СООБЩЕНИЯ
ОБЪЕДИНЕННОГО
ИНСТИТУТА
ЯДЕРНЫХ
ИССЛЕДОВАНИЙ
Дубна

E2 - 5976

E.P. Solodovnikova, A.N. Tavkhelidze,
O.A. Khrustalev

ЛАБОРАТОРИЯ ТЕОРЕТИЧЕСКОЙ ФИЗИКИ

THE BOGOLUBOV TRANSFORMATION IN STRONG COUPLING THEORY

1971

E2 - 5976

E.P. Solodovnikova, A.N. Tavkhelidze,
O.A. Khrustalev

THE BOGOLUBOV TRANSFORMATION IN STRONG COUPLING THEORY

1. Introduction

A consistent dynamic description of a system of interacting fields requires, first of all, knowledge of the symmetry properties of this system. This implies that we ought to indicate the group of transformations G under which the Hamiltonian of the system remains invariant and find the integrals of motion which are generated by G group transformations. Then, any calculation scheme should be construoted so that the corresponding conservation laws might not be violated.

Any Hamiltonian of modern field theory consists, at least, of three parts

$$\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2 + V . \tag{1}$$

If a system is described by the fields ψ and φ then the term $\mathcal{H}_1$ is constructed only out of the field ψ, the term $\mathcal{H}_2$ out of the field φ and the operator V depends simultaneously upon both the fields ψ and φ. The fields ψ and φ realize the representations of the group G, and each term in the right-hand side of (1) is the G group invariant. The operators $\mathcal{H}_1$ and $\mathcal{H}_2$ are naturally invariant under transformations of the fields ψ and φ, separately and are associated with free field Hamiltonians. The operator V which is associated with the interaction operator may be invariant of G group provided that the fields ψ and φ transform in common.

In order to study the properties of the system with Hamiltonian (1) it is necessary, first of all, to find the eigenveotors of the Hamiltonian which realize the representation of the group G. Then, we can reduoe commuting integrals of motion to c - numbers and consider the system states with definite values of the invariants. The arbitrary state of the system can be represented as a superposition of states with definite values of the integrals of motion. Certainly, due to technical difficulties, we imply here a more **or less satisfactory approximation to the true state** vector rather than an exact eigenvector of the operator (1). Nevertheless any approximation to the eigenvector of operator (1) must be the representation of group G since, otherwise, the failure of the required conservation laws makes it impossible to judge the degree of the approximation to the exact state vector. It is thus convenient to choose the initial state-vector such that the individuality of the fields ψ and φ will be preserved as much as possible, that is, the possibly simplest corpuscular interpretation of the field states compatible with the required conservation laws will be provided.

In the case of the Hamiltonian

$$\mathcal{H}_1 + \mathcal{H}_2 \tag{2}$$

such a problem causes no difficulties. The eigenvectors of the operator (2) are constructed out of the products of the eigenvectors $\mathcal{H}_1$ and $\mathcal{H}_2$ each of which realizing a certain representation of the group G. Thus, the eigenvectors of the operator (2) transform under the representation of the group G which ensures the validity of the corresponding conservation laws and, at the same time, the **multiplicative** structure of the eigenvectors provides the conservation of the individuality of the fields

ψ and φ. Certainly this is only a mathematical expression of the physical notion of noninteracting fields.

If the operator V in the Hamiltonian (1) may be thought of as a small perturbation then as the first approximation to the eigenvectors we may take the eigenvectors of the operator (2) and construct the stationary states of the system by perturbation methods. Thus, the problem reduces to the construction of a perturbation theory invariant under the G group. It may be of any difficulty but nevertheless it is a purely technical problem.

The situation is quite another in the case of strong coupling when the operator V is to be taken into account already in the first approximation. The products of the eigenvectors of the operators $\mathcal{H}_1$ and $\mathcal{H}_2$ may not be the ones of the Hamiltonian (1). Therefore the principal problem of strong coupling theory consists in a suitable choice of the first approximation to the ground state vector since it is unclear how to make the covariance of the state vectors under transformation of the group keeping invariant the total Hamiltonian (1) compatible with the individuality of the fields ψ and φ.

This problem was solved in principle by N.N.Bogolubov more than twenty years ago [1]. He has shown how it is possible, not coming into disagreement with conservation laws, to extract the ground state of the system and develop a scheme of successive approximations to the exact vector of the system with Hamiltonian (1) without the assumption on the smallness of the term V compared to both $\mathcal{H}_1$ and $\mathcal{H}_2$.

The Bogolubov method consists in a canonical transformation of Hamiltonian (1) taking into account its group properties. Under this transformation, among new variables describing the system there are some generalized coordinates the canonical momenta of which coincide with the operators of the integrals of motion which are defined by the Hamiltonian group properties. Owing to the fact that the Hamiltonian is invariant under the given group G these coordinates are found to be cyclic which ensures the fulfilment of conservation laws. The introduction of such superfluous variables requires naturally some additional conditions to be imposed on the variables compatible in the new representation with the fields ψ and φ. The choice of the additional conditions is suggested by the following arguments: since the term V in Hamiltonian (1) describes the interaction of two fields it necessarily leads to a violation of the particle number conservation law of one of the field, say φ. Therefore when the interaction is taken into account the stationary state of the system is necessarily the one with indefinite number of particles ψ. By impositing the simplest linear conditions on the field variables φ we change only slightly the structure of the virtual cloud around the field particles ψ the number of which is kept also when the interaction is switched on. After the mentioned canonical transformation has been performed the ground state wave function is the product of three functions depending only on the listed variables separately.

The desired constancy of the integrals of motion and the individuality of particles, the number of which must also be the integral of motion of Hamiltonian (1), has thus been achieved. After this the scheme of successive approximations, the detailed description of which is given in ref.[1], may be developed.

2. Strong Interaction of a Nonrelativistic Particle with a Scalar Field

We consider the simplest examples of realization of the Bogolubov transformation. The systems, in which the classic particle interacting with a scalar field plays the role of the field ψ , are viewed. To the field ψ there corresponds the particle radius-vector. The Hamiltonian of the system is invariant under the Abelian group of translations to which there corresponds the conserved total momentum vector. We assume for simplicity that the system is enclosed in a cube and the periodicity conditions are fulfilled. In the nonrelativistic case, taking into account zero field oscillations in the Schroedinger representation the Hamiltonian has the form

$$\mathcal{H} = \frac{1}{2\mu}\bar{p}^2 + g\sum a_f e^{i\bar{f}\bar{r}} b_f + \overset{*}{a}_f e^{-i\bar{f}\bar{r}} b_f^+ + \frac{1}{2}\sum \omega_f (b_f^+ b_f + b_f b_f^+). \quad (3)$$

To the operator $\mathcal{H}_1$ there corresponds the operator $\frac{1}{2\mu}\bar{p}^2$ and to $\mathcal{H}_2 - \frac{1}{2}\sum \omega_f (b_f^+ b_f + b_f b_f^+)$ the numbers a_f obey the reality condition $\overset{*}{a}_f = a_{-f}$. The system with Hamiltonian (3) is given in ref.[1] in which the case of adiabatic coupling was specially considered when the scalar field frequencies are supposed to be proportional to the

small parametre ε^2 and the coupling constant to ε
In ref. [2] it is shown that the method suggested in ref.[1] is directly extended to the case of a pure strong coupling without the assumption on the smallness of the field frequencies and the constant coupling is put to be muoh larger than unity. In so doing, it turns out that within the strong coupling limit the interaction of a partiole with the field leads to the appearance of discrete oscillator levels. Here we repeat briefly the arguments of refs. [1,2] focusing our attention on the group aspect of the problem.

The variables $\bar{z}$ and b_f realize the representation of the translation group

$$\bar{z} \to \bar{z} + \bar{q}, \quad b_f \to b_f e^{-i f q}, \tag{4}$$

where $\bar{q}$ is a certain constant vector. The Hamiltonian (3) is invariant under transformations (4), therefore the operator

$$\bar{p} = -i\hbar \frac{\partial}{\partial \bar{z}} + \hbar \sum \bar{f} b_f^+ b_f \tag{5}$$

commutes with Hamiltonian (3) which corresponds to the total momentum conservation. The eigenvectors of the operator $-i\hbar\frac{\partial}{\partial \bar{z}}$

$$\psi = V^{-1/2} e^{i\frac{\bar{p}\bar{z}}{\hbar}} \tag{6}$$

realize the translation group representation

$$\psi' = e^{i\frac{\bar{p}\bar{q}}{\hbar}} \psi \tag{7}$$

in just the same way as the eigenvectors of the operator

$$\hbar \sum \bar{f} b_f^+ b_f$$

$$\theta_n = \sum b_{f_1}^+ \dots b_{f_n}^+ |0\rangle\, \delta(\hbar \sum f_e - \bar{p}'). \tag{8}$$

Therefore the products of the vectors (6) and (8) realize also the representation of the group of translations

$$(\theta_n \psi)' = e^{i\frac{p+p'}{\hbar}\bar{q}}\, \theta_n \psi \tag{9}$$

and correspond to the state of systems with definite total momentum $\bar{P} = \bar{p} + \bar{p}'$.

The state-vectors (9) may not be the eigenvectors of operator (3) and in the case of strong coupling they cannot be taken as the first approximation to the true eigenvectors. To find the eigenvector of the operator (3) which realizes the representation of translation group in ref.[1] it was suggested to consider the vector $\bar{q}$ as a new variable independent of $\bar{z}$ and obtained as a result of canonical transformation

$$\{\bar{z}, b_f\} \to \{\vec{\lambda}, \bar{q}, \tilde{b}_f\}. \tag{10}$$

The momentum corresponding to the variable $\bar{q}$ must be the operator of the total momentum of the system

$$-i\hbar \frac{\partial}{\partial \bar{q}} = P. \tag{11}$$

The operators $\bar{P}$, $\bar{z}$, b_f and b_f^+ satisfy the following commutation relations

$$[P_\alpha\ z_\alpha] = -i\hbar\, \delta_{\alpha\beta} \tag{12}$$

$$[\bar{P}, b_f] = -\hbar \bar{f} b_f \quad , [\bar{P}_f, b_f^+] = \hbar \bar{f} b_f^+ . \tag{13}$$

From relations (12) and (13) and condition (11) it follows an explicit dependence of the operators $\bar{z}, b_f, b_f^+$ on the variable $\bar{q}$

$$\bar{z} = \bar{C} + \bar{q} , \tag{14}$$

$$b_f = e^{-i\bar{f}\bar{q}} \tilde{b}_f , \qquad b_f^+ = e^{i\bar{f}\bar{q}} \tilde{b}_f^+ . \tag{15}$$

If the vector $\bar{q}$ is displaced by the constant vector $\bar{\alpha}$ then $\bar{z}$, b_f and b_f^+ transform according to the law (4) which realizes the representation of translation group. The vector $\bar{C}$ which in equality (14) has the meaning of the integration constant is identified with the new independent variable

$$\bar{C} = \frac{1}{\varkappa} \bar{\lambda} . \tag{16}$$

Then the particle momentum operator takes the form

$$-i\hbar \frac{\partial}{\partial \bar{z}} = -i\hbar\varkappa \frac{\partial}{\partial \lambda} . \tag{17}$$

The value of the constant $\varkappa$ will be determined later on. Note that, following the meaning of the transformation (10) the variables $\bar{\lambda}$ and $\tilde{b}_f$ remain unaffected under translations. Before to find expliditly the complete transformation (5), we note that the operators $\tilde{b}_f$ and $\tilde{b}_f^+$ can no longer contain explicitly the variable q since otherwise the commutation relations (13) would be violated. Therefore the total Hamiltonian of the system (3) is also independent explicitly of $\vec{q}$. The eigenfunctions of the operator (3) can be presented in the form

$$\Psi(\bar{\lambda}, q, \tilde{b}_f) = e^{i\frac{Pq}{\hbar}} \varphi(\bar{\lambda}, b_f). \tag{18}$$

These functions realize the representation of translation group and correspond to the state of the system with definite total momentum $\bar{P}$.

To derive an explicit dependence of the operators $\tilde{b}_f$ and $\tilde{b}_f^+$ on the new variables we notice that in the case of strong coupling in the first approximation the energy of the free scalar field may be neglected. In this case the Hamiltonian of the system is linear in the operators b_f and b_f^+

and the Heisenberg equations of motion for these operators do not permit to identify them with the creation and annihilation operators of real particles transferring energy and momentum. This is most easily seen by passing from the operators b_f and b_f^+ to the complex coordinates and momenta

$$q_f = \frac{b_f + b_{-f}^+}{\sqrt{2}}, \quad p_f = i\frac{b_f^+ + b_{-f}}{\sqrt{2}}. \tag{19}$$

In our approximation Hamiltonian (3) gives just this solution for the Heisenberg equations of motion :

$$q_f(t) = u_f = const. \tag{20}$$

The order of magnitude of the coefficients u_f can be estimated by a simple variational calculation performed in ref. [2] . The energy minimum of the system was sought there among the trial functions corresponding to the coherent states of the scalar field. It is found that the energy minimum is achieved in the case when the coefficients u_f become proportional to g . This fact contributes to the choice of the operators $\tilde{b}_f$. It is convenient, first of all, to pass from the operators b_f to complex coordinates (19) normalizing this transformation in such a way that the numbers u_f , the constant component of new variables, could be considered as zero order quantities in the coupling constant and the new variables Q_f taking into account the free scalar field energy effect as small quantities. This is reached by a certain modernization of transformation (19)

$$q_f = \frac{1}{g}\,\frac{b_f + b_{-f}^{+}}{\sqrt{2}} \qquad p_f = ig\,\frac{b_f^{+} - b_{-f}}{\sqrt{2}} \tag{21}$$

and a subsequent replacement of the variables

$$q_f = e^{-i f \bar{q}}\left(u_f + \frac{1}{g} Q_f\right). \tag{22}$$

Three auxiliary conditions should be imposed on the variables Q_f since in transformation (10) instead of the three components of the vector r there appeared six components of the vectors λ and q . Without loss of generality, as the additional conditions we may take the following linear conditions

$$\sum f v_f Q_f = 0 . \tag{23}$$

After this we have only to express the momenta $p_f = -i\frac{\partial}{\partial q_f}$ in terms of the variables $\bar{q}$, $\bar{\lambda}$ and Q_f. This procedure will be described later on in analysing the interaction of a relativistic particle with a scalar field. Now we dwell up on the properties of the ground state of the system with Hamiltonian (3) mentioned in refs. [1,2]. First of all, we note that in the transformed Hamiltonian (3) there appears a large term which plays the role of the potential

$$g^2 \sum a_f \sqrt{2}\, u_f e^{i\frac{f\lambda}{\varkappa}}. \tag{24}$$

The terms containing the operators Q_f and the corresponding momenta p_f enter the Hamiltonian as a series in decreasing powers of g, the highest order in this series being unity. The kinetic particle energy defined by the expression (17) is proportional to $\varkappa^2$.
To take into account the kinetic energy already in the first order the constant should be taken large, Then the exponential in the expression for potential energy (24) can be expanded into a series in inverse powers of $\varkappa$. In so doing, the term linear in $\bar{\lambda}$ vanishes and the variable part of the potential will be of order $g^2/\varkappa^2$. It will be a quantity comparable with the kinetic energy when we choose

$$\varkappa = 1/\sqrt{g}. \tag{25}$$

The terms of order g in the Hamiltonian (3) are [1,2]

$$\mathcal{H}_1(\bar{\lambda}) + \mathcal{H}_2(Q_f, P_f) , \qquad (26)$$

where $\mathcal{H}_2$ is the operator linear in Q_f and P_f .
The expression (26) shows that the wave function of the system in the first approximation allows the separation of the variables, and is of the form

$$\Psi(\bar{\lambda}, \bar{q}, Q_f) = e^{i\frac{\bar{P}\bar{q}}{\hbar}} \varphi_0(\lambda)\, \theta_0(Q_f) . \qquad (27)$$

Thus, the Bogolubov transformation (14), (15) solves really the principal problem of strong coupling theory, namely the separation of the particle and field coordinates with simultaneous conservation of the needed transformation properties of the wave function.

Owing to the choice of the variables in the form (22), where the variables Q_f are small quantities, the operator $\mathcal{H}_2$ in (26) is linear in Q_f and P_f . The problem of the eigenvalues for these operators, provided the eigenfunction is regular, has a solution only when the operator vanishes identically [1] . The condition

$$\mathcal{H}_2(Q_f, P_f) \equiv 0 \qquad (28)$$

allows to find the numbers u_f and define completely the effective particle potential.

In the case of strong coupling when the expansion in $\varkappa^{-1}$ is possible this potential reduces to the oscillator one, since in expression (24) in the first approximation we may restrict ourselves to the terms quadratic in $\bar{\lambda}$.
In ref. [2] it was shown that the account in the Hamiltonian

of the terms of order $g^{1/2}$, which are equal to

$$i g^{1/2} \sum a_f(\lambda f) Q_f , \tag{29}$$

leads to transitions between the states with wave functions (27) and a **broadening of the oscillator levels** by a value comparable with the spacings between them. However, it is not difficult to diagonalize in λ and Q the bilinear form consisting of the operators (26), (29) and the free scalar field energy operator. Due to the smallness of the operator (29) compared to the operator (26), the appropriate canonical transformation leads to small energy level shifts. The spacing between new levels is as before propertional to g, while their width is proportional to the zero power of g. Thus, the wave functions (27) at least in the first approximation reproduce correctly the properties of stationary states, the improvement of the true physical picture after the Bogolubov transformation being achieved by perturbation theory methods even in the case of strong coupling.

3. The Relativistic Particle in the Scalar Field

As the second example of the application of the Bogolubov transformation we oonsider the Lorentz-invariant interaction of a olassic particle with a scalar field. We are interested only in the first approximation with respect to the coupling constant. To preserve Lorentz-invariance it is convenient to start from the Klein-Gordon equation

$$\left[-\frac{\partial^2}{\partial t^2} + \nabla^2 - m^2 + g\varphi(x)\right]\psi = 0 \tag{30}$$

in which $\varphi(x)$ is the free scalar field operator in the Heisenberg representation. Assuming again that the system is enclosed into a cube of finite volume we present $\varphi(x)$ in the form

$$\varphi(x) = \sum_f A_f \left(e^{-ifx} b_f + e^{ifx} b_f^+\right), \tag{31}$$

where the scalar product will be further expressed in terms of the covariant components of the vector x and the countervariant components of the momentum

$$fx = \sum_{\alpha=0}^{3} f^\alpha x_\alpha . \tag{32}$$

The coefficients of expansion (31) A_f contain cutoff factors which ensure in a suitable manner convergence and Lorentz-invariance. The operator acting on the wave function in eq. (30) is invariant under transformations

$$x_\alpha \to x_\alpha + q_\alpha , \; b_f \to e^{ifq} b_f . \tag{33}$$

To find the wave function realizing the representation of the translation group we perform a transformation similar to transformation (14)

$$x_\alpha = \frac{1}{\sqrt{g}} \lambda_\alpha + q_\alpha . \tag{34}$$

The components of the vector q_α are defined by the conditions

$$\frac{\partial}{\partial x_\alpha} = \sqrt{g} \frac{\partial}{\partial \lambda_\alpha} \tag{35}$$

$$-i\frac{\partial}{\partial q_\alpha} = p^\alpha, \qquad (36)$$

where p^α are the components of the four-momentum operator. To find the explicit formulas for the scalar field transformation we introduce, as before, the complex coordinates z_f, $\overset{*}{z}_f$

$$b_f = \frac{1}{\sqrt{2}}\left\{ g z_f + \frac{1}{g}\frac{\partial}{\partial \overset{*}{z}_f} \right\} \qquad (37)$$

$$b_f^+ = \frac{1}{\sqrt{2}}\left\{ g \overset{*}{z}_f - \frac{1}{g}\frac{\partial}{\partial z_f} \right\} \qquad (38)$$

and then pass to the variables Q_f

$$z_f = e^{ifq}\left(u_f + \frac{1}{g} Q_f \right). \qquad (39)$$

The quantities u_f and Q_f in eq. (39) may be thought of as being real

$$\overset{*}{u}_f = u_f \quad \overset{*}{Q}_f = Q_f . \qquad (40)$$

Four additional conditions should be imposed on the variables Q_f . We put

$$\sum_f N_{\alpha f} Q_f = 0 , \qquad (41)$$

where $N_{\alpha f}$ is a certain four-row matrix with real elements. It is always possible to find a four-column matrix $M_{f\alpha}$ obeying the condition

$$\sum_f N_{\alpha f} M_{f\beta} = \delta_{\alpha\beta} . \qquad (42)$$

Of matrices $\mathcal{N}_{\alpha f}$ and $\mathcal{M}_{f\alpha}$ it is possible to construct a quadratic **idempotent** matrix $\mathcal{A}_{ff'}$

$$\mathcal{A}_{ff'} = \delta_{ff'} - \sum_{\beta} \mathcal{M}_{f\beta} \mathcal{N}_{\beta f'} \tag{43}$$

satisfying the relations

$$\sum_{f} \mathcal{N}_{\alpha f} A_{ff'} = 0, \tag{44}$$

$$\sum_{f'} \mathcal{A}_{ff'} \mathcal{M}_{f'\alpha} = 0, \tag{45}$$

$$\sum_{f'} \mathcal{A}_{ff'} \mathcal{A}_{f'e} = \mathcal{A}_{fe}. \tag{46}$$

By means of the matrix $\mathcal{A}_{ff'}$ the variables Q_f can be represented as a linear combination of oertain independent variables $\mathcal{T}_e$

$$Q_f = \sum \mathcal{A}_{fe} \mathcal{T}_e \tag{47}$$

the additional conditions (41) will then be satisfied automatically.

Since in determining the matrices $\mathcal{N}_{\alpha f}$ and $\mathcal{M}_{f\alpha}$ we are interested only in their rank, we can choose, without loss of generality, the matrix $\mathcal{M}_{f\alpha}$ in the form

$$\mathcal{M}_{f\alpha} = \mathcal{U}_f \, \delta^{\alpha}. \tag{48}$$

After this it is not difficult to find a partial derivative Q_f of z_e

$$\frac{\partial Q}{\partial z_e} = g A_{fe} e^{-ilq} + \sum \frac{\partial q_\alpha}{\partial z_e}\left(-i \sum_p l^\alpha Q_p A_{fp}\right) \tag{49}$$

and express the momentum operator $p_e = -i\frac{\partial}{\partial z_e}$ in terms of new variables λ, q and Q_f:

$$p_e = g e^{-ilq} P_e + \sum_\alpha \frac{\partial q_\alpha}{\partial z_e}\left(-i\frac{\partial}{\partial q_\alpha} + i\sqrt{q}\frac{\partial}{\partial \lambda_\alpha} - i \sum t^\alpha Q_f P_f\right), \tag{50}$$

where the operators P_e are the momenta corresponding to the vaiables

$$P_e = -i\sum_f \frac{\partial}{\partial Q_f} A_{fe} . \tag{51}$$

These operators satisfy complementary conditions

$$\sum_e P_e M_{e\alpha} = 0 . \tag{52}$$

To determine completely the operator P_e we have to calculate the derivative $\frac{\partial q_\alpha}{\partial z_e}$. To this end we differentiate the relation (41) writting in the form

$$\sum N_{\alpha f}\left(z_f e^{-ifq} - u_f\right) = 0 . \tag{53}$$

Using the properties of the matrices $N_{\alpha f}$ and $M_{f\alpha}$ it is easy to obtain the relation

$$N_{\alpha e} e^{-ilq} - i\frac{\partial q_\alpha}{\partial z_e} - i\frac{1}{g}\sum_{\beta f} N_{\alpha f} f^\beta \frac{\partial q_\beta}{\partial z_e} Q_f = 0 . \tag{54}$$

Representing the desired partial derivative as

$$\frac{\partial q_\alpha}{\partial z_e} = -i\, B_{\alpha e}\, e^{-i e q} \tag{55}$$

we reduce eq. (54) to

$$N_{\alpha e} - B_{\alpha e} - \frac{1}{g}\sum_{\beta f} N_{\alpha f}\, f^{\beta} B_{\beta e}\, Q_f = 0 \,. \tag{56}$$

The relation (56) allows to find the matrix $B_{\alpha e}$ in the form of a series in inverse powers of the coupling constant. To obtain the equation for the ground state of the particle it is sufficient to take into account two first terms of this series

$$B_{\alpha e} \simeq N_{\alpha e} - \frac{1}{g}\sum_{f\beta} N_{\alpha f}\, f^{\beta} N_{\beta e}\, Q_f \,. \tag{57}$$

Using relation (55) the momentum P_e can be written in the form

$$P_e = e^{-i e q}\left\{ g\, p_e - i\sum_{\alpha} B_{\alpha e}\left(-i\frac{\partial}{\partial q_\alpha} + i\sqrt{g}\,\frac{\partial}{\partial \lambda_\alpha} - i\sum f^{\alpha} Q_f\, p_f\right)\right\}. \tag{58}$$

Defining in a similar way the variable $\overset{*}{z}_e$:

$$\overset{*}{z}_e = e^{-i e q}\left(u_f + \frac{1}{g} Q_f\right) \tag{59}$$

and the corresponding momentum $\overset{*}{P}_e = -i\,\frac{\partial}{\partial \overset{*}{z}_e}$ we get for the momentum $\overset{*}{P}_e$ an analogous expression

$$\overset{*}{P}_e = e^{i e q}\left\{ g\, p_e + i\sum_{\alpha} B_{\alpha e}\left(-i\frac{\partial}{\partial q_\alpha} + i\sqrt{g}\,\frac{\partial}{\partial \lambda_\alpha} + i\sum f^{\alpha} Q_f\, p_f\right)\right\}. \tag{60}$$

The formulas defining the variables z_e, $\overset{+}{z}_e$ and the corresponding momenta in the transition to new variables show that the operator of the left-hand side of eq. (30) does not contain the variable q_α. Therefore the solution of this equation can be presented in the form

$$\psi(\lambda, q, Q) = e^{iP^\alpha \lambda_\alpha} \Theta(\lambda, Q) \tag{61}$$

by identifying P_α with the total four-momentum vector. The wave function (61) realizes the representation of the group of translations of space-time. From definition (37), (38) of the creation and annihilation operators in terms of the variables z_f and $\overset{+}{z}_f$ and the connection of these variables with the variables q_α, λ_α, Q_s it follows that the operators b_f and b_f^+ contain a large c-number component proportional to g therefore the total momentum of the system will be a quantity proportional to g^2. We define

$$P^\alpha = g^2 J^\alpha. \tag{62}$$

Then the variable of q_α is everywhere replaced by the vector $+ig^2 J^\alpha$ and eq. (30) takes the form

$$\Big\{ -g\frac{\partial^2}{\partial\lambda_0^2} + g\frac{\partial^2}{\partial\bar{\lambda}^2} - m^2 + g^2\sqrt{2}\sum_f A_f u_f \cos\frac{f\lambda}{\sqrt{g}} +$$

$$+ g\sqrt{2}\sum_f A_f Q_f \cos\frac{f\lambda}{\sqrt{g}} + g\sqrt{2}\sum_f A_f P_f \sin\frac{f\lambda}{\sqrt{g}} + \tag{63}$$

$$+ \sqrt{2}\sum_f A_f \cos\frac{f\lambda}{\sqrt{g}} B_{\alpha f}\Big(g^2 J^\alpha - i\sqrt{g}\frac{\partial}{\partial\lambda_\alpha}\Big)$$

$$+ \sqrt{2}\sum_{f e_\alpha} A_f \sin\frac{f\lambda}{\sqrt{g}} B_{\alpha f} e^\alpha Q_e P_e \Big\} \Theta(\lambda, Q) = 0 .$$

Expanding the trigonometric functions into power series in $g^{-1/2}$, taking as $B_{\alpha e}$ expression (57) and retaining only higher powers of $g^{-1/2}$ we get in the first approximation the following equation

$$\Big\{ -\frac{\partial^2}{\partial \lambda_0^2} + \frac{\partial^2}{\partial \lambda^2} - \frac{m^2}{g} + g\sqrt{2}\sum A_f u_f - \frac{1}{2}\sqrt{2}\sum A_f u_f (f\lambda)^2 + \sqrt{2}\sum A_f Q_f + g\sqrt{2}\sum A_f N_{\alpha f} J^\alpha - \frac{1}{2}\sqrt{2}\sum A_f N_{\alpha f} J^\alpha (f\lambda)^2 - \tag{64}$$

$$- \sqrt{2}\sum A_f N_{\alpha e} J^\alpha e^\beta N_{\beta f} Q_e - \frac{1}{2}\sqrt{2}\sum A_f (f\lambda)^2 N_{\alpha f} J^\alpha \Big\} \theta(\lambda,\theta) = 0.$$

Eq. (64) allows the separation of the variables λ_α and Q_f, i.e. the function $\theta(\lambda, Q)$ may be represented as

$$\theta(\lambda, Q) = \varphi_0(\lambda)\,\theta_0(Q). \tag{65}$$

Since the variables Q_f **enter** eq. (64) only linearly then for the function $\theta_0(Q)$ to be regular we should vanish the terms containing the variables Q_f.
The function $\theta_0(Q)$ thus remains completely indefinite in the first approximation, but instead we have secceeded in obtaining a relation allowing to determine u_f numbers. Assuming all the coefficients for Q_f in the left-hand side of eq. (64) to be equal to zero we get the condition

$$A_e - \sum_{f\beta} A_f N_{\alpha e} e^\beta N_{\beta f} J^\alpha = 0 \tag{66}$$

which can be rewritten in the form

$$A_e - \sum_\beta v_e\, e^\beta c_\beta = 0, \tag{67}$$

where the numbers $\mathcal{V}_e$ and C_β are defined by the relations

$$\mathcal{V}_e = \sum_\alpha \mathcal{N}_{\alpha e} \mathcal{J}^\alpha , \tag{68}$$

$$C_\beta = \sum \mathcal{A}_f \mathcal{N}_{\beta f} . \tag{69}$$

Let us require that the numbers $\mathcal{V}_e$ coincide with the numbers $\mathcal{U}_e$. Then from eqs. (48) and (68) we can get the expression for the total momentum of the system

$$\mathcal{J}^\alpha = \sum_f f^\alpha \mathcal{U}^\ell_f . \tag{70}$$

If now we take the numbers C_β as a new parameter then eq. (67) makes it possible to determine the numbers $\mathcal{U}_e$ by means of the coefficients $\mathcal{A}_e$ and C_β

$$\mathcal{U}_e = \frac{\mathcal{A}_e}{\sum_\beta e^\beta C_\beta} . \tag{71}$$

Let us clarify the tensor dimensionality of the quantities C_β . The matrices $\mathcal{N}_{\alpha f}$ and $\mathcal{M}_{f\alpha}$ were first introduced regardless of the metric **properties** of space-time. However, the definition of the matrix $\mathcal{M}_{f\alpha}$ by relation (48) ascribes to it the transformation properties of the counter**variant** vector. After this it is natural to define the matrix $\mathcal{N}_{\alpha f}$ as a covariant vector with respect to the index α . This makes it possible to attribute to the condition of orthogonality of the matrices $\mathcal{N}_{\alpha f}$ and $\mathcal{M}_{f\alpha}$ (42) a Lorentz-invariant meaning and makes additional condition (41) Lorentz-invariant. As **functions** of Latin indices these matrices should be regarded to be invariant, at least, under the orthochronous proper Lorentz group. Under this condition the coefficients define the covariant vector.

Eq. (64) taking into account relations (70) and (71) takes the form

$$\left\{-\frac{\partial^2}{\partial\lambda_0^2}+\frac{\partial^2}{\partial\bar{\lambda}^2}+\frac{2\sqrt{2}}{g}\sum_\beta c_\beta\,\rho^\beta-\frac{m^2}{g}-\sqrt{2}\sum A_f u_f (f\lambda)^2\right\}\varphi_0(\lambda)=0 \quad (72)$$

Eq. (72) is the one of the four-dimensional oscillator. This fact forces us to pay a special attention to the way of regularization of the scalar field. Indeed, in spite of the fact that the quadratic form

$$\sum_{\alpha\beta}\lambda_\alpha\lambda_\beta\left(\sum_f\frac{A_f^2}{(fc)}f^\alpha f^\beta\right) \quad (73)$$

is formally Lorentz-invariant it may play the role of a physically admissible potential only with a special choice of the ooefficients A_f . Otherwise the proper oscillator frequency may depend on the vector c_β , which is inadmissible in a relativistically invariant theory. Therefore as A_f we should choose some functions of the invariant (fc) . Then the sum over the momenta in eq. (73) can be presented as

$$\sum_f\frac{A_f^2}{(fc)}f^\alpha f^\beta=\frac{\partial^2}{\partial c^\alpha\partial c^\beta}\sum_f F_f(fc)\,. \quad (74)$$

The sum in the right-hand side of (74) is a relativistic invariant therefore it must be a function of the Lorentz square of the vector C_β:

$$\sum_f F_f(fc) = +v(g^{\alpha\beta}C_\alpha C_\beta) \tag{75}$$

To ensure independence of the oscillator frequency of the vector C_β, we should choose the function $\mathcal{A}_f$ in such a fashion that the equality

$$v(g^{\alpha\beta}C_\alpha C_\beta) = \frac{\omega^2}{\sqrt{2}} g_{\alpha\beta} C^\alpha C^\beta \tag{76}$$

be satisfied. Then eq. (72) takes the form

$$\left[-\frac{\partial^2}{\partial\lambda_0^2} + \frac{\partial^2}{\partial\vec{\lambda}^2} - \frac{m^2}{g} + \frac{2\sqrt{2}}{g}\sum_\beta C_\beta P^\beta + \omega^2 g_{\alpha\beta}\lambda^\alpha\lambda^\beta\right]\varphi_0(\lambda) = 0 . \tag{77}$$

Physically admissible solutions for eq. (77) can be obtained following the method of ref. [3]. There it was suggested to put

$$\lambda_0 = i\lambda_4 \tag{78}$$

after which eq. (77) transforms to the equation

$$\left\{\sum_{\alpha=1}^4 \frac{\partial^2}{\partial\lambda_\alpha^2} - \omega^2\sum_{\alpha=1}^4 \lambda_\alpha^2 - \frac{m^2}{g} + \frac{2\sqrt{2}}{g}\sum C_\beta P^\beta\right\}\varphi_0(\lambda) = 0 . \tag{79}$$

The latter allows normalizable covariant solutions with positive equidistant squared mass spectrum

$$\mathcal{M}^2 = m^2 + g\omega(2n+1) . \tag{80}$$

The quantities $2\sqrt{2}C_\beta$ are the covariant components of the total momentum. In ref.[3] it is shown that such a mean of treating eq. (77) is equivalent to the description [4] of the four-dimensional oscillator on the basis of the use of an additional condition excluding time quanta. From equality (8o) it follows that the spacings between some values from the mass spectrum are proportional to the first degree of the coupling constant g . Certainly the solution for eqs. (79) are not the exact stationary states of the system. To judge the reliability of the approximation realized by eq. (79) we should consider the following terms of the expansion of the operator in the left-hand side of (63) into a power series in powers of $g^{1/2}$

The terms proportional to $g^{1/2}$ are

$$\sqrt{2}\sum A_f (f\lambda) P_f - i\sqrt{2}\sum_{\beta} C_\beta \frac{\partial}{\partial \lambda_\beta} \; . \tag{81}$$

The second term in this expression can be eliminated by a simple transformation of the wave function which leads to a shift of all the mass spectrum defined by equality (80) by a value of zero order in the coupling constant. The account of the first term in (80) leads to transitions between stationary states of the system in the first approximation, the transition probabilities being proportional to the first power of g , i.e. the account of terms (81) leads to a broadening of levels (80) by a value comparable with the spacings between these levels. It is not difficult to indicate more steady states of the system. Notice that the first term in (8o) leading to transitions is similar to the operator (29) which is a bilinear form in the variables λ_α and P_f . Adding to (81) the terms quadratic in variables P_f and Q_f which are proportional to the zero power of g we get an equation of the type (79) the operator of which is the quadratic form in the variables λ_+ , P_+ and Q_+ . In this approximation the variables λ_+ and Q_+ are already not separable. However the appropriate quadratic form is easily diagonalized by the method developed in ref. [5] . For the nonrelativistic case such a problem was solved in ref.[2]. It was shown there that the account of the interaction of a particle with the field (81) leads to a small shift of levels (80) **and the spacings** between the levels which are now assumed to be stationary are, as before, proportional to the first power of the coupling constant g . The subsequent terms in the expansion of the operator (63) begin with the zero power g , therefore the width of new levels is also proportional to the zero power of the coupling constant.

4. Conclusion

We have shown that the application of the Bogolubov transformation to the description of strong interaction of a classic particle with a scalar field allows to extract the motion of the particle in the field taking into account explicitly the conservation law of the total momentum. This transformation allows to draw a consistent picture of creation by the particle of a potential well moving together with the particle, in the case of strong interaction this well being reduced to the oscillator one and the ground state of the system being described by a set of shifted oscillators constructed on the particle and field variables.

It should be noted that recently the experimental data have led to the creation of various **kinds of dynamical models** in which strong **interacting** particles are regarded as certain complexes consisting of either truly elementary particles or some quasielementary excitations (quars, coherent complexes [4], droplet formations [5], partons [6]). Recently a very interesting paper appeared [3] where the connection between the coherent state method and the quark models **was first established, i.e.** dynamic model of generation of quasi-elementary excitations of the coherent type was first constructed. The suggested scheme of generation of the ground state is in its idea very close to the considerations of ref.[3] . Thus, the simple problem considered here may serve as a model of the oscillator interaction of particles at high energies [3,4,7] . The formalism used here allows to indicate a certain internal mechanism of generation of oscillator levels.

Finally, it is necessary of recall the deeper physical sense of the singling out of the variable q_α associated with the total momentum of the system. In ref.[1] it was indicated that this variable describes the translational motion of the particle interacting with the field while the quantities λ_α describe the vibrational motion of the particle inside the potential well. This should be taken into account in generalizing the method suggested to the many-particle case where there must appear independent variables describing the translational motions inside each potential well separately.

The present paper is a result of numerous discussions with N.N.Bogolubov who drew our attention to the deep meaning of the rigorous account of exact conservation laws in the strong coupling theory. The authors are grateful to A.A.Logunov for stimulating discussions. Very fruitful are the discussions with B.A.Arbusov, D.I.Blokhintsev, V.A.Matveev, R.M.Muradian, M.K.Polivanov, L.D.Soloviev, R.N.Faustov. We are very grateful to all of them.

Received by Publishing Department

on August 2, 1971.