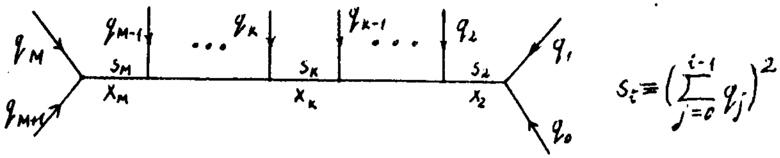
$$T(\alpha,\beta,\gamma)\, f(za^{+})|0\rangle = f\Big(-\sum_{\mu=0}^{3} z^{\mu} a_{\mu}^{+} + \frac{e^{\frac{\alpha}{2}} z_4 + \beta}{\gamma z_4 + (1+\beta\gamma) e^{-\frac{\alpha}{2}}}\, a_4^{+}\Big)|0\rangle ,$$

where $f(za^+)$ is an arbitrary function of the variable za^+. However it is easy to see that the requirement for the operators $T(\alpha,\beta,\gamma)$ to be independent of z_i extracts from the whole set of states $f(za^+)|0\rangle$ only coherent states

$f(za^+) \equiv c\, exp(za^+)$.

2.

In[1,2] it has been shown, that the dual multiperipherical diagram (Fig. 2.1):



F i g. 2.1.

in terms of the coherent states can be represented in the form:

$$B_{M+2} = \langle \Psi_{M-K+2} | \frac{D}{H - \alpha(s_K)} | \Psi_K \rangle, \qquad (2.1)$$

where*

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*The first index lebels the vectors, the second one their components.

$$H=\sum_{i=1}^{M} a_i^+ a_i = \sum_{i=1}^{M} \sum_{m,n=0}^{4} G^{mn} a_{im}^+ a_{in} \qquad (2.2)$$

is the Hamiltonian of the whole system of M oscillators and the operator D is given by

$$D = \prod_{i=1}^{K-1} \prod_{n=K}^{M} \left(1 - a_{i4} a_{n4}^+\right)^{2\sum_{\mu=0}^{3} \frac{a_{i\mu} a_{n\mu}^+}{a_{i4} a_{n4}^+} - \rho_{in}}$$

$$\rho_{in} = (1-\alpha(0))\,\delta_{i+1,n}\,. \qquad (2.3)$$

Eq. (2.1) shows, that B_{M+2} is a matrix element of the operator

$$\frac{D}{H-\alpha(s_K)} = D\int_0^1 x_K^{\,H-\alpha(s_K)-1}\, dx_K \qquad (2.4)$$

evaluated between the coherent states $\langle \Psi_{M-K+2}|$ and $|\Psi_K\rangle$ which are defined as follows:

$$|\Psi_K\rangle = \int d\varphi_{K+1}(\tau,q)\, \exp\Big\{\sum_{i=1}^{K-1} Q_i a_i^+ \frac{\tau_{K-1}}{\tau_i}\Big\}|0\rangle$$

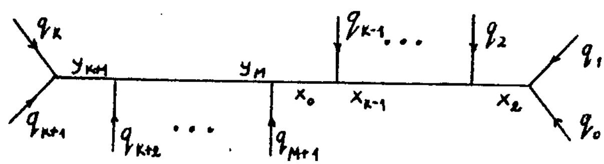
$$\langle \Psi_{M-K+2}| = \int d\varphi_{M-K+3}(\sigma,q)\langle 0|\exp\Big\{\sum_{i=K}^{M} Q_i a_i \sigma_i\Big\} \qquad (2.5)$$

$$\tau_i \equiv \prod_{j=1}^{i} x_j \qquad \sigma_i \equiv \prod_{j=K+1}^{i} x_j\,.$$

Here $Q_i \equiv (-i\sqrt{\alpha'}q_\mu, 1)$ are 5-dimensional vectors and $d\varphi_N(\sigma,q)$ denotes the integrand of the Bardakci-Ruegg formula:

$$d\varphi_{M-K+3}(\sigma,q) = \prod_{i=K+1}^{M} d\sigma_i\, \sigma_i^{-2\alpha' q_i \sum_{j=i+1}^{M+1} q_j - \alpha' m^2 - 1} \prod_{i=K}^{M-1} \prod_{n=i+1}^{M} \left(1-\frac{\sigma_n}{\sigma_i}\right)^{-2\alpha' q_i q_n - \rho_{in}} \qquad (2.6)$$

According to duality both diagrams of Fig. 2.1 and Fig. 2.2.



F i g. 2.2.

are equivalent. The amplitude B^D_{M+2} (Fig. 2.2.) is obtained from the integrand of the expression (2.1) by the following change of variables

$$\sigma_i = \frac{1-\frac{\mu_M}{\mu_{i-1}}}{1-x_0\frac{\mu_M}{\mu_{i-1}}} \equiv \sigma(\mu) \tag{2.7}$$

$$x_0 \equiv x_K \qquad \mu_i = \prod_{j=K+1}^{i} y_j \, .$$

We shall show, that the amplitude B^D_{M+2},too,can be represented as a matrix element of some operator between the appropriate coherent states. The dual transformation (2.7) affects only those variables, which stand to the left of the variable $x_0 \equiv x_K$. Thus the state $|\psi_K\rangle$ remains unchanged and $\langle\psi_{M-K+2}|$ turns into:

$$\langle \Psi'_{M-K+2} | \int d\varphi_{M-K+3}(\sigma(\mu), q) \langle 0 | \exp\left\{ \sum_{i=K}^{M} a_i a_i \frac{1-\frac{\mu_M}{\mu_i}}{1-x_0 \frac{\mu_M}{\mu_i}} \right\}. \quad (2.8)$$

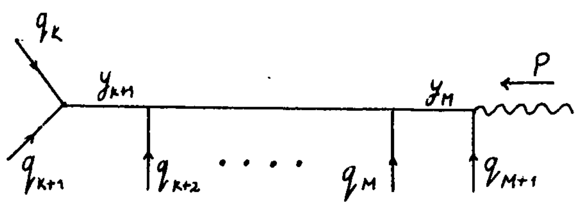
If we denote $P = \sum_{i=0}^{K-1} q_i$, then

$$d\varphi_{M-K+3}(\sigma(\mu), q) = d\varphi^D_{M-K+3}(\mu, q)(1-x_0)^{-\alpha((P+q_{M+1})^2)} \cdot$$

$$\cdot \prod_{i=K+1}^{M} \left(1 - x_0 \frac{\mu_M}{\mu_{i-1}}\right)^{-2\alpha' P_0 q_i - \varrho_{iP}} \quad (1.9)$$

$$\varrho_{MP} = -\alpha(q^2_{M+1}) \qquad \varrho_{iP} = 0 \quad i \neq M$$

where $d\varphi^D_{M-K+3}(\mu, q)$ is the integrand of the state (Fig. 2.3):



F i g. 2.3.

We remind, that this state has the form:

$$\langle \Psi^D_{M-K+2} | = \int d\varphi^D_{M-K+3}(\mu, q) \langle 0 | \exp\left\{ \sum_{i=K+1}^{M+1} a_i a_i \frac{\mu_M}{\mu_{i-1}} \right\}, \quad (2.10)$$

where

$$d\varphi^{D}_{M-k+3}(\mu,q)=\prod_{i=k+1}^{M} d\mu_i \, \mu_M^{-\alpha'((P+q_{M+1})^2)-1} \prod_{i=k+1}^{M-1} \mu_i^{-2\alpha' q_{i+1}\left(\sum_{j=i+2}^{M+1} q_j+P\right)-\alpha' m^2-1}$$

$$\prod_{i=k}^{M-1} \prod_{n=i+1}^{M} \left(1-\frac{\mu_n}{\mu_i}\right)^{-2\alpha' q_{i+1} q_{n+1}-\rho_{in}} \qquad (2.11)$$

Let us find out the connection between the states (2.8) and (2.10). It is easy to see, that a projective transformation with the matrix

$$g=\begin{pmatrix} \frac{1}{\sqrt{1-x_0}} & \frac{-1}{\sqrt{1-x_0}} \\ \frac{-x_0}{\sqrt{1-x_0}} & \frac{1}{\sqrt{1-x_0}} \end{pmatrix} = g\left(-\ln(1-x_0), \frac{-1}{\sqrt{1-x_0}}, \frac{-x_0}{\sqrt{1-x_0}}\right) \qquad (2.12)$$

transforms the coefficient $\frac{\mu_M}{\mu_i}$ from (2.10) exactly into the coefficient $\frac{1-\frac{\mu_M}{\mu_i}}{1-x_0\frac{\mu_M}{\mu_i}}$ from eq. (2.8). Therefore, taking into account eqs (1.35) and (1.13) we get:

$$\langle 0|\exp\left\{\sum_{i=k+1}^{M+1} Q_i a_i \frac{1-\frac{\mu_M}{\mu_{i-1}}}{1-x_0\frac{\mu_M}{\mu_{i-1}}}\right\} = \langle 0|\exp\left\{\sum_{i=k+1}^{M+1} Q_i a_i \frac{\mu_M}{\mu_{i-1}}\right\} \theta^{+}(x_0)(-1)^{\sum_{i=k+1}^{M+1} H_i} \qquad (2.13)$$

where by definition

$$\theta^{+}(x_0) \equiv \prod_{i=k+1}^{M+1} \theta_i^{+}\left(-\ln(1-x_0), \frac{-1}{\sqrt{1-x_0}}, \frac{-x_0}{\sqrt{1-x_0}}\right) \qquad (2.14)$$

and the index "i" means that a given operator acts in the space of the i-th oscillator. The k^{-th} oscillator is absent in the state $\langle \Psi^{D}_{M-k+2}|$, while it appears in $\langle \Psi'_{M-k+2}|$.
Introducing the operator:

$$\Lambda_k^+(P) = (L^+)^{-1} \exp\Big\{-i\sqrt{\alpha'}\sum_{\mu=0}^{3} P^\mu a_{k\mu} + \sum_{i=k+1}^{M+1}\sum_{\mu=0}^{3} a_{i\mu}^+ a_k^\mu + a_{k4}\Big\} L^+$$

$$\Big(L^+ \equiv \prod_{i=k+1}^{M+1} L_i^+\Big) \qquad (2.15)$$

we get

$$\langle 0|\exp\Big\{\sum_{i=k}^{M} Q_i a_i \frac{1-\frac{\mu_M}{\mu_{i-1}}}{1-x_0\frac{\mu_M}{\mu_{i-1}}}\Big\} = \langle 0|\exp\Big\{\sum_{i=k+1}^{M+1} Q_i a_i \frac{\mu_M}{\mu_{i-1}}\Big\}\Lambda_k^+(P)\Theta^+(x_0)(-1)^{\sum_{i=k+1}^{M+1} H_i}. \qquad (2.16)$$

Thus the connection between the states (2.8) and (2.10) has the form

$$\langle \Psi'_{M-k+2}| = \langle \Psi^D_{M-k+2}|\prod_{i=k+1}^{M+1}(1-x_0 a_{i4})^{-2\alpha'\sum_{\mu=0}^{3} P^\mu \frac{a_{i\mu}^+}{a_{i4}^+} - \rho_{iP}} \cdot$$

$$\cdot \Lambda_k^+(P)\,\Theta_k^+(x_0)\,(-1)^{\sum_{i=k+1}^{M+1} H_i} \qquad (2.17)$$

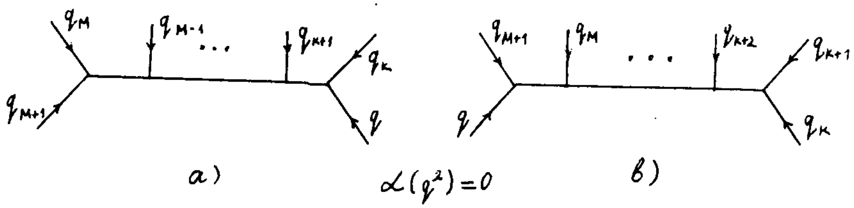
$$\rho_{M+1\,P} = 1+\alpha(0)+\alpha'(P^2+q_{M+1}^2).$$

The parameter x_0 in eq. (2.17) can be identified with the integration variable x_k from eq. (2.4). The presence of x_0 in eq. (2.17) leads to a re-definition of the propagator in eq. (2.4). In order to find out the new propagator we rewrite eq. (2.17) in the form:

$$\langle \Psi'_{M-k+2}| = \langle \Psi^D_{M-k+2}|\Lambda_k^+(P)\Theta^+(x_0)(-1)^{\sum_{i=k+1}^{M+1} H_i}(1-x_0)^{\alpha' P^2} \cdot$$

$$(1-x_0 a_{k4}^+)^{2i\sqrt{\alpha'}\sum_{\mu=0}^{3} P^\mu \frac{a_{k\mu}^+}{a_{k4}^+}} \prod_{i=k+1}^{M+1}(1-x_0 a_{i4}^+)^{2i\sqrt{\alpha'}\sum_{\mu=0}^{3} P^\mu \frac{a_{i\mu}^+}{a_{i4}^+} + \rho_{iP}}. \qquad (2.18)$$

The dual transformation (2.7) for the case $x_0 = 0$ corresponds to the transition from the amplitude(Fig. 2.4a)to the amplitude(Fig. 2.4b)



F i g. 2.4.

We call the dual transformation of variables (2.7) with $x_0 = 0$ simple. It follows from (2.9), that for this transformation holds the equality:

$$d\varphi_{M-K+3}(\sigma(\mu), q) = d\varphi^{D}_{M-K+3}(\mu, q). \qquad (2.19)$$

The operator $\Theta^{+}(0)$, that performs the simple dual transformation corresponds to the matrix

$$h = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}. \qquad (2.20)$$

Every element $g \in SL(2,R)$ of the type (2.11) can be represented in the form:

$$g = g_1 h \qquad g_1 = \begin{pmatrix} \frac{1}{\sqrt{1-x_0}} & 0 \\ \frac{-x_0}{\sqrt{1-x_0}} & \sqrt{1-x_0} \end{pmatrix}. \qquad (2.21)$$

Therefore we can write down

$$\theta^{+}(x_0) = \theta^{+}(0)\,\theta_1^{+}(x_0)$$

$$\theta_1^{+}(x_0) \equiv \prod_{i=k+1}^{M+1} \Theta_i^{+}\left(-\ln(1-x_0), 0, \frac{-x_0}{\sqrt{1-x_0}}\right). \qquad (2.22)$$

If we denote

$$\Theta_{1S}(x_0) \equiv (-1)^{\sum_{i=k+1}^{M+1} H_i}\, \Theta_1^{+}(x_0)\,(-1)^{\sum_{i=k+1}^{M+1} H_i} \qquad (2.23)$$

eq. (2.19) finally takes the form

$$\langle \Psi'_{M-k+2}| = \langle \Psi^{D}_{M-k+2}|\Lambda_k^{+}(P)\,\Theta^{+}(0)\,(-1)^{\sum_{i=k+1}^{M+1} H_i}\,(1-x_0)^{\alpha' P^2}\,.$$

$$\Theta_{1S}^{+}(x_0).\left(1-x_0 a_{k\nu}^{+}\right)^{2i\sqrt{\alpha'}\sum_{\mu=0}^{3} P^{\mu}\frac{a_{k\mu}^{+}}{a_{k\nu}^{+}}} \prod_{i=k+1}^{M+1}\left(1-x_0 a_{i\nu}^{+}\right)^{2i\sqrt{\alpha'}\sum_{\mu=0}^{3} P^{\mu}\frac{a_{i\mu}^{+}}{a_{i\nu}^{+}}+\rho_{iP}}. \qquad (2.24)$$

Hence the transition from $\langle \Psi^{D}_{M-k+2}|$ to $\langle \Psi'_{M-k+2}|$ is performed by means of two operators. The first of them

$$\Lambda_k^{+}(P)\,\Theta^{+}(0)\,(-1)^{\sum_{i=k+1}^{M+1} H_i} \qquad (2.25)$$

performs a simple dual change of variables. The second one

$$(1-x_0)^{\alpha' P^2}\,\Theta_{1S}^{+}(x_0)\left(1-x_0 a_{k\nu}^{+}\right)^{2i\sqrt{\alpha'}\sum_{\mu=0}^{3} P^{\mu}\frac{a_{k\mu}^{+}}{a_{k\nu}^{+}}} \prod_{i=k+1}^{M}\left(1-x_0 a_{i\nu}^{+}\right)^{2i\sqrt{\alpha'}\sum_{\mu=0}^{3} P^{\mu}\frac{a_{i\mu}^{+}}{a_{i\nu}^{+}}+\rho_{iP}} \qquad (2.26)$$

depends on X_0 and should be considered together with the propagator (2.4). Taking into account eqs (2.24), (2.1), (2.3) and (2.4) we see, that all the x_0 dependence is concentrated in the term

$$d x_0 (1-x_0)^{\alpha' p^2} \theta^+_{1s}(x_0)(1-a^+_{k4})^{2i\sqrt{\alpha'}\sum_{\mu=0}^{3} p^\mu \frac{a^+_{k\mu}}{a^+_{k4}}} \prod_{i=k+1}^{M} (1-x_0 a^+_{i4})^{2i\sqrt{\alpha'}\sum_{\mu=0}^{3} p^\mu \frac{a^+_{i\mu}}{a^+_{i4}} + p_i p}$$

$$x_0^{-\alpha(p^2)-1+H} \prod_{i=1}^{k-1} \prod_{n=k}^{M} \left(1-a^+_{n4} a_{i4}\right)^{2\sum_{\mu=0}^{3} \frac{a^+_{n\mu} a_i^\mu}{a^+_{n4} a_{i4}} - p_{in}} . \tag{2.27}$$

If we pick the multiplier $(1-a^+_{k4} a_{k-14})^{-p_{k\,k-1}}$ out and move it further to the left we get

$$\frac{d x_0}{(1-x_0)^2} \cdot \left(\frac{x_0}{1-x_0}\right)^{-\alpha(p^2)-1+H} \theta^+_{2s} \prod_{n=k}^{M} (1-a^+_{n4} a_{k-14})^{2i\sqrt{\alpha'}\sum_{\mu=0}^{3} \frac{p^\mu a^+_{n\mu}}{a_{k-14} a^+_{n4}}}$$

$$\prod_{i=1}^{k-1} \prod_{n=k}^{M} \left(1-a^+_{n4} a_{i4}\right)^{2\sum_{\mu=0}^{3} \frac{a^+_{n\mu} a_i^\mu}{a^+_{n4} a_{i4}} - p_{in}} . \tag{2.28}$$

Here $p_{k-1\,k}=0$ and θ^+_{2s} denotes the operator:

$$\theta^+_{2s} = exp\left\{-\sum_{i=k+1}^{M+1} a^+_{i4} a^+_i a_i\right\}. \tag{2.29}$$

We mention two significant facts:

1) If one introduces the variable

$$y_0 = \frac{x_0}{1-x_0} \tag{2.30}$$

then

$$\frac{d x_0}{(1-x_0)^2} \cdot \left(\frac{x_0}{1-x_0}\right)^{-\alpha(p^2)-1+H} = d y_0 \, y_0^{-\alpha(p^2)-1+H} \tag{2.31}$$

and y_0 varies from 0 to ∞.

2) To the operator D a term is added, that shifts $a_{k-1\,\mu}$ to the vector P_μ. Instead of this we can shift $q_{k-1\,\mu}$ by $-P_\mu$. Thus eq. (2.28) takes the form:

$$dy_0\, y_0^{-\alpha(P^2)-1+H}\, \Theta^+_{\lambda s} \prod_{i=1}^{k-1} \prod_{n=k}^{M} \left(1 - a^+_{n4} a_{i4}\right)^{2\sum\limits_{\mu=0}^{3} \frac{a^+_{n\mu} a_i^{\mu}}{a^+_{n4} a_{i4}} - \rho_{in}} . \qquad (2.32)$$

The constants ρ_{in} are obtained as before, but one has to take into account, that now they refer to the diagram of Fig. 2.2, where the neighbourhood of the external particles does not correspond to their numeration. *E.g.*, q_{k-1} and q_k are no longer neighbours and therefore $\rho_{k-1\,k} = \rho_{k-1\,k+1} = \rho_{k-2\,k} = 0$. Simultaneously q_{k-1} and q_{M+1} are separated by one propagator only and hence

$$\rho_{M+1\;k-1} = 1 - \alpha(0)$$

$$\rho_{M\;k-1} = \rho_{MP} = -\alpha\left(q^2_{M+1}\right) . \qquad (2.33)$$

All these equalities are automatically carried out in eq. (2.32).

In order to get the final form of the above considered expression we move the operator $\Theta^+_{\lambda s}$ to the right. (The vacuum state is invariant with respect to $\Theta^+_{\lambda s}$). Thus we get

$$\theta^+_{2S}\prod_{i=1}^{k-1}\prod_{n=k}^{M}\left(1-a^+_{n4}a_{i4}\right)^{2\sum\limits_{\mu=0}^{3}\frac{a^+_{n\mu}a_i^{\mu}}{a^+_{n4}a_{i4}}\cdot\rho_{in}} \exp\left\{-i\sqrt{\alpha'}\sum_{\mu=0}^{3}P^{\mu}a^+_{k-1,\mu}\right\}\exp\left\{\sum_{i=1}^{k-1}Q_i a_i^+\frac{\tau_{k-1}}{\tau_i}\right\}|0\rangle=$$

$$=\prod_{i=1}^{k-1}\prod_{n=k}^{M+1}\left(1-a^+_{n4}a_{i4}\right)^{2\sum\limits_{\mu=0}^{3}\frac{a^+_{n\mu}a_i^{\mu}}{a^+_{n4}a_{i4}}-\rho_{in}}\prod_{n=k}^{M+1}\left(1+a^+_{n4}\right)^{-2\sqrt{\alpha'}i\sum\limits_{\mu=0}^{3}q_0^{\mu}\frac{a^+_{n\mu}}{a^+_{n4}}}\cdot$$

$$\exp\left\{\sum_{i=1}^{k-1}\left(i\sqrt{\alpha'}\sum_{\mu=0}^{3}a^+_{i\mu}q_i^{\mu}+a^+_{i4}\right)\right\}\exp\left\{\sum_{i=1}^{k-1}Q_i a_i^+\frac{\tau_{k-1}}{\tau_i}\right\}|0\rangle. \tag{2.34}$$

Further by analogy with eq. (2.16) using the operator

$$\Lambda_0(P)=L\exp\left\{i\sqrt{\alpha'}\sum_{\mu=0}^{3}P^{\mu}a^+_{0\mu}+\sum_{i=1}^{k-1}\sum_{\mu=0}^{3}a_i^{\mu}a^+_{0\mu}+a^+_{04}\right\}L^{-1} \tag{2.35}$$

one can introduce the oscillator $a^+_{0\mu}$, which permits to unite all operational terms. Thus, the operator $\mathcal{D}$ takes its final form:

$$\mathcal{D}=\prod_{i=0}^{k-1}\prod_{n=k}^{M+1}\left(1-a^+_{n4}a_{i4}\right)^{2\sum\limits_{\mu=0}^{3}\frac{a^+_{n\mu}a_i^{\mu}}{a^+_{n4}a_{i4}}-\rho_{in}}. \tag{2.36}$$

Finally the action of the first exponential in the right-hand side of eq. (2.34) can be replaced by the operator $(-1)^{\sum_{i=1}^{k-1}H_i}\theta(0)$ where

$$\theta(0)=\prod_{i=1}^{k-1}\theta_i(0,-1,0). \tag{2.37}$$

Thus the expression for B^{D}_{M+2} takes a symmetrical form/5/:

$$B^{D}_{M+2}=\langle\Psi^{D}_{M-k+2}|\Lambda^+_k(P)\theta^+(0)(-1)^{H_k}\int_0^{\infty}dy_0\,y_0^{-\alpha(P^2)-1+H}\cdot$$

$$\mathcal{D}\cdot(-1)^{\sum_{i=1}^{k-1}H_i}\theta(0)\Lambda_0(P)|\Psi_k\rangle. \tag{2.38}$$

In order to find the region of convergence of the integral over y_0 one has first to evaluate the matrix element (2.38). Then this integral takes the form:

$$\int_0^\infty dy_0\, y_0^{-\alpha(p^2)-1} \prod_{i=0}^{k-2}\prod_{n=k}^{M}\left[1+y_0\left(1-\frac{\mu_M}{\mu_{n-1}}\right)\left(1-\frac{\tau_{k-1}}{\tau_i}\right)\right]^{-2\alpha' q_i q_n} . \qquad (2.39)$$

The domain in which the integral (2.39) converges is determined by two inequalities:

$$\alpha(p^2) < 0$$

$$\alpha\left((p+q_{M+1}-q_{k-1})^2\right) < 0 . \qquad (2.40)$$

Outside this domain the integral is obtained by means of analitycal continuation.

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APPENDIX

We will note two significant properties of the operator $\mathcal{L}$ (1.29). First of them is expressed in the following equality:

$$:e^{\sum_{\mu=0}^{3} a_\mu^+ a^\mu (1-\alpha)}::e^{\sum_{\mu=0}^{3} a_\mu^+ a^\mu (1-\beta)}:=:e^{\sum_{\mu=0}^{3} a_\mu^+ a^\mu (1-\alpha\beta)}: \quad (A.1)$$

Here α and β commute with each other and also with a_μ and a_μ^+ $(\mu = 0,1,2,3)$.

In order to prove (A1) we rewrite its left-hand side in the following manner

$$\prod_{\mu=0}^{3} \mathcal{L}_\mu \equiv \prod_{\mu=0}^{3} :e^{a_\mu^+ a^\mu (1-\alpha)}::e^{a_\mu^+ a^\mu (1-\beta)}:$$

Next we can expand $\mathcal{L}_\mu$ in power series of a_μ^+ and a^μ and go back to the normal form of the operators;

$$\mathcal{L}_\mu =:\sum_{m,n,k} \frac{(-1)^k (1-\alpha)^n (1-\beta)^m}{k!(m-k)!(n-k)!} (a_\mu^+ a^\mu)^{m+n-k}:=:e^{a_\mu^+ a^\mu (1-\alpha\beta)}:$$

Multiplying all four $\mathcal{L}_\mu$ we get the r.h.s. of the equality (A.1).

The second property of the operator $\mathcal{L}$ is

$$\mathcal{L}\, e^{-\gamma a_4^+ a_4^2} \mathcal{L}^{-1} = e^{-\gamma a^+ a\, a_4} \quad (A.2)$$

(a^+a – the five-dimensional scallar product). Using eq.(1.10), (1.12) and (1.30) we get

$$\mathcal{L}\, e^{-\gamma a_4^+ a_4^2} \mathcal{L}^{-1} e^{z a^+} |0\rangle = e^{\frac{1}{1+\gamma z_4} z a^+} |0\rangle,$$

(A.3)

To find the action of the r.h.s. of eq. (A.2) on a coherent state we use the formula

$$e^{\gamma a^{+} a a_4} a_i e^{-\gamma a^{+} a a_4} = \frac{a_i}{1-\gamma a_4} \qquad i=0,1,2,3,4 \tag{A.4}$$

Applying the same procedure as in the case of eq. (1.15) and (1.16) we get

$$e^{-\gamma a^{+} a a_4} e^{z a^{+}} |0\rangle = C(z_4,\gamma) e^{\frac{z a^{+}}{1+\gamma z_4}} . \tag{A.5}$$

The value of the coefficient $C(z_4,\gamma)$ is obtained by means of the equality

$$e^{\gamma a^{+} a a_4} a_\nu^{+} e^{-\gamma a^{+} a a_4} = a_\nu^{+} (1+\gamma a_4) . \tag{A.6}$$

As before, we get

$$C(z_4,\gamma) \equiv 1 .$$

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