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ЛАБОРАТОРИЯ ТЕОРЕТИЧЕСКОЙ ФИЗИКИ

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# BOUNDS FOR $K_{l3}$ DECAY PARAMETERS

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Ограничения для параметров распада $K_{\ell 3}$

Определен экстремум интеграла $\int_1^\infty f(x)|F(x)|^2 dx$, если значения функции $F(x)$ известны в трех заданных точках. Результат применяется к анализу $K_{\ell 3}$ распада, причем функция $F(x)$ выражается через $f_+(q^2)+f_-(q^2)q^2/(m_k^2-m_\pi^2)$, где $f_\pm(q^2)$ есть формфакторы, описывающие матричный элемент от дивергенции несохраняющего странность векторного тока.

**Препринт Объединенного института ядерных исследований.**
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Bounds for $K_{\ell 3}$ Decay Parameters

The extremum of the integral $\int_1^\infty f(x)|F(x)|^2 dx$, is determined for the case when the values taken by the function $F(x)$ at three points are known. The result is applied to the $K_{\ell 3}$ problem, the function $F(x)$ being replaced by the form factor $f_+(q^2)+f_-(q^2)q^2/(m_k^2-m_\pi^2)$ of the divergence of the strangeness changing vector current.

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## I. Introduction

The present experimental data concerning the parameters $f_+(0)$, $\lambda_+$ and $\xi$ /1,2/ from the $K_{\ell 3}$ decay problem are still uncertain, large experimental errors being reported for the parameters $\lambda_+$ and $\xi$. For this reason, it would be useful to restrict, in some way, through theoretical consideration, the possible values of these parameters.

Firstly, Li and Pagels /3/, using the fact that the Fourier transform $\Delta(q^2)$ of the vacuum expectation value $<0|D^+(x)D^-(0)+|0>$ is known /4/ at the point $q^2=0$ obtained that $|12.3\lambda_+ + \xi| \leq 0.29$. Later Okubo /5/ and Li and Pagels /6/, using the method of Meiman /7/, gave a bound for the parameter $f_+(0)$. The method was improved by Okubo /8/ who obtained a lower bound for $\Delta(0)$ in terms of the form factor $f_+(q^2) + f_-(q^2)\, q^2/(m_K^2 - m_\pi^2)$ and of an arbitrary number of its derivatives all taken at zero momentum transfer.

Recently Radescu /9/, by using a method based on a direct application of the principle of maximum for holomorphic functions, has obtained new bounds and rederived the results from papers /5,6/.

The Gallan-Treiman relation /10/ gives for zero mass mesons the form factor of divergence of strangeness changing vector current at $q^2 = m_K^2$ momentum transfer. For physical $\pi$ meson this formula is

$$f_+(m_K^2) + f_-(m_K^2) = \frac{f_K}{f_\pi} + \text{on mass shell corrections,} \qquad (I)$$

where the unknown on shell corrections are expected to be of order $m_\pi^2 / m_K^2$. Since the experimental errors assigned to $\lambda_+$ and $\xi$ and the errors involved in the evaluation of $\Delta(0)$ /8/ are larger than these unknown on shell corrections, it would be useful to take into account eq. (I) in a bound formula.

The purpose of this paper is twofold, firstly, to rederive by using the method of Meiman /7/, all known bounds /5,6,9/, secondly to improve these bounds by taking into account the Gallan-Treiman formula in the form of eq. (I). Section 2 deals with the form factor for $K_{\ell 3}$ decay and with the vacuum expectation value giving the basic inequality from which all the bounds are obtained. In section 3 we obtain a formula for the bound when the form factor is known at three points. In section 4 the comparison with the experimental data and with the result from papers /5,6,9/ is made.

## 2. Form Factors and Vacuum Expectation Value

The Cabibbo theory describes leptonic decays of $K$ mesons in terms of the form factors $f_{\pm}(q^2)$ defined by the matrix elements of strangeness changing vector currents

$$<\pi^+(p)|\,V_\mu^+(0)\,|\bar{K}^0(k)> = -\frac{1}{\sqrt{2}}\,[f_+(q^2)(k_\mu+p_\mu)+f_-(q^2)(k_\mu-p_\mu)]\,, \quad (2)$$

where $k$ and $p$ represent the momentum of $\bar{K}^0$, respectively, $\pi^+$ meson and $q^2=(k-p)^2$ is the square of momentum transfer.

The form factor $(m_K^2-m_\pi^2)\,f(q^2)$ of the divergence of vector currents $D^{\pm}(x) = \partial_\mu V_\mu^{\pm}(x)$ used in this paper is expressed in terms of the form factors $f_{\pm}(q^2)$ as

$$f(q^2) = f_+(q^2) + \frac{q^2}{m_K^2-m_\pi^2}\,f_-(q^2)\,. \quad (3)$$

The experimental data concerning these form factors are presented as the first two terms in their series expansion around $q^2=0$ value

$$f_{\pm}(q^2) = f_{\pm}(0)\,[\,1 + \frac{q^2}{m_\pi^2}\,\lambda_{\pm}] \quad (4)$$

$$f(q^2) = f_+(0)\,[1 + \frac{q^2}{m_K^2-m_\pi^2}\,(12.3\lambda_+ + \xi)\,]\,, \quad (5)$$

where $\xi = f_-(0)/f_+(0)$.

In order to obtain bounds for these parameters it is necessary to introduce the vacuum expectation value of the time ordered product $(D^+(x)\;D^-(0))_+$ whose Fourier transform is $\Delta(q^2)$

$$\Delta(q^2) = -i\int e^{iqx}<0|\,(D^+(x)\;D^-(0))_+|\,0>d^4x = \int\frac{\rho(q'^2)}{q'^2-q^2}\,dq'^2 \quad (6)$$

with the spectral function $\rho(q^2)$ given by

$$\rho(q^2) = (2\pi)^3 \sum_n \langle 0 | D^+(0) | n \rangle \langle n | D^-(0) | 0 \rangle \delta(q - p_n) . \tag{7}$$

If we retain in eq. (6) only $\pi, K$ system as intermediate state $| n \rangle$ the spectral function $\rho^{\pi K}(q^2)$ is given by

$$\rho^{\pi K}(q^2) = \frac{3}{64\pi^2} \frac{1}{q^2} [q^2 - (m_K + m_\pi)^2]^{1/2} [q^2 - (m_K - m_\pi)^2]^{1/2} [(m_K^2 - m_\pi^2) f(q^2)]^2 , \tag{8}$$

The contribution of each intermediate state in eq. (7) being positive, it results

$$\rho(q^2) > \rho^{\pi K}(q^2) . \tag{9}$$

Combining this result with eqs (6) and (8) we find the inequality

$$\frac{\Delta(0)}{(m_K^2 - m_\pi^2)^2} > \frac{3}{64\pi^2} \int_{(m_K+m_\pi)^2}^{\infty} \frac{dq}{q^4} [q^2 - (m_K + m_\pi)^2]^{1/2} [q^2 - (m_K - m_\pi)^2]^{1/2} | f(q^2) |^2 . \tag{10}$$

It is convenient to rewrite it, by a change of variable, in the form

$$\frac{\Delta(0)}{(m_K^2 - m_\pi^2)^2} > \frac{3}{64\pi^2} \int_1^{\infty} \frac{dx}{x^2} (x-1)^{1/2} (x - \omega^2)^{1/2} | F(x) |^2 , \tag{11}$$

where

$$\omega = \frac{m_K - m_\pi}{m_K + m_\pi} = 0.57 , \tag{12}$$

$$F(x) = f((m_K + m_\pi)^2 x) . \tag{13}$$

Mathur and Okubo /4/, using the symmetry breaking Hamiltonian, introduced by Gell-Mann, Oakes and Renner /11/ obtained a value for $\Delta(q^2)$ at $q^2 = 0$ momentum transfer

$$\Delta^{1/2}(0) = 1.01\, m_\pi f_\pi \tag{14}$$

but other evaluations [8] give larger values. For this reason it is convenient to write

$$\Delta^{1/2}(0) = 1.01\, m_\pi f_\pi M \ , \tag{15}$$

where $M$ shows the departure from the value (14). All evaluations from paper [8], give $M$ smaller than 2.3.

The present experimental data concerning the decay parameters are:

Chounet and Gaillard [1] report

$$\begin{aligned} \lambda_+ &= 0.045 \pm 0.015 \ , \\ \xi &= -0.85 \pm 0.20 \ . \end{aligned} \tag{16}$$

Collaboration [2], with fixed $\lambda_+$ finds the value for

$$\begin{aligned} \lambda_+ &= 0.029 \ \text{as world average value,} \\ \xi &= -0.65 \pm 0.13 . \end{aligned} \tag{17}$$

X2 Collaboration [2], with $\lambda_+$ free parameter, gives

$$\begin{aligned} \lambda_+ &= 0.060 \pm 0.019 \ , \\ \xi &= -1.0 \pm 0.5 \ . \end{aligned} \tag{18}$$

At zero momentum transfer $f_+(0)$ has a value between 0.85 and 0.95. From experimental data [1] we have

$$\frac{f_K}{f_\pi \ f_+(0)} = 1.28 \tag{19}$$

in the approximation of a single Cabibbo angle.

## 3. Derivation of Bound Formula

The integral (11) for which it is necessary to find a lower bound has the form

$$I = \frac{1}{\pi} \int_1^\infty (x + a_1)^{a_1} \ldots (x + a_n)^{a_n} |F(x)|^2 dx \ , \tag{20}$$

where it is supposed that the integral exists and the integrand is positive. In addition $F(x)$ is a function satisfying the following conditions:

analytic in all $x$ plane except for a cut from 1 to $\infty$,
bounded at infinity by some power of $x^{1/2}$,
nonzero on the cut excepting a finite number of points,
its value is known at some points $x_i$.

The conformal transformation

$$z = -\frac{t-i}{t+i} \tag{21}$$

with

$$t = (x-1)^{1/2} \tag{22}$$

$$tg\,\frac{\theta}{2} = t \tag{23}$$

brings the cut plane into the interior of the unit circle $|z| < 1$ and the integral (20) becomes

$$I = \frac{1}{\pi}\int_{-\pi}^{\pi} p(\theta)\,|G(e^{i\theta})|^2\,d\theta\,, \tag{24}$$

where

$$G(e^{i\theta}) = F(x) \tag{25}$$

$$p(\theta) = \frac{1}{2}\,|t|\,(t^2+1)\,[1+a+t^2]^{a_1}\ldots[1+a+t^2]^{a_n}\,. \tag{26}$$

According to ref. /12/ the function $G(z)$ can be developed in a series of orthogonal polynomials $p_n(z)$

$$G(z) = \sum_{n=0}^{\infty} G_n\,p_n(z)\,, \tag{27}$$

where the coefficients $G_n$ are given by

$$G_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} p(\theta)\,p_n^*(e^{i\theta})\,G(e^{i\theta})\,d\theta \tag{28}$$

because the polynomials $p_n(z)$ are chosen to satisfy

$$\int_{-\pi}^{\pi} p(\theta)\, p^*_m(e^{i\theta})\, p_n(e^{i\theta})\, d\theta = 2\pi\, \delta_{mn} . \tag{29}$$

In terms of the coefficients $G_n$, the integral (24) takes the form of a series

$$I = 2 \sum_{n=0}^{\infty} G^*_n\, G_n . \tag{30}$$

When we look for the extremal value of the right-hand side of eq. (30) we must take into account that the function is not arbitrary but it takes some value $g_i$ at the points $z_i$ i.e.

$$\sum_{n=0}^{\infty} G_n\, p_n(z_i) = g_i , \tag{31}$$

where the index $i$ takes the values 0, 1, and 2. We restricted ourselves to three points because the experimental data for the form factor $f(q^2)$ are not more than three.

The values $G_n$ for which the extremum of the right-hand side of eq. (30) holds are obtained by using the Lagrange method, multiplying by $\lambda_i$ the relations (31), adding them and their complex conjugate relations to eq. (30), and equating to zero the derivative with respect to $G_n$ and $G^*_n$ of the result. Thus one obtains

$$G_n = \sum_i \lambda_i\, p_n(z_i) . \tag{32}$$

The Lagrange multipliers $\lambda_i$ can be found by introducing the values (32) into eqs (31) and finally a linear system of equations for unknown $\lambda_i$ is obtained. The coefficients appearing in these equations are of the form $\sum_{n=0}^{\infty} p^*_n(z_i)\, p_n(z_k)$ and it can be expressed in terms of the weight function $p(\theta)$ from eq. (26) /12/ according to

$$\sum_{n=0}^{\infty} p^*_n(a)\, p_n(z) = [\,(1-a^*z)\, D^*(a)\, D(z)\,]^{-1} \tag{33}$$

where

$$D(z) = \exp\left[\frac{1}{4\pi}\int_{-\pi}^{\pi} \log p(\theta)\, \frac{e^{i\theta}+z}{e^{i\theta}-z}\, d\theta\right] . \tag{34}$$

For the particular form (26) of weight functions $p(\theta)$ we have

$$D(z) = \sqrt{2}(1-z)^{\frac{1}{2}} (1+z)^{-\frac{3}{2}-\sum_{i=1}^{n} a_i} \prod_{i=1}^{n} [1-z+(1+z)\sqrt{1+a_i}] . \quad (35)$$

If we take $z_0 = 0$, $z_1 = a$ and $z_2 = b$, where $a$ and $b$ are on the real axis inside the unit circle, the extremal value $I(g_0, g_1, g_2)$ is

$$\frac{1}{2} I(g_0, g_1, g_2) = \gamma_0^2 + (1-ab)\Big[\frac{(1-a^2)(1-b^2)}{ab(a-b)^2}(\gamma_1 - \gamma_2)^2 + \frac{1-b^2}{ab^2(a-b)}(\gamma_2 - \gamma_0)^2 - \frac{1-a^2}{a^2 b(a-b)}(\gamma_1 - \gamma_0)^2\Big], \quad (36)$$

where

$$\gamma_i = D(z_i)\, g_i . \quad (37)$$

In the limit $b = 0$ the extremal (36) transforms into the extremal $I(g_0, g_0', g_1)$ that refers to the case when we know the function $g(z)$ at the points $z_0 = 0$, $z_1 = a$ and its derivative $g_0'$ at the point $z_0 = 0$

$$\frac{1}{2} I(g_0, g_0', g_1) = \gamma_0^2 + \frac{1}{a^2}\Big[\gamma_0' + \frac{1-a^2}{a}(\gamma_0 - \gamma_1)\Big]^2 + \frac{1-a^2}{a^2}(\gamma_0 - \gamma_1)^2, \quad (38)$$

where

$$\gamma_0' = \frac{d}{dz}(D(z)\, G(z))\Big|_{z=0} . \quad (39)$$

If in the extremum (38) the derivative $g_0'$ is not known its minimum value with respect to $g_0'$ gives the extremum $I(g_0, g_1)$ when we know the function $G(z)$ at two points

$$\frac{1}{2} I(g_0, g_1) = \gamma_0^2 + \frac{1-a^2}{a^2}(\gamma_0 - \gamma_1)^2 . \quad (40)$$

In the limit $a = 0$ the extremum (38) becomes

$$\frac{1}{2} I(g_0, g_0', g_0'') = \gamma_0^2 + \gamma_0'^2 + \frac{1}{4}\gamma_0''^2 \quad (41)$$

which gives

$$\frac{1}{2} I(g_0, g_0') = \gamma_0^2 + \gamma_0'^2 \tag{42}$$

when the second derivative $g_0''$ is not known. Taking the minimum of eq. (42) for $\alpha g_0 + \beta g_0'$ fixed, one obtains the extremum

$$I(\alpha g_0 + \beta g_0') = 2 |\alpha g_0 + \beta g_0'| (D(0))^2 [(\alpha + \beta \frac{D'(0)}{D(0)})^2 + \beta^2]^{-1} \tag{43}$$

which gives for $\alpha = 1$, $\beta = 0$

$$I(g_0) = 2 g_0^2 (D(0))^2 \tag{44}$$

and

$$I(g_0') = 2 g_0'^2 (D(0))^2 [1 + (\frac{D'(0)}{D(0)})^2]^{-1} \tag{45}$$

for $\alpha = 0$, $\beta = 1$.

## 4. Application to $K_{\ell 3}$ Decay Parameters

Bounds on the $K_{\ell 3}$ decay parameters are obtained from ineq. (11) when its right-hand side is replaced through one expression from (38), (40)-(45). The form of the function $D(z)$ corresponding to the integral (11) is obtained from eq. (35).

$$D(z) = \frac{1}{2\sqrt{2}} \frac{1-z}{\sqrt{1+z}} [1 - z + (1+z)\sqrt{1-\omega^2}]^{1/2} . \tag{46}$$

In the bound formulae $D(z)$ appears in the following combinations

$$D(0) = \frac{1}{2\sqrt{2}} (1 + \sqrt{1-\omega^2})^{1/2} = 0.477 , \tag{47}$$

$$\frac{D'(0)}{D(0)} = -[1 + (1 + \sqrt{1-\omega^2})^{-1}] = -1.55 , \tag{48}$$

$$\frac{D(a)}{D(0)} = 0.61 , \tag{49}$$

where $a$ is the point corresponding to $q^2 = m_K^2$ according to eq. (21)

$$a = 0.235 \ . \tag{50}$$

The quantities $g_0$, $g_0'$ and $g_1$ are:

$$g_0 = f_+(0) \tag{51}$$

$$g_0' = \frac{4}{\omega} f_+(0) \ (12.3 \lambda_+ + \xi) \tag{52}$$

$$g_1 = 1.28 \ f_+(0) + \text{on mass shell corrections.} \tag{53}$$

The extremum (44) gives the bound first obtained by Okubo [5] and later by Li and Pagels [6] and Radescu [9]

$$\frac{\Delta^{1/2}(0)}{m_K^2 - m_\pi^2} \geq \frac{1}{16} \sqrt{\frac{3}{\pi}} \left(1 + \sqrt{1-\omega^2}\right)^{1/2} |f_+(0)| \ . \tag{54}$$

According to eqs (15) and (47), relation (54) becomes

$$M \geq |f_+(0)| \ . \tag{55}$$

The extremum (45) gives an improvement of the bound found in paper [3] and now it is

$$M \geq \frac{4}{\omega} \{ 1 + [1 + (1 + \sqrt{1-\omega^2})^{-1}]^2 \}^{-1/2} . \tag{56}$$

The extremum (38) gives the new bound

$$\frac{\omega a}{4} \left[ \frac{M^2}{f_+^2(0)} - 1 - \frac{1-a^2}{a^2} \left(1 - 1.28 \frac{D(a)}{D(0)}\right)^2 \right]^{1/2} \geq$$
$$\geq \left| 12.3 \lambda_+ + \xi + \frac{\omega}{4} \left[ \frac{D'(0)}{D(0)} + \frac{1-a^2}{a} \left(1 - 1.28 \frac{D(a)}{D(0)}\right) \right] \right| . \tag{57}$$

where the left-hand side is smaller than 0.1 for all values of $M$ [8]. The right-hand side is 12.3 $\lambda_+ + \xi - 0.09$ but when the on mass shell corrections of order 10% are added to eq. (53) the term - 0.09 must be replaced by some value in the interval - 0.04 to - 0.16. Finally we have

$$-0.06 \leq 12.3 \lambda_+ + \xi \leq 0.26 \ . \tag{58}$$

This bound can be improved firstly, giving for $M$ a realistic value, not the largest one as we did in the expression (58), and secondly, by evaluating the on shell corrections to the Gallan-Treiman relation (1).

The bound corresponding to the extremum (42) can be obtained from the bound (57) in the limit $a = 1$. This bound was discovered first by Okubo [8].

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