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ЛАБОРАТОРИЯ ТЕОРЕТИЧЕСКОЙ ФИЗИКИ

D.I. Blokhintsev

THE STOCHASTIC SPACES

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1. Introduction

This work is an extension of the method suggested earlier in my paper [1] for calculating the wave propagation in a medium with random characteristics. A more perfect method of integration of the random phase equation is proposed, the averaging over the random phases is improved, and an application to quantum field theory is given.

In some cases it is reasonable to introduce the notion of stochastic space [2]. This just explains the title of the article.

2. Propagation of the Plane Wave

As the initial object we consider the plane wave:

$$\Psi_p(x) = u(p)\, e^{ipx} \quad , \tag{1}$$

where $px = p^{a} x_{a} = Et - \vec{p}\vec{x}$, $p(E, \vec{p})$, $E = \sqrt{p^2 + M^2}$ is the wave momentum, in quantum theory the parameter M playing the role of the particle mass, $u(p)$ is the wave amplitude.

If this wave is propagated in a medium with random characteristics then as the first approximation we introduce a correction in the wave phase (1) which is assumed to be equal to

$$\hat{S}(x) = px + \hat{\sigma}(x), \tag{2}$$

where σ is a linear function of a certain random field $\hat{\phi}(x)$. It is not difficult to show that this additional phase in the linear approximation satisfies the equation /1/ x):

$$\frac{d\hat{\sigma}(x)}{d\tau} + \hat{F}(x) = 0 \tag{3}$$

here $\tau = nx$ and n is the vector with components $n^{a} = \frac{p^{a}}{M}$. It is obvious that $n^2 = 1$. Therefore τ is the proper time of the wave or, in quantum theory, the proper time of the particle with momentum p and mass M. $\hat{F}(x)$ is the linear function of the random field $\hat{\phi}(x)$ which in general case can be written as

$$\hat{F}(x) = g\, n^{a} n^{\beta} \ldots \hat{\phi}_{a\beta}(x), \tag{4}$$

where g is a coupling constant and $\phi_{a\beta\ldots}(x)$ are the components of the field $\hat{\phi}(x)$ which may be a tensor of different rank xx).

This field is expanded in the Fourier series:

$$\hat{\phi}_{a\beta\ldots}(x) = \frac{1}{\sqrt{V}} \sum_{k} \frac{1}{\sqrt{2\omega_k}} e^{\lambda}_{a\beta} \{ \hat{a}^{+}_{k\lambda} e^{ikx} + a_{k\lambda} e^{-ikx} \}, \tag{5}$$

x) As compared with ref. /1/, here the notations are somewhat changed and a proper time τ is introduced instead of the time t.

xx) Here we do not consider the spinor field, when $\hat{F}(x)$ could not be a linear function of the spinor $\hat{\phi}(x)$.

where $e^{\lambda}_{\alpha\beta}$ is the tensor defining the polarization of the wave (λ) and $\frac{\hat{a}^{+}_{k\lambda}}{\sqrt{2\omega_k}}$ and $\frac{\hat{a}_{k\lambda}}{\sqrt{2\omega_k}}$ are the random Fourier series amplitudes. The vector k has components $k=(\omega_k\,\vec{k})$, $\omega_k=\omega(\vec{k})$. In particular, for the field obeying the Klein equation $\omega_k=\sqrt{k^2+\mu^2}$, where μ -is the particle mass for the field $\phi(x)$, $V=L^3$ is the normalization volume. Below, this is assumed to be infinitely. By expressing x as $x=x_{||}+x_{\perp}$ and $x_{||}=n(nx)=n\tau$ by means of eqs. (4) and (5), from eq. (1) we obtain

$$\hat{\sigma}(x)=ig\frac{1}{\sqrt{V}}\sum_k\frac{1}{\sqrt{2\omega_k}}\frac{1}{\Omega_k(p)}(n^{\alpha}n^{\beta}\ldots e^{\lambda}_{\alpha\beta})\times \tag{6}$$

$$\times\{\hat{a}^{+}_{k\lambda}e^{ikx}-\hat{a}_{k\lambda}e^{-ikx}\}f(\tau_0,x_{\perp}),$$

where $f(\tau_0,x_{\perp})$ is an arbitrary function of $x_{\perp}$ depending on the choise of the initial conditions at $\tau=\tau_0$. In what follows we assume that $\tau_0=-\infty$ and $f(-\infty,x_{\perp})=0$. The quantity $\Omega_k(p)$ is an invariant frequency:

$$\Omega_k(p)=\frac{1}{M}(kp). \tag{7}$$

In all the cases but the case of scalar field the quantity $\hat{\sigma}(x)$ may be represented in the form

$$\hat{\sigma}(x)=p^{\alpha}\hat{\xi}_{\alpha}(x), \tag{8}$$

where $\xi_{\alpha}(x)$ has the meaning of a random displacement of the coordinate x. This fact allows us to consider, besides the initial space $R_4(x)$ which we call the reference space, the <u>stochastic</u> space $R_4(\hat{X})$. The coordinates of this space are connected with the coordinates of the points of the reference space by the transformation

$$\hat{X} = x + \hat{\xi}(x) \tag{9}$$

and are random quantities depending on the random field. Owing to this fact the labelling of events in the stochastic space is probabilistic.

The meaning of the concept of stochastic space or generally stochastic geometry may go beyond the scope of the present article (see ref. /2/, § 41, 44, 45).

3. Calculation of Averages

We represent the Fourier amplitude $\hat{a}_{k\lambda}$ of the field in the form

$$\hat{a}^{+}_{k\lambda} = A_{k\lambda}\, e^{i\hat{\theta}_{k\lambda}}, \qquad \hat{a}_{k\lambda} = \hat{A}_{k\lambda}\, e^{-i\hat{\theta}_{k\lambda}}; \tag{10}$$

where $Z\hat{A}_{k\lambda}$ and $\hat{\theta}_{k\lambda}$ are real quantities. Then eq. (16) can be rewritten in the form:

$$\hat{\sigma}(x) = -2 \sum_{k} N^{\lambda}_{k}\, \hat{A}_{k\lambda} \sin(kx + \hat{\theta}_{k\lambda}), \tag{11}$$

where

$$N^{\lambda}_{k} = g \frac{1}{\sqrt{V}} \frac{1}{\sqrt{2\omega_k}} \frac{1}{\Omega_k(p)} (n^{\alpha} n^{\beta} \ldots e^{\lambda}_{\alpha\beta}). \tag{12}$$

In the theory of classic fields the random field distribution is specified by the functional $dw\{\hat{\phi}(x)\} \geq 0$. This functional is assumed to have the form:

$$dw\{\hat{\phi}(x)\} = \prod_{k} e^{-\frac{A_k^2}{\Delta_k^2}} \frac{dA_k}{\sqrt{\pi}\,\Delta_k} \frac{d\theta_k}{2\pi}, \tag{13}$$

i.e. the normal distribution of amplitudes A_k with dispersion Δ_k and the uniform distribution of phases θ_k. (For the sake of simplicity we have omitted cep Λ and the polarization index for the amplitudes $\hat{A}_{k\lambda}$ and phases $\hat{\theta}_{k\lambda}$). From the definition of $dw\{\hat{\phi}(x)\}$ we have for the average values of a) the wave $\Psi_p(x)$:

$$< \Psi_p(x) > = u_p\, e^{ipx} \int e^{i\hat{\sigma}(x)}\, dw\{\hat{\phi}(x)\} \tag{14}$$

and b) interference correlation of two waves $\Psi_{p'}(y)$ and $\Psi_p(x)$

$$< \bar{\Psi}_{p'}(y)\Gamma\Psi_p(x) > = (u_{p'}\Gamma u_p)\exp\{-i(p'y-px)\}\times$$

$$\times\int \exp\{-i\hat{\sigma}_{p'}(y)\}\exp\{-i\hat{\sigma}_p(x)\}\,dw\{\phi(x)\}, \tag{15}$$

where Γ is any spinor operator. Note that the correlation (15) at $x=y$ and $\Gamma=\gamma^\mu$ (γ^μ –is the Dirac matrix) coincides with the so-called "vertex" part known from quantum field theory.

Now inserting in eq. (14) the distribution (13) and integrating over A_k and θ_k we get:

$$\int \exp\{i\hat{\sigma}(x)\}dw\{\phi(x)\} = \exp - Q_p(x) = \prod_k R_k, \tag{16}$$

where

$$R_k = \exp\{-\frac{g^2}{2}\Delta_k^2 N_k^2 + \ln I_0(\frac{\Delta_k^2 N_k^2}{4})\}. \tag{17}$$

Here $I_0(z)$ is the Bessel function. According to eq. (12), $N_k \approx \frac{1}{\sqrt{V}}$. Therefore when $V\to\infty^3$ the sum of $\ln I_0$ over k has the order of magnitude $0(\frac{1}{V^2})$ and may be omitted.

This is a justification for the native operation of replacement of $\cos\theta_k$ and $\sin\theta_k$ in the exponential by their average values which are equal to zero. Inserting (17) in eq. (16) and making the transition to the limit $V \to \infty^3$ we obtain

$$Q_p(x) = \frac{g^2}{2} \int \frac{d^3k}{2\omega_k} \frac{\Delta_k^2}{\Omega_k^2(p)} (n^\alpha n^\beta \ldots e^\lambda_{\alpha\beta})^2 . \tag{18}$$

This equation is relativistically invariant if $\omega_k = +\sqrt{k^2+\mu^2}$ and if the dispersion Δ_k^2 is an invariant. If no special direction in the Minkovsky space $R_4(x)$ is connected with the field $\phi(x)$ then the only possibility is to assume that $\Delta_k^2 = \text{const}$, is constant, i.e. the amplitudes A_k obey one and the same distribution law independently of the vector k. This supposition leads immediately to divergences in the integral (18), namely

$$Q_p(x) \approx \frac{g^2}{2} \ln \frac{p_{max}}{M} \tag{19}$$

and $p_{max} \to \infty$. At $\mu = 0$ there appears a divergency on the lower limit, too.

The correlation (15) is calculated in a similar manner. To make the calculations simpler we give the result for $x = y$

$$\int \exp\{-i\hat{\sigma}_{p'}(x)\} \exp\{i\hat{\sigma}_p(x)\} dw\{\hat{\phi}(x)\} = \exp\left[-Q_{pp'}(x)\right] \tag{20}$$

in this case

$$Q_{p'p}(x) = \frac{g^2}{2} \int \frac{d^3k}{2\omega_k} \Delta_k^2 \sum_\lambda \{ e^\lambda_{\alpha\beta\ldots} [\frac{n'^\alpha n'^\beta}{\Omega_k(p')} - \frac{n^\alpha n^\beta}{\Omega_k(p)}] \}^2 . \tag{21}$$

This correlation like the quantity (18) turns out to be divergent for $\Delta_k^2 = const$. Note that Δ_k^2 may be considered as a form factor ensuring convergence of the integrals in (18) and (21). If it is considered as a function of the invariant $\Omega_k(p)$ (the remaining invariants from k and p are constant) then the quantities $Q_p(x)$ and $Q_{p'p'}(x)$ are invariant too (which must just be the case in relativistically invariant theory). The quantity $Q_p(x)$ is simply a number and $Q_{p'p}(x)$ a function of only q^2, $q = p' - p$ (if p' and p are on the mass shell, the invariant $qp = 0$ is zero).

4. The Quantum Field

In the case under consideration the random amplitudes $\hat{a}^+_{k\lambda}$ and $\hat{a}_{k\lambda}$ are the operators obeying the commutation relations:

$$[\hat{a}_{k\lambda}\, \hat{a}^+_{k'\lambda'}] = \delta_{kk'}\, \delta_{\lambda\lambda'}\,. \tag{22}$$

Next, the average over the measure $dw\{\phi(x)\}$ should now be replaced by the average over the wave functional $\Omega_0\{\phi(x)\}$9 which is an analog of the quantity $\sqrt{dw_0\{\phi(x)\}}\exp iS_0(\hat{\phi})$, where $S(\hat{\phi})$ is the functional phase and the mark o implies the vacuum state of the field $\phi(x)$. Using the usual notations we rewrite eqs. (16) and (20) in the form:

$$e^{-Q_p(x)} = \langle 0 | e^{i\hat{\sigma}_p(x)} | 0 \rangle, \tag{16}$$

$$e^{-Q_{p'p}(x)} = \langle 0 | e^{-i\hat{\sigma}_{p'}(x)} \cdot e^{i\hat{\sigma}_p(x)} | 0 \rangle, \tag{20}$$

where $\langle 0 | \hat{L} | 0 \rangle$ means the vacuum-expectation value of the field $\hat{\phi}(x)$. Further calculations are based on the relations:

$$< 0 | \exp (\hat{A}^+_k + \hat{A}_k) | 0 > = \exp \frac{1}{2} [\hat{A}_k , \hat{A}^+_k]$$

$$< 0 | \exp (\hat{A}^+_k + \hat{A}_k) \exp (\hat{A}^+_{k'} + \hat{A}_{k'}) | 0 > = \tag{23}$$

$$= \exp \frac{1}{2} ([\hat{A}_k , \hat{A}^+_k] + \frac{1}{2} [\hat{A}_{k'} \hat{A}^+_{k'}] + [\hat{A}_k , \hat{A}^+_{k'}]), \tag{23'}$$

where $[\hat{A}, \hat{B}]$ denotes the Poisson bracket, which is assumed to be the c-number. The calculation of the averages in eqs. (16) and (20) with the help of eqs. (23) and (23') and the expression (12) for $N_{k\lambda}$ leads exactly to the classic formula (18) and (21), if $\Delta^2_k = 1$.

Thus, the divergent result is a consequence of the assumption on vacuum isotropy ($\Delta^2_k = const$) which follows from the requirement of relativistic invariance.

The formula (21) is tightly connected with the theory of the so-called "coherent states" /5/. The model of the vertex in this latter differs from the considered fluctuations of the spinor particle coordinate $\xi(x)$ by the supposition that the zero harmonic is predominant so that the dispacement $\xi(x)$ is independent of x.

It is useful to note that in the case of the vector field $\hat{\phi}_a(x)$ the mean square displacement $\xi(x)$ defined by eq. (8) is

$$< \xi^2_a (x) > = \frac{g^2}{M^2} \int \frac{d^3k}{2\omega_k} \frac{1}{\Omega^2_p(p)} \sum_\lambda (e^\lambda_a)^2 . \tag{24}$$

For $|\vec{p}| \ll M$, we obtain:

$$< \xi^2_a (x) > = \frac{g^2}{M^2} \int \frac{d^3k}{2\omega^3_k} \sum_\lambda (e^\lambda_a)^2 . \tag{24'}$$

If $g = e$ is assumed to be equal to the electron charge and implies its mass eq. (24') coincides with the value of the square displacement of the electron coordinate $\langle \xi_a^2(x) \rangle$ which defines the Lamb shift of the level in the hydrogen atom (see refs. /5,6/).

5. Gravitation

In ref. /3/ it was shown that if one considers the gravitational field fluctuations, which arise from the fluctuations of the energy-momentum tensor of the fields in the vacuum, one arrives at strongly divergent expressions for the fluctuations of the metric (length and time).

The method developed here makes it possible to calculate the fluctuations of the wave phase which arise from zero-point fluctuations of the free gravitational field.

It is not difficult to show that in this case the function $\hat{F}(x)$ is

$$\hat{F}(x) = \frac{M}{2} \hat{h}^{\alpha\beta}(x) n_\alpha n_\beta \ , \qquad (25)$$

where

$$\hat{h}^{\alpha\beta}(x() = \hat{g}^{\alpha\beta}(x) - g_0^{\alpha\beta} \qquad (26)$$

and $g_0^{\alpha\beta}$ is the metric tensor in the absence of gravitational waves. Among the quantities $\hat{h}^{\alpha\beta}(x)$ only four of them are independent. Bearing in mind these components we represent $\hat{h}^{\alpha\beta}(x)$ as a series:

$$\hat{h}^{\alpha\beta}(x) = \frac{\gamma}{\sqrt{V}} \sum_k e_\lambda^{\alpha\beta}(a_{k\lambda}^{+} e^{ikx} + a_{k\lambda} e^{-ikx}). \qquad (27)$$

In contrast to (5), a constant γ is here introduced which is defined in such a fashion that the energy of a gravitation wave is equal to $\sum_k n_k h \omega_k$, where n_k are integers and $h\omega_k$ the gravitation energy. On the other hand, this energy is expressed in terms of the integral of the energy-momentum pseudo-tensor (see e.g. /7/). This fact allows to determine the constant $\gamma^2 = \frac{8\pi k}{c^2}$; where k is the gravitational constant ($k = 6.67 \cdot 10^{-8} cm^3\, g^{-1} sec^{-2}$).

Performing the calculations analogous to those described in § 3 we are led to formulas (18) and (21) at $\Delta_k^2 = 1$ and $g^2 = \gamma\, M^2$. If the usual dimensionality is restored it is easy to make oneself sure that in the case of gravitational field the quantities $Q_p(x)$ and $Q_{p'p}(x)$ are proportional to Λ_{gq}^2, where $\Lambda_g = \sqrt{\frac{8\pi k}{c^2}\,\frac{k}{c}} = 0.82 \cdot 10^{-32}$ cm is the characteristic length, which defines the limits, outside which the metric fluctuations may become essential.

It is seen from these calculations that without artificial introducing a "cut-off" form-factor fluctuations of the phase $\hat{\sigma}(x)$ turn out to be indefinite. The same may be said about the fluctuations of the stochastic coordinate X (9). This extreme dispersion of the averages $\langle \hat{\sigma}_p^2(x) \rangle$ and $\langle \xi_p^2(x) \rangle$ is due to the requirement of homogeneity of vacuum ($\Delta_k^2 = const$). Physically it is clear that the above fluctuations can be restricted only by taking into account the effect of the particle itself on the vacuum, in other words, by taking into account a possible deformation of the vacuum in the vicinity of the particle. The introduction of the "cut-off" factor may be thought of as a formal procedure of taking this effect into account.

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