25/x-71

F-52

ОБЪЕДИНЕННЫЙ
ИНСТИТУТ
ЯДЕРНЫХ
ИССЛЕДОВАНИЙ

Дубна

E2 - 6043

A.T. Filippov, V.S. Gogokhia

APPROXIMATE METHODS FOR SOLVING SCHROEDINGER EQUATION WITH MARGINALLY SINGULAR POTENTIAL

ЛАБОРАТОРИЯ ТЕОРЕТИЧЕСКОЙ ФИЗИКИ

1971

E2 - 6043

A.T. Filippov, V.S. Gogokhia*

APPROXIMATE METHODS FOR SOLVING SCHROEDINGER EQUATION WITH MARGINALLY SINGULAR POTENTIAL

Submitted to *ЯФ*

* Tbilisi State University

S U M M A R Y

The renormalization theory for the Schroedinger equation with marginally singular potentials is outlined. The singular potentials are shown to be classified naturally into renormalizable and nonrenormalizable ones by the structure of the perturbative series (with a small-distance cut-off) analogous to that of field theories.

Approximate methods for solving the Schroedinger equation with the singular potential $g_o V(r)$ are discussed. These enable one to find the wave function as a series of powers of $g_o^{\nu} (\ln g_o)^n$ where n is an integer and ν is in general non-integer (a modified perturbation theory). The most powerful method for constructing such a series is the "asymptotic" perturbation theory in which the first approximation for the wave function has the exact asymptotic behaviour for $r = 0$. By developing more simple method which may be used in field theory problems as well, we suggest the new method for reconstruction of modified perturbation theory employing only finite number of perturbation theory terms.

I. Introduction

After the general recipes for eliminating divergences from the perturbation expansion in renormalizable field theories (R-theories) have been formulated, the main point in quantum field theory investigations became the problem of finding the way out of the framework of perturbation theory /1/. On the one hand, this was necessary for applications to strong interaction physics, on the other, the standard renormalization procedure failes to eliminate divergences in the theory of weak interaction and in many other field theories, which are of the physical interest but nonrenormalizable (N-theories). In this connection, many times a conjecture was expressed that the very new ideas are required to describe N-theories or, alternatively, they do not exist at all in Nature. It is quite possible, however, that for some N-theories one can learn to calculate the higher-order corrections, if one either abandons perturbation theory or finds certain modification of it. To this end it is useful to study first some primitive exactly solvable model, in which it is possible to understand the nature of the difference between N and R-theories and to compare the exact solutions with perturbative ones. The starting point of our work is the formulation of this model. After the general discussion of the model we will present the method for constructing a modified perturbation expansion. We hope that this one can be used for calculating higher-order approximations in the theory of weak interactions as well as in other N-theories.

Let us consider nonrelativistic Schroedinger equation

$$\frac{d^2u}{d r^2} + \left[\kappa^2 - \frac{\ell(\ell+1)}{r^2} - g_o V(r) \right] u(r) = 0, \qquad (1.1)$$

where a potential at sufficiently small r is finite, nonvanishing and monotinic. The potential $g_o V(r)$ is called regular if $\int_0^{r_o} dr\, r\, |V(r)| < \infty$. If $\int_0^{r_o} dr\, r\, |V(r)| = \infty$, the potential is singular by definition. Let

$$u_o^{(1)}(r) = j_\ell(\kappa r) \equiv (\kappa r)^{1/2} J_{\ell+1/2}(\kappa r) \underset{r\to 0}{\sim} r^{\ell+1}, \qquad (1.2a)$$

$$u_o^{(2)}(r) = n_\ell(\kappa r) \equiv (\kappa r)^{1/2} N_{\ell+1/2}(\kappa r) \underset{r\to 0}{\sim} r^{-\ell} \qquad (1.2b)$$

be two linearly-independent solutions of Eq. (1.1) for g_o=0. Then the model can be obtained by replacing Eq. (1.1) and the boundary condition $u(r) \underset{r\to 0}{\longrightarrow} 0$ by the integral equation

$$u(r) = Z u_o^{(1)}(r) - g_o \int_0^r d\rho\, V(\rho) \frac{u_o^{(1)}(r) u_o^{(2)}(\rho) - u_o^{(1)}(\rho) u_o^{(2)}(r)}{W[u_o^{(1)}, u_o^{(2)}]} u(\rho). \qquad (1.3)$$

Here Z is the normalization factor, and

$$W[u_o^{(1)}, u_o^{(2)}] = u_o^{(1)} u_o^{(2)\prime} - u_o^{(1)\prime} u_o^{(2)} = 2\kappa/\pi .$$

For κ=0 this equation has the form

$$u(r) = Z r^{\ell+1} + \frac{g_0}{2\ell+1} \int_0^r d\rho \, V(\rho) \left[r^{\ell+1} \rho^{-\ell} - \rho^{\ell+1} r^{-\ell} \right] u(\rho) \quad (1.4)$$

(a model of such a kind has been first suggested in Ref./2/ on the basis of studying the Logunov-Tavkhelidze quasipotential equation/3/). Integrating Eq. (1.3) or (1.4), one may derive the solution $u(r)$ in the form of a series of powers of the coupling constant g_0 . In the case of the regular potential $V(r)$ each term of the expansion is expressed in terms of convergent integrals, futhermore the solution proves to be an analytic function of g_0 in some vicinity of the point $g_0=0$. The perturbation expansion converges uniformly at sufficiently small values of g_0 and r/4/. The asymptotic form of $u(r)$ for $r \to 0$ coincides with that of the zeroth approximation $Z u_0^{(1)}(r)$. In the case of singular potentials one easily finds (e.g. from Eq. (1.4)) that even the first iteration produces the divergent expressions. One can try to remove these divergences by renormalizations (as in field theory). With this aim, we replace the singular potential $V(r)$ by the regularized one $V_\varepsilon(r) = \theta(r-\varepsilon)\, V(r)$ and try to eliminate all divergences from perturbation expansion by a suitable choice of the renormalization constant Z. In the next section we will describe the potentials for which it is possible.

2. Renormalizable and Nonrenormalizable Singular Potentials

As was shown by one of us (A.T.F./5/), all the singular potentials are classified into two groups. Those singular potentials for which all divergences can be eliminated by renormalizations are called "renormalizable", otherwise "nonrenormalizable". For renormalizable potentials, the perturbation expansion of $u_\varepsilon(r)$ can be represented in the form

$$u_\varepsilon(r) = Z_\varepsilon \left\{ \sum_{m=0}^{\infty} g_0^m\, w_m(\varepsilon) \right\} \left\{ \sum_{n=0}^{\infty} g_0^n\, v_n^\varepsilon(r) \right\}, \tag{2.1}$$

where the functions $v_n^\varepsilon(r)$ tend to the finite limits $v_n(r)$ as $\varepsilon \to 0$ and $w_m(\varepsilon)$ do not depend on r and become infinite as $\varepsilon \to 0$. Choosing the renormalization constant Z_ε as (Z does not depend on ε):

$$Z_\varepsilon = Z \left\{ \sum_{m=0}^{\infty} g_0^m\, w_m(\varepsilon) \right\}^{-1} = Z \left\{ 1 - g_0 w_1 - g_0^2 (w_2 - w_1^2) + \dots \right\}, \tag{2.2}$$

we get the finite solution as a series of powers of g_0 :

$$u(r) = Z \sum_{n=0}^{\infty} g_0^n\, v_n(r) \tag{2.3}$$

(on passing to the limit $\varepsilon \to 0$). As it turned out it is possible to obtain the solutions in the form (2.3) only for the potentials satisfying the renormalization criterion. This is as follows. The potential V(r) is renormalizable if and only if $r^2 |V(r)| < C r^{-\delta}$ as $r \to 0$, where δ is an arbitrarily small number*. The exact solution $u_\varepsilon(r)$ of Eq. (1.3) with the potential $V_\varepsilon(r)$ in any case can be represented in the form

$$u_\varepsilon(r) = Z_\varepsilon \left[w_1(\varepsilon)\, u_1(r) + w_2(\varepsilon)\, u_2(r) \right], \tag{2.4}$$

where $u_1(r)$ and $u_2(r)$ are linearly independent solutions of Eq.(1.1) and w_1 , w_2 can be expressed in terms of u_1, u_2 , u_1' and u_2'. For the repulsive potential the solutions u_1 and u_2 can be chosen in such a way that $u_1(r) \xrightarrow[r\to 0]{} 0$, $u_2(r) \xrightarrow[r\to 0]{} \infty$, $w_1(\varepsilon) \xrightarrow[\varepsilon\to 0]{} \infty$, $w_2(\varepsilon) \xrightarrow[\varepsilon\to 0]{} 0$. Then in the limit $\varepsilon \to 0$ we get $u_\varepsilon(r) \to [Z_\varepsilon\, w_1(\varepsilon)]\, u_1(r)$. Setting $Z_\varepsilon = Z / w_1(\varepsilon)$, we find that $u_\varepsilon(r) \xrightarrow[\varepsilon\to 0]{} Z u_1(r)$, where $u_1(r)$ is the solution of Eq. (1.1) obeying the boundary condition $u_1(r) \xrightarrow[r\to 0]{} 0$. This assertion is true for any repulsive singular potential (for attractive one $w_1(\varepsilon)/w_2(\varepsilon)$

* The renormalization procedure produces the unique solution (2.3) not only for the repulsive renormalizable potentials but for attractive ones, as well.

does not approach any limit as $\varepsilon \to 0$). For the renormalizable potentials all functions u_1 , u_2 , w_1 , w_2 can be expanded in a series of powers of g_0 , each term of the expansion of $w_2(\varepsilon)$ approaching zero as $\varepsilon \to 0$. The latter makes it possible to remove all divergences from the perturbative expansion. In the case of nonrenormalizable potentials the functions u , w prove to be nonexpansible in a series of powers of g_0 , since the expansion of these functions for small g_0 contains, for example, the terms like $g_0^{\nu} \ln^n g_0$ In the paper[6] one may find some exact solutions of Eq. (1.1)(with $\kappa = 0$) for singular potentials

$$V = \tau^{-2}\left(\ln\frac{\tau_0}{\tau}\right)^{-1},\ \tau^{-2}\left(a\ln\frac{\tau_0}{\tau} + b\ln^2\frac{\tau_0}{\tau}\right),\ a\tau^{-4} + b\tau^{-3}, \dots,$$

the studying of which enables one to understand better the connection between perturbation theory and the exact solutions and learn the character of a singularity in g_0 at $g_0 = 0$. For R-potentials the exact solution has an essential singularity in g_0 at $g_0 = 0$. If the potential is nonrenormalizable the solutions of Eq.(1.1) possess the branch-point, in g_0 and so they cannot be expanded even in the asymptotic series of powers of g_0 . Finally, we write down the exact solutions of Eq.(1.1) (with $\kappa = 0$) for N-potentials of the form $g_0 V = g^2 \tau^{-2(n+1)}$

$$u^{(1)}(\tau) = c_1\sqrt{\tau}\, K_{\lambda/n}(g/n\tau^n), \tag{2.5a}$$

$$u^{(2)}(\tau) = c_2\sqrt{\tau}\, I_{\lambda/n}(g/n\tau^n), \quad \lambda \equiv \ell + 1/2. \tag{2.5b}$$

3. The "Asymptotic" Perturbation Theory

The wave function singularity in g_0 at $g_0 = 0$ is closely connected with the presense of the essential singularity in τ at $\tau = 0$. The asymptotic form of the exact solution $u(\tau)$ as $\tau \to 0$,

$$u(\tau) \sim C\sqrt{\tau}\left[\lambda^2 + \tau^2 g_0 V(\tau)\right]^{-1/4} \exp\left\{\int_{\tau_0}^{\tau}\frac{d\rho}{\rho}\left[\lambda^2 + \rho^2 g_0 V(\rho)\right]^{1/2}\right\}, \tag{3.1}$$

differs from that of unperturbed wave function $u_0^{(1)}(r) \sim C r^{\ell+1}$.
If the approximation method reproduces correctly the asymptotic behaviour of the wave function as $r \to 0$, then, at the same time, it makes it possible to get the expansion of the solution for small g_0, which gives correctly singularities of the exact solution at $g_0 = 0$. An idea of the "asymptotic" perturbation theory is essentially based on this comment. If the potential V(r) is broken up into two parts $V(r) = V_S(r) + V_R(r)$ in such a way that $V_S(r)/V_R(r) \xrightarrow[r\to 0]{} \infty$ then the solutions $u^{(1)}, u^{(2)}$ of Eq. (1.1), in which $V(r)$ is replaced by $V_S(r)$ and the term $\kappa^2 u(r)$ is neglected, have the correct asymptotic form. These are reasonable (for small r) first approximations to the exact solution, and the latter can be derived by iterating the equation

$$u(r) = u^{(1)}(r) + \int_0^r d\rho \left[\kappa^2 - g_0 V_R(\rho)\right] \frac{u^{(1)}(r)u^{(2)}(\rho) - u^{(1)}(\rho)u^{(2)}(r)}{W[u^{(1)}, u^{(2)}]}, \quad (3.2)$$

where $u^{(1)}(r) \xrightarrow[r\to 0]{} 0$ and $u^{(2)}(r) \xrightarrow[r\to 0]{} \infty$. The resulting series converges uniformly for sufficiently small g, r and κ if[5]

$$\int_0^{r_0} dr\, V_R(r)\left[V_S(\rho)\right]^{-1/2} < \infty, \qquad r^2 V(r) \xrightarrow[r\to 0]{} +\infty.$$

Now, let us demonstrate how the "asymptotic" perturbation theory does work, for example, in the case of the potential $g_0 V(r) = g^2 r^{-4}$. Solutions of Eq. (1.1) for the S-wave have the form of (2.5) with $\lambda = 1/2$ and $n = 1$. Then, iterating Eq. (3.2) we get

$$\varphi(\kappa, r) = c\Big\{1 - \frac{g}{r} + \frac{g^2}{2r^2} + \kappa^2\Big[\frac{2}{3}g^2 \ln g - \frac{2}{3}g^2 \ln r + g^2\Big(\frac{2}{3}\gamma + \frac{2}{3}\ln 2 - \frac{5}{12}\Big) + \frac{1}{2}g r - \frac{1}{6}r^2 + \dots\Big] + \dots\Big\}. \quad (3.3)$$

For $\kappa^2 = 0$ this expansion coincides with the exact one, given by Eq. (25a). (Here and in what follows $\varphi \equiv u/u_0^{(1)}$).

To conclude this Section we briefly discuss the approximate methods commonly used for the partial summation of the perturbative expansion in quantum field theory. In R-theoreies the method is often used of summing the leading logarithmic terms[7]. In the model under consideration it is not difficult to sum up all highest divergencies for any R-potential. To this end, it suffices to neglect the last term in the right member of Eq. (1.3) with the potential V_ε. (This term does not give contribution to the highest divergences in each order of perturbation theory). The solutuion of this "shortened" equation is

$$u_\varepsilon^{\rho}(r) = Z_\varepsilon u_o^{(1)}(r) \exp\left\{-\frac{g_o}{W}\int_\varepsilon^{r} d\rho\, V(\rho)\, u_o^{(1)}(\rho)\, u_o^{(2)}(\rho)\right\}. \qquad (3.4)$$

This solutuion is an entire function of g_o and its asymptotic form differs decidedly from Eq. (3.1). Further, the approximation (3.4) can be tested in applying to N-potentials. In N-theories such an approximation was used by the authors of Ref.[8], and is known as the "peratization" method. However, in this case as well, Eq.(3.4) has no relation to the exact solution.

4. Differential Interpolation Method

The idea of the differential interpolation method (DIM) is based on the existence of the representation (2.4) for the exact solution $u_\varepsilon(r)$. We will show first that $w_1(\varepsilon)$ and $w_2(\varepsilon)$ satisfy a simple differential equation with respect to the variable ε. This equation is tightly connected with the Schroedinger equation (1.1). To formulate the method in the form which is assumed to be applicable to more complicated problems of field theory, only a finite number of the terms of the perturbative expansion for $u_\varepsilon(r)$ are supposed to be known. Then we guess the differential interpolating equation (the cut-off parameter, $\varepsilon \equiv D^{-1}$, being the independent variable) which is satisfied by the known terms of the perturbative expansion. Comparing term by term the expansion of the general solution of the equation with the known perturbative expansion we obtain an expression for arbitrary coefficients (depending on r) included in the general solution of interpolating equation. In the expression derived in such a way we perform the renormalization and limiting process $\varepsilon \to 0$. The limit renormalized expression includes terms nonanalytic in g and provides us with the finite number of the modified perturbation theory terms.

By comparing the above results to the expansions of exact solutions we will demonstrate the efficiency of the method. However, we have not yet found the general proof of convergence of the DIM approximations to the exact solution*.

Now let us precisely formulate DIM in the case k=0**. To this end we first derive the exact differential equation with respect to ε for $u_\varepsilon(r)$. For this purpose we cut integral of Eq. (1.4) at the lower limit. Then, for $K=0$ the equation

$$u_\varepsilon(r) = Z_\varepsilon r^{\ell+1} + \frac{g_0}{2\ell+1}\int_\varepsilon^r d\rho\, V(\rho)\left[r^{\ell+1}\rho^{-\ell} - \rho^{\ell+1} r^{-\ell}\right] u_\varepsilon(\rho) \qquad (4.1)$$

is equivalent to the differential equation (1.1) with the boundary conditions

$$u_\varepsilon(\varepsilon) = Z_\varepsilon \varepsilon^{\ell+1}, \quad du_\varepsilon/dr \underset{r=\varepsilon}{=} Z_\varepsilon(\ell+1)\varepsilon^{\ell}. \qquad (4.2)$$

Now, it follows that the exact solution $u_\varepsilon(r)$ can be written in the form of (2.4). This is as follows:

$$u_\varepsilon(r) = Z_\varepsilon\left\{w_1(\varepsilon)\,u_1(r) + w_2(\varepsilon)\,u_2(r)\right\}, \qquad (4.3)$$

---

*For some simplest cases DIM gives directly the exact solutions.

**The case $K \neq 0$ is more complicated and requires a special consideration; nevertheless, as will be shown below, a modification of DIM exists which allows to get an interpolating solution even for $K \neq 0$.

where

$$w_1(\varepsilon) = \varepsilon^{\ell+1} u_2'(\varepsilon) - (\ell+1)\varepsilon^{\ell} u_2(\varepsilon), \tag{4.4a}$$

$$w_2(\varepsilon) = \varepsilon^{\ell+1} u_1'(\varepsilon) - (\ell+1)\varepsilon^{\ell} u_1(\varepsilon). \tag{4.4b}$$

Further, using the Schroedinger equation (1.1), the solutions of which are $u_1(\tau)$ and $u_2(\tau)$, and (4.4), one may easily show that $w_1(\varepsilon)$ and $w_2(\varepsilon)$ satisfy the equation

$$\frac{d^2 w}{d\varepsilon^2} - \left[2\,\frac{\ell+1}{\varepsilon} + \frac{V'(\varepsilon)}{V(\varepsilon)}\right]\frac{dw}{d\varepsilon} = g_0 V(\varepsilon)\, w(\varepsilon). \tag{4.5}$$

Hence it follows that $u_\varepsilon(\tau)$ satisfies this equation as well and so DIM can be easily proved. Indeed, Eq. (4.5) is equivalent to the recurrence relations

$$\frac{d^2 u_\varepsilon^{(n+1)}}{d\varepsilon^2} - \left[2\,\frac{\ell+1}{\varepsilon} + \frac{V'(\varepsilon)}{V(\varepsilon)}\right]\frac{du_\varepsilon^{(n+1)}}{d\varepsilon} = g_0 V(\varepsilon)\, u_\varepsilon^{(n)} \tag{4.6}$$

between the terms of the perturbative expansion

$$u_\varepsilon(\tau) = u_\varepsilon^{(0)}(\tau) + u_\varepsilon^{(1)}(\tau) + u_\varepsilon^{(2)}(\tau) + \ldots \tag{4.7}$$

So, to find the interpolating differential equation it is sufficient to know the recurrence relation (4.6). In our simple case two first terms of the expansion (4.7) are sufficient for reconstructing the exact equation (4.5). In other, more complicated cases (see e.g. DIM for $\kappa \neq 0$), the recurrence relation has more complicated form, depends on the number of perturbative terms taken into account and defines only the approximate interpolating equation. With one constant of dimension of length available, the reccurrence relation for all problems of practical importance has the form

$$(gD)^n \sum_{\kappa=0}^{N_n} c_\kappa^{(0)} D^\kappa \frac{\partial^\kappa u^{(0)}}{\partial D^\kappa} + (gD)^{n-1} \sum_{\kappa=0}^{N_{n-1}} c_\kappa^{(1)} D^\kappa \frac{\partial^\kappa u^{(1)}}{\partial D^\kappa} + \ldots + \sum_{\kappa=0}^{N_0} c_\kappa^{(n)} D^\kappa \frac{\partial^\kappa u^{(n)}}{\partial D^\kappa} = 0. \tag{4.8}$$

As soon as the relation is found, one immediately obtains the interpolating equation for u by dropping indices of $u^{(i)}$ in Eq. (4.8). Then u may be represented as a sum $u = \sum w_i(D)\, u_i(\tau)$, where

$w_i(D)$ are exact solutions of the interpolating equation and can be found by comparison of the sum with perturbative expansion (as explained above). Next two sections illustrate this general procedure by simple examples.

5. The Potential $g_0 V(\tau) = g^2 \tau^{-4}$.

Consider first the application of DIM to the nonrenormalizable singular potential $g_0 V(\tau)=g^2\tau^{-4}$ for $\kappa = \ell = 0$. In this case the perturbation series (4.1) is

$$\varphi(D,\tau) = Z_D\left\{1 + g^2\left[\frac{D^2}{2} - \frac{D}{\tau} + \frac{1}{2\tau^2}\right] + \dots\right\} = \varphi^{(0)} + \varphi^{(1)} + \dots \qquad (5.1)$$

and Eq. (4.8) reduces to the equation

$$(gD)^2 c_0^{(0)}\varphi^{(0)} + c_0^{(1)}\varphi^{(1)} + c_1^{(1)} D\frac{\partial\varphi^{(1)}}{\partial D} + c_2^{(1)} D^2\frac{\partial\varphi^{(1)}}{\partial D} + \dots = 0. \qquad (5.2)$$

The terms of Eq. (5.1) satisfy Eq. (5.2) if $c_0^{(1)} = c_1^{(1)} = 0$, $c_0^{(0)} = -c_2^{(1)}$. Hence, the terms of the series (5.1), obey the equation

$$D^2\frac{\partial^2\varphi}{\partial D^2} - g^2 D^2\varphi = 0. \qquad (5.3)$$

Two linearly independent solutions of this equation are $(gD)^{1/2} I_{1/2}(gD)$ and $(gD)^{1/2} K_{1/2}(gD)$. The relation (4.3) can be written here as follows

$$\varphi(D,\tau) = Z_D\left\{(gD)^{1/2} I_{1/2}(gD)\, u_1(\tau) + (gD)^{1/2} K_{1/2}(gD)\, u_2(\tau)\right\}. \qquad (5.4)$$

Now let us expand both terms in the right member of (5.4) in powers of g . Then to the g^2-order terms we get:

$$\varphi(D,\tau) = Z_D\left\{[gD + \dots][u_1^{(0)} + g u_1^{(1)} + g^2 u_1^{(2)} + \dots] + \right.$$
$$\left. + [1 - gD + \tfrac{1}{2}(gD)^2 + \dots][u_2^{(0)} + g^2 u_2^{(1)} + \dots]\right\}. \qquad (5.5)$$

The perturbation expansion (5.1) satisfies Eq. (5.5) under the following conditions $u_1^{(0)} = u_2^{(0)} = 1$, $u_1^{(1)} = -\frac{1}{\tau}$, $u_1^{(2)} = u_2^{(1)} = 1/2\tau^2$.

inserting these relations into (5.4) and taking the limit $D \to \infty$ we get finally

$$\varphi(\infty,\tau) = Z\left\{1 - \frac{g}{\tau} + \frac{g^2}{2\tau^2} + \dots\right\}, \tag{5.6}$$

where $Z = Z_D(gD)^{1/2} I_{1/2}(gD)$. It follows from Eq. (2.5a), with $n=1, \lambda=1/2$, that decreasing exact solution has the form

$$\varphi(\tau) = c\left\{1 - \frac{g}{\tau} + \frac{g^2}{2\tau^2} + \dots\right\}. \tag{5.7}$$

Normalizing (5.6) and (5.7) in the same way we get that square-root branch point (the terms of the order $g = \sqrt{g^2}$) is found by DIM correctly, and, up to the g^2-order terms, the interpolating solution $\varphi(\infty,\tau)$ coincides with the exact solution $\varphi(\tau)$.

Next, let us consider DIM in the case: $l=0, \kappa \neq 0$. When $\kappa \neq 0$ it is necessary to use Eq. (1.3) which takes the form

$$u(D,\kappa,\tau) = Z_D j_0(\kappa\tau) - \frac{g^2\pi}{2\kappa}\int_{D^{-1}}^{\tau} \frac{d\rho}{\rho^4}\left[j_0(\kappa\tau)n_0(\kappa\rho) - j_0(\kappa\rho)n_0(\kappa\tau)\right]u(\rho), \tag{5.8}$$

on substituting $u_0^{(1)}(\tau)$ and $u_0^{(2)}(\tau)$ from Eq. (1.2) and making a cut-off at the lower limit. Iterating this equation within an accuracy of the g^2-order terms, the following perturbation expansion at large but finite cut-off parameter D is obtained:

$$\varphi(D,\kappa,\tau) = Z_D\left\{1 + g^2\left[\tfrac{1}{2}D^2 + \kappa D\, n_0(\kappa\tau)/j_0(\kappa\tau) - \right.\right.$$
$$\left.\left. - \tfrac{2}{3}\kappa^2 \ln D + \Phi(\kappa,\tau)\right] + \dots\right\} = \varphi^{(0)} + \varphi^{(1)} + \dots \tag{5.9}$$

where

$$\Phi(\kappa,\tau) = -\kappa\frac{n_0(\kappa\tau)}{j_0(\kappa\tau)}\left[\tau^{-1} + \tfrac{1}{3}\kappa^2\tau + \dots\right] - \left[\frac{1}{2\tau^2} + \frac{2}{3}\kappa^2 \ln\tau + \dots\right].$$

Proceeding further as in the case k=0 we might get the interpolating differential equation of the third order, which could be solved in terms of the Meyer functions /10/. In the case k=0 it is suitable, however, to change somewhat the interpolating equation (4.8). In particular, we may expand the coefficient of the zeroth approximation in a series of κ^2. Then we arrive at the following interpolating equation

$$C_0^{(0)}(g\mathcal{D})^2\varphi^{(0)} + \tilde{C}_0^{(0)} g^2\kappa^2\varphi^{(0)} + C_0^{(1)}\varphi^{(1)} + C_1^{(1)}\frac{\partial\varphi^{(1)}}{\partial\mathcal{D}} + C_2^{(1)}\mathcal{D}^2\frac{\partial^2\varphi^{(1)}}{\partial\mathcal{D}^2} + \ldots = 0 . \tag{5.10}$$

The expansion (5.9) obeys (5.10) only if

$$C_0^{(1)} = C_1^{(1)} = 0 ; \quad C_0^{(0)} = -C_2^{(1)} ; \quad \tilde{C}_0^{(0)} = -\tfrac{2}{3} C_2^{(1)} .$$

Thus, $\varphi(\mathcal{D}, \kappa, \tau)$ satisfies the second-order interpolating equation

$$\mathcal{D}^2\frac{\partial^2\varphi}{\partial\mathcal{D}^2} - g^2(\mathcal{D}^2 + \tfrac{2}{3}\kappa^2)\,\varphi = 0, \tag{5.11}$$

the solutions of which are $(g\mathcal{D})^{1/2} I_\nu(g\mathcal{D})$ and $(g\mathcal{D})^{1/2} K_\nu(g\mathcal{D})$, where

$$\nu = \tfrac{1}{2}\left(1 + \tfrac{8}{3}g^2\kappa^2\right)^{1/2} = \tfrac{1}{2}\left(1 + \tfrac{4}{3}g^2\kappa^2 + \ldots\right).$$

We represent now $\varphi(\mathcal{D},\kappa,\tau)$ in the form (4.3):

$$\varphi(\mathcal{D},\kappa,\tau) = Z_{\mathcal{D}}\left\{(g\mathcal{D})^{1/2} I_\nu(g\mathcal{D})\, u_1(\tau) + (g\mathcal{D})^{1/2} K_\nu(g\mathcal{D})\, u_2(\tau)\right\}. \tag{5.12}$$

Expanding both terms in the right member of (5.12) in powers of g, we get

$$\begin{aligned}\varphi(\mathcal{D},\kappa,\tau) = Z_{\mathcal{D}}\Big\{&\left[g\mathcal{D} + g^3\left(\tfrac{1}{6}\mathcal{D}^3 + \tfrac{2}{3}\kappa^2(\ln 2g\mathcal{D} + \gamma - 2)\right) + \ldots\right]\left[u_1^{(0)} + g u_1^{(1)} + g^2 u_1^{(2)} \ldots\right]\\ &+\left[1 - g\mathcal{D} + g^2\left(\tfrac{1}{2}\mathcal{D}^2 - \tfrac{2}{3}\kappa^2(\ln 2g\mathcal{D} + \gamma)\right) - g^3\left(\tfrac{1}{6}\mathcal{D}^3 + \tfrac{2}{3}\kappa^2\mathcal{D}(\ln 2g\mathcal{D} + \gamma - 2)\right)\right]\cdot\\ &\cdot\left[u_2^{(0)} + g u_2^{(1)} + g^2 u_2^{(2)} + g^3 u_2^{(3)} + \ldots\right].\end{aligned} \tag{5.13}$$

The comparison of (5.13) with (5.9) leads to the following relations between the terms of expansions $u_1(\tau)$ and $u_2(\tau)$

$$u_1^{(0)} = u_2^{(0)} = 1\,,\quad u_2^{(1)} = u_2^{(3)} = 0\,,\quad u_1^{(1)} = -\kappa\,\mathrm{ctg}\,\kappa\tau\,,$$

$$u_1^{(2)} = u_2^{(2)} = \left(\frac{1}{\tau}\kappa\,\mathrm{ctg}\,\kappa\tau - \frac{1}{2\tau^2}\right) + \\ + \frac{2}{3}\kappa^2\left(\ln 2\frac{g}{\tau} + \gamma + \frac{1}{2}\kappa\tau\,\mathrm{ctg}\,\kappa\tau\right) + O(\kappa^4\tau^2). \tag{5.14}$$

Substituting these relations into (5.12) and passing to the limit $\mathcal{D}\to\infty$ we finally get

$$\varphi(\infty,\kappa,\tau) = Z\left\{1 - \frac{g}{\tau}\left(1 - \frac{1}{3}\kappa^2\tau^2 + O(\kappa^4\tau^4)\right) + \\ + \frac{g^2}{2\tau^2}\left[1 + \frac{4}{3}\kappa^2\tau^2\left(\ln 2\frac{g}{\tau} + \gamma\right) + O(\kappa^4\tau^4)\right] + o(g^2)\right\}. \tag{5.15}$$

(here $Z = Z_{\mathcal{D}}(g\mathcal{D})^{1/2} I_\nu(g\mathcal{D})$). In (5.15) all terms have been expanded in κ^2 . Comparing (5.15) with the asymptotically exact solution (3.3), we find that the difference

$$u(\infty,\kappa,\tau) - u(\tau) = Z\cdot\kappa\tau\cdot\frac{1}{3}g^2\kappa^2$$

is small for sufficiently small g and κ .

6. The Potential $g_0V(\tau) = g\tau^{-3}$

In the case of nonrenormalizable singular potentials the perturbative expansion of Eq.(4.1) with $\ell=\kappa=0$ looks as follows

$$\varphi(\mathcal{D},\tau) = Z_{\mathcal{D}}\left\{1 + g\left(\mathcal{D} - \frac{1}{\tau}\ln\frac{\mathcal{D}}{\tau} - \frac{1}{\tau}\right) + \ldots\right\} = \varphi^{(0)} + \varphi^{(1)} + \ldots \tag{6.1}$$

Then Eq.(4.8) has the form

$$(g\mathcal{D})C_0^{(0)}\varphi^{(0)} + C_0^{(1)}\varphi^{(1)} + C_1^{(1)}\mathcal{D}\frac{\partial\varphi^{(1)}}{\partial\mathcal{D}} + C_2^{(1)}\mathcal{D}^2\frac{\partial^2\varphi^{(1)}}{\partial\mathcal{D}^2} + \ldots = 0. \tag{6.2}$$

It is quite obvious that (6.1) satisfies (6.2) if $C_0^{(1)} = 0$, $C_1^{(0)} = C_2^{(1)} = -C_0^{(0)}$. So, the terms of perturbative expansion (6.1) obey the equation

$$\mathcal{D}^2\frac{\partial^2\varphi}{\partial\mathcal{D}^2} + \mathcal{D}\frac{\partial\varphi}{\partial\mathcal{D}} - g\mathcal{D}\varphi = 0, \tag{6.3}$$

the solutions of which are $I_0(2\sqrt{gD})$ and $K_0(2\sqrt{gD})$.
The representation (4.3) in this case can be written as

$$\varphi(D,\tau) = Z_D\left\{I_0(2\sqrt{gD})\, u_1(\tau) + K_0(2\sqrt{gD})\, u_2(\tau)\right\}. \qquad (6.4)$$

Expanding now the right-hand side of this representation in small g we find:

$$\varphi(D,\tau) = Z_D\left\{\left[1 + gD + \dots\right]\left[u_1^{(0)} + g u_1^{(1)} + g \ln g\, \tilde{u}_1^{(1)} + \dots\right] + \right.$$
$$\left. + \left[\left(-\gamma - \tfrac{1}{2}\ln gD\right) + gD\left(1 - \gamma - \tfrac{1}{2}\ln gD\right) + \dots\right]\left[u_2^{(0)} + g u_2^{(1)} + \dots\right]\right\}. \qquad (6.5)$$

Eq. (6.5) coincides with (6.1) under the condition that
$u_1^{(0)} = 1,\ u_2^{(0)} = 0,\ u_1^{(1)} = -\tau^{-1}(\ln\tau + 1) + \tau^{-1}2\gamma,\ \tilde{u}_1^{(1)} = \frac{1}{2}u_2^{(1)},\ u_2^{(1)} = 2\tau^{-1}$.
Inserting, further, these relations into (6.4) and passing to the limit $D \to \infty$ we finally get:

$$\varphi(\infty,\tau) = Z\left\{1 + g\left[\tfrac{1}{\tau}\ln g + \tfrac{2}{\tau}\left(\gamma - \tfrac{1}{2}\ln\tau - \tfrac{1}{2}\right)\right] + \dots\right\} \qquad (6.6)$$

(here $Z = Z_D\, I_0(2\sqrt{gD})$). The exact solution of the Schroedinger equation is derived from Eq. (2.5a) with $n = \lambda = 1/2$:

$$\varphi(\tau) = C\left\{1 + g\left[\tfrac{1}{\tau}\ln g + \tfrac{2}{\tau}\left(\gamma - \tfrac{1}{2}\ln\tau - \tfrac{1}{2}\right)\right] + \dots\right\}. \qquad (6.7)$$

Comparing (6.6) with (6.7) we see that the difference,

$$\varphi(\tau) - \varphi(\infty,\tau) = o(g),$$

is small for sufficiently small g . Thus in this case, the DIM approximation also is sufficiently close to the exact solution.

7. Conclusion

In conclusion we would like to make a remark on possible applications of the methods considered above. The asymptotic perturbation theory may obviously be used for solving singular Bethe-Salpeter and Edwards equations[11,12] but, for the purpose of obtaining modified perturbation series, it is unreasonably comlicated. This enforces us to use the simpler differential-interpolation method. which can be expected, to give modified perturbation-theory terms by use of finite number terms of usual perturbation theory with a cut-off. We have shown the method to be well-founded for the model theory of Schroedinger equation with nonrenormalizable potential. We hope it may be useful for calculating higher-order corrections in nonrenormalizable field theories as well.

It is a pleasure for the authors to express their gratitude to Prof. A.N.Tavkhelidze for his interest in the work and to Prof. B.A.Arbuzov for helpful discussions.

References

1. N.N.Bogolyubov and D.V.Shirkov. Introduction to the Theory of Quantized Fields, Moscow, 1967.
2. A.T.Filippov. Phys.Lett., 9, 78 (1964).
3. A.A.Logunov, A.N.Tavkhelidze. Nuovo Cim., 29, 380 (1963), 30, 134 (1963).
4. V.de Alfaro, T.Regge. Potential Scattering, North-Holland Publishing Company, Amsterdam, 1965.
5. A.T.Filippov. Proc. of the Symposium on non-local field theories at Azau. USSR, 1970. Preprint JINR, 2-5400, Dubna (1970).
6. W.M.Frank, D.I.Land, R.M.Spector Rev.Mod.Phys., 43, 36 (1971).
7. L.D.Landau, A.A.Abrikosov, I.M.Khalatnikov. Docl.Acad.Nauk SSSR, 95, 773; 95, 1177; 96, 261 (1954).
8. G.Feinberg, A.Pais. Phys.Rev., 131, 2724 (1963), 133 B477 (1964).
9. B.L.Ioffe. Žurn. Eksp. Teor. Fiz., 38, 1608 (1960).
10. H.Bateman, A.Erdelyi, Higher Transcendental Functions, v.1., New-York, Toronto, London, 1953.
11. B.A.Arbuzov, A.T.Filippov. Nuovo Cim., 38, 796 (1965).
12. B.A.Arbuzov, A.T.Filippov. Žurn.Eksp.Teor Fiz., 49, 990 (1965).

Received by Publishing Department on September 20, 1971.