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E2 - 6017

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# LADDER APPROXIMATION FOR VERTEX FUNCTION IN NONRENORMALIZABLE THEORY OF $\omega\rho\pi$ INTERACTION

ЛАБОРАТОРИЯ ТЕОРЕТИЧЕСКОЙ ФИЗИКИ

1971

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# LADDER APPROXIMATION FOR VERTEX FUNCTION IN NONRENORMALIZABLE THEORY OF $\omega \rho \pi$ INTERACTION

Submitted to ЯФ

## 1. Introduction

It is well-known that the commonly used classification of field theories into renormalizable and nonrenormalizable ones is obtained by use of perturbation theory, and one may assume that the failure of perturbation theory for nonrenormalizable interactions is due to the impossibility of expanding S-matrix in a series of powers of coupling constant (see e.g. /1/). It is therefore quite possible that by summing perturbative series one may obtain either finite S-matrix or that which may be made finite by adding a number of counterterms to the Lagrangian. So the distinction between renormalizable and nonrenormalizable theories may appear to be the peculiar feature of perturbation theory. This possibility was clearly demonstrated in refs. /2-6/ which considered the ladder-type Edwards /7/ equation for the vertex function. Due to a rather complicated structure of the equation these papers deal with the special case of vanishing 4-momentum of one of the three particles. Here, the most general case of nonvanishing momenta of all three particles is considered, with the example of $\omega\rho\pi$ vertex. The solution of the integral equation is shown to be obtainable by using an iteration method. All the iterations are finite and enable one to obtain the expression for the vertex in terms of a series which in fact is a proper modification of the usual perturbative expansion ( taking into account the nonanalytic dependence on the coupling constant).

## 2. The solution of the equation for the vertex function

The Lagrangian of the $\omega\rho\pi$ interaction has the form

$$\mathcal{L}_{\omega\rho\pi} = g\,\varepsilon_{ijk\ell}\,\partial_i\omega_j\,\partial_k\rho_\ell^\alpha\pi^\alpha\,, \tag{1}$$

where $\alpha$ is the isospin index. The most general representation for the $\omega\rho\pi$ vertex can be written as

$$\Gamma_{ij}^{\alpha\beta}(p,k) = (2\pi)^4 i\,g\,F(p,k)\,\delta^{\alpha\beta}\,\varepsilon_{ijk\ell}\,p_j k_\ell\,. \tag{2}$$

For notations see the figure. Our choice of variables is the most convenient due to the symmetry of F(p,k) under the reflection $k\to -k$.

The ladder-type Edwards equation is represented in the form of diagrams in the figure. The factor $Z$ defines the normalization of the vertex and in what follows it is omitted. The analytic representation of the equation in the Euclidean metrics is

$$F(p,k) = 1 + \int d^4q\, K(p,q;k)\,F(q,k)\,,$$

$$K = \frac{g^2}{(2\pi)^4}\,\frac{[q^2(pk)-(pq)(kq)][p^2(kq)-(pq)(pk)]+\frac{1}{4}[(kq)(k,p+q)-k^2(q,p+q)][(pk)(k,p+q)-k^2(p,p+q)]}{[(pk)^2-p^2k^2][M^2+(k-q/2)^2][M^2+(k+q/2)^2][m^2+(p+q)^2]}\,, \tag{3}$$

where $M_\omega = M_\rho = M$ is the vector boson mass and $m$ is the mass of the pion.

In the Euclidean momentum space we choose the coordinate system with the forth axis along $K_i$. Then

$$d^4q = q^3 dq\, d\Omega_q\,, \quad d\Omega_q = \sin^2\psi_q \sin\vartheta_q\, d\psi_q\, d\vartheta_q\, d\varphi_q\,;$$

$$0 \leqslant \psi_q, \vartheta_q \leqslant \pi\,, \quad 0 \leqslant \varphi_q \leqslant 2\pi\,, \quad 0 \leqslant q \leqslant \infty\,;$$

$$\cos\psi_q \equiv \frac{(Kq)}{\sqrt{K^2q^2}}\,;$$

and the invariant function $F(p,k)$ depending on $p^2$, $(pk)$ and $k^2$ can be written as $F(p,k) = F(p^2, \psi_p)$ ( dropping $k^2$ ).

For $k_1=0$ eq. (3) is quite analogous to that of ref.[2] and can be solved by the same method the essence of which is as follows. The kernel $K$ is broken up into two parts $K = K_0 + K'$, where $K_0$ is the most singular for large p and q:

$$K_0(p,q) = \frac{g^2}{(2\pi)^4}\,\frac{[q^2(pk)-(pq)(kq)][p^2(kq)-(pq)(pk)]}{[(pk)^2-p^2k^2]\, q^4\, (p+q)^2}\,. \qquad (4)$$

Then we solve the equation

$$F(p^2) = f(p^2) + \int d^4q\, K_0\, F\,, \qquad (5)$$

with

$$f(p^2) = 1 + \int d^4q\, K'F \,, \quad K' = K - K_0 \,. \tag{6}$$

Introducing the notation

$$\int d\Omega_q\, K_0(p,q) = K_{00}(p^2,q^2) \tag{7}$$

one may rewrite eq. (5) as

$$F(p^2) = f(p^2) + \int_0^\infty dq\, q^3\, K_{00}(p^2,q^2)\, F(q^2)\,, \tag{8}$$

with

$$\int_0^\infty dq\, q^3\, K_{00}(p^2,q^2) F(q^2) = \frac{\lambda^2}{12}\left\{ \int_0^x dy \left(\frac{y^2}{x^2} - 2\frac{y}{x}\right) F(y) + \int_x^\infty dy \left(\frac{x^2}{y^2} - 2\frac{x}{y}\right) F(y) \right\}, \tag{9}$$

$$\lambda^2 = \frac{g^2}{32\pi^2}\,, \; p^2 = x\,, \; q^2 = y\,.$$

By differentiating, eq. (8) is reduced to the differential equation, the Green's function of which can be easily found. The resolvent kernel, $R(p^2,q^2)$, of the integral equation (8) may be easily expressed in terms of the Green's function/2/and so eq. (8) is equivalent to

$$F(p^2) = \int_0^\infty dq^2\, R(p^2,q^2)\, f(q^2)\,,$$

or

$$F(p^2) = F_0(p^2) + \int_0^\infty dq^2\, R(p^2,q^2)\, K'F\,. \tag{10}$$

Here $F_0(p^2)$ is the solution of eq. (8) for $f(p^2)=1$.

$$F_0(x) = 4/\lambda^2 x \; - 2\, G^{30}_{04}(\lambda^2 x \,|\, 2,1,-1,-2)\,, \tag{11}$$

and $G^{30}_{04}$ is the Meijer function [8]. Unlike eq. (3), the integral equation (10) is regular enough to be solved by iterations. For large $p^2$ all iterations decrease faster than $F_0(p^2)$ and so the asymptotic behaviour of $F(p^2)$ (as $p^2 \to \infty$) coincides with that of $F_0(p^2)$ ( for the details see ref. [2]) .

In the case $K_1 \neq 0$ eq. (3) can be solved by the same method.

The simplest way to do so is to choose the most singular kernel in the form $K_{oo}(p^2, q^2)\, \delta(\psi_p - \psi_q)$, i.e. to rewrite eq. (3) as follows

$$F(p^2, \psi_p) = f(p^2, \psi_p) + \int dq\, d\psi_q\, q^3 K_{oo}(p^2, q^2)\, F(q^2, \psi_q)\, \delta(\psi_p - \psi_q). \qquad (12)$$

The inhomogeneous term now is

$$f(p^2, \psi_p) = 1 + \int d^4q\, K_1 F \, , \qquad (13)$$

where

$$\int d^4q\, K_1 F = \int d^4q\, K F - \int dq\, q^3 K_{oo} F \Big|_{\psi_p = \psi_q} =$$

$$= \int d^4q\, K' F + \int d^4q\, K_o F - \int dq\, q^3 K_{oo} F \Big|_{\psi_p = \psi_q} . \qquad (14)$$

The angular variables $\psi_p, \vartheta_p, \varphi_p$ in eq. (12) are considered to be free parameters and therefore this integral equation is in fact one-dimensional. Thus, the method developed for the case $k_1=0$ is applicable and eq. (12) can be solved by reducing it to the differential equation. It is not hard to verify that its solution is

$$F(p,\kappa) = \int dq^2\, R(p^2,q^2)\, f(q,\kappa)\Big|_{\psi_p=\psi_q} .$$

Putting in here eq. (13) we find the integral equation.

$$F(p,\kappa) = F_o(p^2) +$$
$$+ \int d^4q'\, dq^2\, R(p^2,q^2)\, K_1(q,q')\Big|_{\psi_q=\psi_p} \cdot F(q',\kappa) , \qquad (15)$$

which is equivalent to eq. (3) but much less singular. In fact, all iterations of eq. (15) are finite due to the nice asymptotic properties of $F_o(p^2)$ and of $R(p^2,q^2)$. The iterative solution of eq. (15) can be written as

$$F(p,k) = F_o(p^2) + \sum_{n=1}^{\infty} F_n(p,k) , \qquad (16)$$

where for $p^2 \to \infty$ $F_n$ decrease faster than $F_o(p^2)$. So it is reasonable to expect that, at least for $p^2 \to \infty$, the series (16) converges and the asymptotic behaviour of $F(p,k)$ is identical to that of $F_o(p^2)$ .

3. Expansion of F(p,k) in a modified perturbation series

It is not hard to verify ( cf. ref.[2]) that all corrections of the order of $\lambda^2$ are included in the zeroth and first iterations

$$F(p,k) \simeq F_o(p^2) + F_1(p,k) =$$
$$= F_o(p^2) + \int d^4q'\, dq^2\, R K_1 F_o . \qquad (17)$$

Since $F_0(p^2)$ does not depend on angular variables one easily finds ( using eqs. (14) and (17) ) that

$$\int d^4q' \, K_1 F = \int d^4q' \, K' F_0 \, ,$$

where $K' = K - K_0$ . It is not difficult to check that

$$\int d^4q' dq^2 R(p^2,q^2) K'(q,q') \Big|_{\varphi_q=\varphi_p} F_0(q'^2) = \int d^4q \, K'(p,q) F_0(q^2) + o(\lambda^2), \tag{18}$$

and using eq. (17) we find

$$F(p,k) = F_0(p^2) + \int d^4q \, K' F_0 + o(\lambda^2) \, . \tag{19}$$

Now, employing eq. (11) we obtain

$$F_0(x) = 1 + \frac{\lambda^2 x}{6}\left(\ln \lambda^2 x + 4\gamma - \frac{10}{3}\right) + o(\lambda^2), \tag{20}$$

where $\gamma = 0.577...$ is the Euler constant. The second term in the right member of eq. (19) diverges logarithmically if we replace $F_0$ by 1. So, to calculate its contribution to the $\lambda^2$- order term of the modified perturbation theory we employ the following trick. First, we extract from $K'$ the most singular part $K'_\infty$,

$$K' = K'_\infty + K'', \tag{21}$$

and calculate $\int d^4q \, K'_\infty F_0$ exactly. Then the term $\int d^4q \, K'' F_0$ is calculated approximately by substituting $F_0$ by 1. Taking $K'_\infty$ as

$$K'_\infty = -\frac{\lambda^2}{6\pi^2}\cdot\frac{7}{4}\kappa^2(q^2+M^2)^{-2}, \qquad (22)$$

one may easily verify that, for calculating $\int d^4q\, K'_\infty F_0$, it is sufficient to find the integral /8/

$$\int_0^\infty dy \frac{y}{(y+M^2)^2}\, G^{30}_{04}(\lambda^2 y \,|\, 2,1,-1,-2) = G^{41}_{15}\left(\lambda^2 M^2 \,\middle|\, {-1 \atop 0,2,1,-1,-2}\right). \qquad (23)$$

For small $\lambda^2$

$$G^{41}_{15}\left(\lambda^2 M^2 \,\middle|\, {-1 \atop 0,2,1,-1,-2}\right) = \frac{2}{\lambda^2 M^2} + \frac{1}{2}\ln\lambda^2 M^2 + 2\gamma - \frac{5}{4} + O(\lambda^2), \qquad (24)$$

and finally we get

$$\int d^4q\, K'_\infty F_0 = \frac{7}{24}\lambda^2\kappa^2\left(\ln\lambda^2 M^2 + 4\gamma - \frac{5}{2}\right) + o(\lambda^2). \qquad (25)$$

Now, substituting eqs. (20), (21) and (25) into eq. (19) we arrive at the modified perturbation series

$$F(p,k) = 1 + \frac{\lambda^2 p^2}{6}\left(\ln\lambda^2 p^2 + 4\gamma - \frac{10}{3}\right) + \frac{7}{24}\lambda^2\kappa^2\left(\ln\lambda^2 M^2 + 4\gamma - \frac{5}{2}\right) + \int d^4q\, K''(p,q) + o(\lambda^2). \qquad (26)$$

It is of interest to compare this expression with that of triangle diagram, calculated with a cut-off for $q^2 \geqslant \Lambda$. The latter is identical to the first iteration of eq. (3):

$$\int^{\Lambda} d^4q\, K(p,q) = \int^{\Lambda} d^4q\, K_0 + \int^{\Lambda} d^4q\, K'_{\infty} + \int^{\Lambda} d^4q\, K'' =$$

$$= \frac{\lambda^2 p^2}{6}\left(\ln\frac{p^2}{\Lambda} + \frac{1}{6}\right) + \frac{7}{24}\lambda^2 K^2\left(\ln\frac{M^2}{\Lambda} + 1\right) + \int^{\Lambda} d^4q\, K'' + O(1/\Lambda). \tag{27}$$

Comparing eq. (26) with eq. (27) we find that

$$F(p,k) = 1 + \int^{\Lambda_0} d^4q\, K(p,q) + o(\lambda^2), \tag{28}$$

where

$$\Lambda_0 = \frac{1}{\lambda^2} \exp\left(\frac{7}{2} - 4\gamma\right). \tag{29}$$

Eq. (29) is in fact the "unitary cut-off parameter" which, from general considerations, must have the form $\Lambda = C/\lambda^2$. The value of C obtained above enables us to calculate the terms $\sim\lambda^2$ exactly and to estimate the higher-order terms [x]. ( We suppose to exploit this possibility elsewhere).

Now, it is not hard to prove the unitarity relation for the $\lambda^2$- order correction. Indeed, the imaginary part of eq. (28) is equal to that of the triangle diagram and thus it is given by the unitarity relation. It does not make sense to discuss the unitarity relations for higher-order terms since we considered only ladder type diagrams.

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[x] The usual unitary cut-off gives only the term $\sim\lambda^2 \ln\lambda^2$ which is not sufficient to estimate the $\lambda^2$- order correction.

Eq. (26) may be expressed as a series of powers of external momenta. Introducing the notations $p_1^2 = (p-k/2)^2 = x$ , $p_2^2 = (p+k/2)^2 = x$ , we get ( with $m^2 = 0$, $M^2 = 1$)

$$F(x_1, x_2, k^2) = \frac{\lambda^2}{12}\Big[\Big(\ln\lambda^2 + 4\gamma - \frac{17}{6}\Big)(x_1 + x_2) + \Big(3\ln\lambda^2 + 12\gamma - \frac{13}{2}\Big)k^2 +$$

$$+ \frac{1}{4}(x_1 + x_2)^2 + \frac{1}{12}(x_1 - x_2)^2 + \frac{1}{2}(x_1 + x_2)k^2 + \frac{1}{3}k^4 + \ldots\Big] + o(\lambda^2),$$

where $\lambda^2 M^2 = g^2 M^2 / 32\pi^2 \cong 0.8$. From this expression the important conclusion concerning a rather rapid variation of the vertex function ( with external momenta) is drawn. This fact was used in ref. /10/ for improving the vector-dominance model for the decays

$$\omega \to 3\pi, \quad \omega \to \pi\gamma \;, \quad \pi^0 \to \gamma\gamma \,.$$

The useful discussions with Prof. B.A. Arbuzov are kindly acknowledged.

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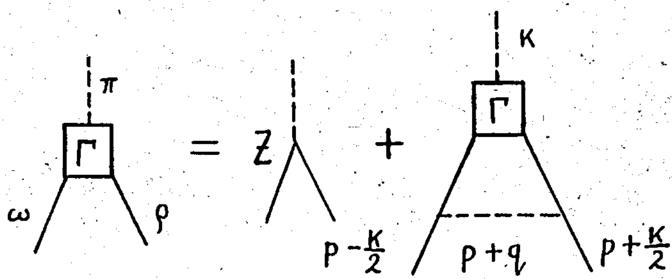
Received by Publishing Department

on September 1, 1971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