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RELATIVISTIC THREE-PARTICLE PROBLEM IN TERMS OF THREE-DIMENSIONAL VARIABLES

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RELATIVISTIC THREE-PARTICLE PROBLEM IN TERMS OF THREE-DIMENSIONAL VARIABLES

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The quantum field theory equations for the Green functions and various transition amplitudes /1-3/ based on the analysis of the Feynman diagram structure are formulated in terms of the non-Euclidean four-dimensional variables. The presence of relative time variables (or relative energies in the momentum representation) leads to the well-known difficulties of mathematical character and those connected with the physical interpretation.

The quasipotential method of Logunov and Tavkhelidze /4/ which is essentially based on the equal-time description of the two-body system does not suffer from such defects and is formulated only in terms of three-dimensional variables. In this sense the quasipotential equations are a direct generalization of the two-particle non-relativistic equations to the relativistic case. Because of the specific features of the many-body systems a similar generalization rigorously proved in the framework of quantum field theory has not been made up to now and all the attempts undertaken dealt only with the solution of some particular problems.

In the papers /5-7/ the approximations have been used to obtain three-dimensional integral equations for the relativistic three-body problem. For example, a basic approximation made in /7/ is that the two-particle solutions for each subsystem are dominated by a finite number of bound states and resonances.

In the paper /8/ on the basis of the quasipotential approach relativistically covariant equations for the N-particle wave functions are proposed.

In[9] general properties of the quasipotential equation for many-body bound system are considered.

The papers[11] are devoted to the derivation of an equation for the three-particle scattering amplitude in the framework of a three-dimensional formulation of quantum field theory[10].

The equal-time description of a three-particle system allowed one to derive equations in terms of the three-dimensional variables which determine the physical relativistic amplitudes of all 16 possible transitions[12]. An analogy with the non-relativistic Faddeev equations [13,14] can be established, the properties of the two-particle subsystems being given by the Logunov-Tavkhelidze quasipotential equations. Nevertheless, in addition to the purely relativistic features there is one more difference from the Faddeev equations that the two-particle solutions are contained in the three-particle equations[12] as kernels in a rather complicated form. This is due to the presence of the advanced part of the two-time Green function[4].

An approach to the three-particle relativistic problem proposed below is based on the considerations of the retarded part of the two-time Green function (RGF).

By means of similar considerations in the two-particle case the quasipotential-type equations have been obtained[15].

In what follows these equations play the same role for our approach as the Schrödinger equations for that of Faddeev. We will show that the three-particle RGF contains the necessary information about all the possible scattering processes and bound states in the relativistic three-particle system.

A direct generalization of the Faddeev equations to the relativistic case is obtained. For the sake of simplicity we will consider the scalar particle case.

2. Two-Particle Quasipotential Equations

In this section we recall briefly the results of the paper /15/ required in what follows.

The total Green function of two interacting particles with masses m_1 and m_2

$$\hat{G}(x_1, x_2, y_1, y_2) = \langle 0 | T\{\psi_1(x_1)\,\psi_2(x_2)\,\psi_1(y_1)\,\psi_2(y_2)\} | 0 \rangle \tag{2.1}$$

in the momentum representation has the form

$$(2\pi)^8 \delta(P-Q)\, G(P,p,q) = \int \hat{G}(x_1,x_2,y_1,y_2)\, e^{ip_1x_1 + ip_2x_2 - iq_1y_1 - iq_2y_2}\, dx_1\, dx_2\, dy_1\, dy_2 \,, \tag{2.2}$$

where

$$P = p_1 + p_2$$
$$p = \mu_2 p_1 - \mu_1 p_2 \qquad \mu_i = \frac{m_i}{m_1 + m_2} \,. \tag{2.3}$$

Then for the two free particles we have

$$G_0(P,p,q) = \frac{i}{p_1^2 - m_1^2 + i\varepsilon}\;\frac{i}{p_2^2 - m_2^2 + i\varepsilon}\,\delta(p-q) \,. \tag{2.4}$$

Now let us introduce the two-time Green function and its Fourier transform as

$$\tilde{G}(t-t',\vec{x}_1,\vec{x}_2,\vec{y}_1,\vec{y}_2)=\langle 0|T\{\Psi_1(t,\vec{x}_1)\Psi_2(t,\vec{x}_2)\Psi_1^+(t',\vec{y}_1)\Psi_2^+(t',\vec{y}_2)\}|0\rangle \tag{2.5}$$

$$(2\pi)^5\tilde{G}(P_0\vec{P},\vec{q})\delta(\vec{P}-\vec{Q})=\int dt\,d\vec{x}_1 d\vec{x}_2 d\vec{y}_1 d\vec{y}_2\,\hat{\tilde{G}}(t,\vec{x}_1,\vec{x}_2,\vec{y}_1,\vec{y}_2)e^{i[P_0t-\vec{p}_1\vec{x}_1-\vec{p}_2\vec{x}_2+\vec{q}_1\vec{y}_1+\vec{q}_2\vec{y}_2]}. \tag{2.6}$$

Hence

$$\tilde{G}(P_0\vec{P},\vec{q})=\int_{-\infty}^{\infty}G(P_0P,q)\,dp_0\,dq_0\,. \tag{2.7}$$

According to (2.5) and (2.7) the symbol $\sim$ will denote the equal-time operation in the coordinate representation and integration over relative energies in the momentum representation.

The two-time Green function can be expressed by the sum of the retarded $\tilde{G}^r$ and advanced $\tilde{G}^a$ parts. For each of them we have the following spectral representation[4]

$$\tilde{G}^a(P_0\vec{P},\vec{q})=\int_0^{\infty}\frac{I_a(E,\vec{P},\vec{p},\vec{q})\,dE}{P_0+E-i\varepsilon} \tag{2.8}$$

$$\tilde{G}^r(P_0\vec{P},\vec{q})=\int_0^{\infty}\frac{I_r(E,\vec{P},\vec{p},\vec{q})\,dE}{P_0-E+i\varepsilon}\,, \tag{2.9}$$

where

$$I_r(E,\vec{P},\vec{p},\vec{q})=\frac{i\pi}{2E}\sum_n\delta(E-E_n)\delta(\vec{P}-\vec{K}_n)\chi_{0n}(\vec{p})\otimes\chi_{0n}^+(\vec{q}) \tag{2.9a}$$

$\chi_{on}(\vec{p})$, $\chi^{+}_{on}(\vec{p})$ are the Fourier transforms of the equal--time wave functions

$$\delta(\vec{P}-\vec{K}_n)\chi_{on}(\vec{p})=\frac{2\sqrt{E_n}}{(2\pi)^3}\int d\vec{x}_1 d\vec{x}_2\, e^{-i\vec{p}_1\vec{x}_1-i\vec{p}_2\vec{x}_2}\langle 0|\psi_1(0,\vec{x}_1)\,\psi_2(0,\vec{x}_2)|n\rangle \tag{2.10}$$

$$\delta(\vec{P}-\vec{K}_n)\chi^{+}_{on}(\vec{p})=\frac{2\sqrt{E_n}}{(2\pi)^3}\int d\vec{x}_1 d\vec{x}_2\, e^{i\vec{p}_1\vec{x}_1+i\vec{p}_2\vec{x}_2}\langle n|\psi^{+}_1(0,\vec{x}_1)\,\psi^{+}_2(0,\vec{x}_2)|0\rangle .$$

Iterating the Bethe-Salpeter equation

$$G=G_0+G_0KG \tag{2.11}$$

we get the expansion

$$G=G_0+G_0KG_0+G_0KG_0KG_0+\cdots \; . \tag{2.12}$$

Integrating both sides of the expansion (2.12) over the relative energies and picking out the retarded parts afterwards we find

$$\tilde{G}^{r}=\tilde{G}_0^{r}+(\widetilde{G_0KG_0})^{r}+(\widetilde{G_0KG_0KG_0})^{r}+\cdots \; . \tag{2.13}$$

It is easy to see that

$$\tilde{G}_0^{r}(P,\vec{p},\vec{q})=\frac{\pi i\,\delta(\vec{p}-\vec{q})}{2\omega_1(\vec{p}_1)\omega_2(\vec{p}_2)[P_0-\omega_1(\vec{p}_1)-\omega_2(\vec{p}_2)+i\varepsilon]} , \tag{2.14}$$

where $\omega_i(\vec{p}_i)=\sqrt{m_i^2+\vec{p}_i^{\,2}}$,

$\tilde{G}^{r}$ can be treated as an operator in "two-particle Hilbert space" of functions of the vector $\vec{p}$. The operators involved will depend on the total four-momentum P as external parameter. Then we can write for the inverse operator $(\tilde{G}^{\eta})^{-1}$

$$(\tilde{G}^{\eta})^{-1} = (\tilde{G}_{0}^{\eta})^{-1} - (\tilde{G}_{0}^{\eta})^{-1} (\widetilde{G_0 K G_0})^{\eta} (\tilde{G}_{0}^{\eta})^{-1} + \dots \quad . \tag{2.15}$$

Introducing the quasipotential K^{η}

$$\frac{i}{\pi} K^{\eta} = (\tilde{G}_{0}^{\eta})^{-1} (\widetilde{G_0 K G_0})^{\eta} (\tilde{G}_{0}^{\eta})^{-1} + (\tilde{G}_{0}^{\eta})^{-1} (\widetilde{G_0 K G_0 K G_0})^{\eta} (\tilde{G}_{0}^{\eta})^{-1} + \dots \tag{2.16}$$

one obtains the following equation for $\tilde{G}^{\eta}$

$$\tilde{G}^{\eta} = \tilde{G}_{0}^{\eta} + \frac{i}{\pi} \tilde{G}^{\eta} K^{\eta} \tilde{G}_{0}^{\eta} \quad . \tag{2.17}$$

Further let us write the term which corresponds to the bound state pole contribution to $\tilde{G}^{\eta}$.

$$\tilde{G}^{\eta}(P,\vec{p},\vec{q}) \approx \frac{i\pi}{2E_B} \frac{\chi_B(\vec{p}) \otimes \chi_B(\vec{q})}{P_0 - E_B + i\varepsilon} \tag{2.18}$$

here $E_B = \sqrt{\vec{P}^2 + M_B^2}$

Now it can be easily checked that the bound state equation and normalization condition have the following form

$$\left(\omega_1(\vec{p}_1) + \omega_2(\vec{p}_2) - E_B\right) \chi_B(\vec{p}) = \frac{1}{2\omega_1(\vec{p}_1)\omega_2(\vec{p}_2)} \int K^{\eta}(P,\vec{p},\vec{k})\, d\vec{k}\, \chi_B(\vec{k}) \quad , \tag{2.19}$$

$$\int 2\omega_1(\vec{p}_1)\omega_2(\vec{p}_2) \chi_B^{+}(\vec{p}) \chi_B(\vec{p})\, d\vec{p} + \int \chi_B(\vec{p}) \frac{\partial K^{\eta}(P,\vec{p},\vec{q})}{\partial P_0} \chi_B(\vec{q})\, d\vec{p}\, d\vec{q} = 2E_B \quad , \tag{2.20}$$

where $P_0 = E_B$.

If we define the off-shell amplitude $T^{\eta}(P,\vec{p},\vec{q})$ as

$$\tilde{G}^{\tau} = \tilde{G}_{10}^{\tau} + \frac{i}{\pi} \tilde{G}_{10}^{\tau} T^{\tau} \tilde{G}_{0}^{\tau} \tag{2.21}$$

then the physical relativistically invariant scattering amplitude $T(s,t)$ is defined by the relation

$$T(s,t) = 16\pi^3 \, T^{\tau}(P_0, \vec{p}, \vec{q}) \Big|_{P_0 = \omega_1(\vec{p}_1) + \omega_2(\vec{p}_2) = \omega_1(\vec{q}_1) + \omega_2(\vec{q}_2)} .$$

Inserting (2.21) into (2.17) we get the equation for the off-shell amplitude

$$T^{\tau}(P_0, \vec{p}, \vec{q}) = K^{\tau}(P_0, \vec{p}, \vec{q}) + \int \frac{T^{\tau}(P_0, \vec{p}, \vec{k}) \, K^{\tau}(P_0, \vec{k}, \vec{q}) \, d\vec{k}}{2\omega_1(\vec{k}_1)\omega_2(\vec{k}_2)\left[\omega_1(\vec{k}_1) + \omega_2(\vec{k}_2) - P_0 - i\varepsilon\right]} . \tag{2.22}$$

3. Three-Particle 2-Time Green Function and Its Retarded Part

Let us consider here the total Green function of three particles with masses m_1, m_2, m_3 which are described by the fields Ψ_1, Ψ_2, Ψ_3

$$\hat{G}(x,y) = \langle 0 | T \{ \Psi_1(x_1) \Psi_2(x_2) \Psi_3(x_3) \Psi_1^{+}(y_1) \Psi_2^{+}(y_2) \Psi_3^{+}(y_3) \} | 0 \rangle \tag{3.1}$$

$X=(x_1, x_2, x_3)$ and $y=(y_1, y_2, y_3)$ denote the sets of four-coordinates of the final and initial states. Here and below the conventional notations are taken for the three-particle quantities, the two-particle quantities will be denoted by special indices (in case of need).

Taking into account the translation invariance the Fourier transform of (3.I) can be written as

$$(2\pi)^{12}\delta(P-Q)G_1(P,\tilde{p},\bar{p},\tilde{q},\bar{q})=\int \hat{G}_1(x,y)\, e^{i\sum_{m=1}^{3}(p_m x_m - q_m y_m)}(dx)(dy) , \tag{3.2}$$

where $(dx)= dx_1\, dx_2\, dx_3$,

$P, \tilde{P}_i, \bar{P}_i$ are four-momentum variables of the i-th Jacoby system

$$P=p_1+p_2+p_3 ; \qquad p_i=\frac{m_i}{M}P-\tilde{p}_i ; \tag{3.3}$$

$$\tilde{p}_i=\frac{m_i}{M}(p_e+p_k)-\frac{m_k+m_e}{M}p_i ; \qquad p_k=\frac{m_k}{M}P+\frac{m_k}{m_k+m_e}\tilde{p}_i+\bar{p}_i ;$$

$$\bar{p}_i=\frac{m_e p_k - m_k p_e}{m_k+m_e} ; \qquad p_e=\frac{m_e}{M}P+\frac{m_e}{m_k+m_e}\tilde{p}_i-\bar{p}_i ;$$

$$M=m_1+m_2+m_3 .$$

Here i, k, ℓ represent any cyclic permutation of (1,2,3).

According to (3.1) and (3.2) the three-particle free Green function has the form

$$G_0(P,\tilde{p}_i,\bar{p}_i,\tilde{q}_i,\bar{q}_i)=\frac{(i)^3\,\delta(\tilde{p}_i-\tilde{q}_i)\,\delta(\bar{p}_i-\bar{q}_i)}{(p_1^2-m_1^2+i\varepsilon)(p_2^2-m_2^2+i\varepsilon)(p_3^2-m_3^2+i\varepsilon)}, \tag{3.4}$$

The total Green function (3.1) satisfies the Bethe-Salpeter-type equation /2,3/

$$G=G_0+G_0KG, \tag{3.5}$$

where K is the sum of all irreducible Feynman graphs which can be represented in the form

$$K=K_1+K_2+K_3+K_T \tag{3.6}$$

with

$$K_i(P,\tilde{p}_i,\bar{p}_i,\tilde{q}_i,\bar{q}_i)=\delta(\bar{p}_i-\bar{q}_i)(-i)(p_i^2-m_i^2)K_i^{(2)}(P_i,\bar{p}_i,\bar{q}_i) \tag{3.6a}$$

K_T is the sum of all irreducible connected graphs;

$K_i^{(2)}$ is the kernel of Eq. (2.11) for the i-th two-particle subsystem which consists of the k-th and ℓ-th particles.

P_i is the i-th subsystem total four-momentum

$$P_i = p_\ell + p_k = \frac{m_k + m_\ell}{M} P + \tilde{P}_i \; . \tag{3.7}$$

Let us introduce the 2-time Green function of three particles

$$\tilde{\tilde{\hat{G}}}_1(t-t', \vec{x}, \vec{y}) = \langle 0| T\{\Psi_1(t,\vec{x}_1)\Psi_2(t,\vec{x}_2)\Psi_3(t,\vec{x}_3)\Psi_1^+(t',\vec{y}_1)\Psi_2^+(t',\vec{y}_2)\Psi_3^+(t',\vec{y}_3)\}|0\rangle \tag{3.8}$$

$\vec{X} = (\vec{x}_1, \vec{x}_2, \vec{x}_3)$ and $\vec{y} = (\vec{y}_1, \vec{y}_2, \vec{y}_3)$ are the sets of three-dimensional coordinates. The Fourier transform of (3.8) is defined as

$$(2\pi)^7 \tilde{\tilde{\hat{G}}}_1(P, \vec{\tilde{p}}_i, \vec{\bar{p}}_i, \vec{\tilde{q}}_i, \vec{\bar{q}}_i)\, \delta(\vec{P}-\vec{Q}) = \int \tilde{\tilde{\hat{G}}}_1(t,\vec{x},\vec{y})\, e^{i[P_0 t - \sum_m^3 (\vec{p}_m \vec{x}_m - \vec{q}_m \vec{y}_m)]} dt\, (d\vec{x})(d\vec{y}) \tag{3.9}$$

$$(d\vec{x}) = d\vec{x}_1\, d\vec{x}_2\, d\vec{x}_3 \; .$$

Therefore it is not difficult to verify that the symbol $\approx$ denotes the following integration in the momentum representation

$$\tilde{\tilde{G}}_1(P, \vec{\tilde{P}}, \vec{\bar{P}}, \vec{\tilde{q}}, \vec{\bar{q}}) = \int_{-\infty}^{\infty} d\tilde{p}^\circ d\bar{p}^\circ d\tilde{q}^\circ d\bar{q}^\circ\, G_1(P, \tilde{p}, \bar{p}, \tilde{q}, \bar{q}) \; . \tag{3.10}$$

By analogy with (2.8) and (2.9) the following spectral representation holds for (3.10)

$$\tilde{\tilde{G}} = \tilde{\tilde{G}}^a + \tilde{\tilde{G}}^r \; , \tag{3.11}$$

where

$$\tilde{\tilde{G}}_1^{a}(P,\vec{\tilde{p}},\vec{p},\vec{\tilde{q}},\vec{q})=\int_0^{\infty}\frac{I_a(E,\vec{P},\vec{\tilde{p}},\vec{p},\vec{\tilde{q}},\vec{q})\,dE}{P_0+E-i\varepsilon} \tag{3.12a}$$

$$\tilde{\tilde{G}}_1^{r}(P,\vec{\tilde{p}},\vec{p},\vec{\tilde{q}},\vec{q})=\int_0^{\infty}\frac{I_r(E,\vec{P},\vec{\tilde{p}},\vec{p},\vec{\tilde{q}},\vec{q})\,dE}{P_0-E+i\varepsilon} \tag{3.12b}$$

and, in particular,

$$I_r(E,\vec{P},\vec{\tilde{p}},\vec{p},\vec{\tilde{q}},\vec{q})=\frac{i\pi^2}{2E}\sum_n \delta(E-E_n)\,\delta(\vec{P}-\vec{K}_n)\,\chi_{on}(\vec{\tilde{p}},\vec{p})\otimes\chi^{+}_{cn}(\vec{\tilde{q}},\vec{q}) \tag{3.12c}$$

$$\delta(\vec{P}-\vec{K}_n)\,\chi_{on}(\vec{\tilde{p}},\vec{p})=\frac{2\sqrt{2E_n}}{(2\pi)^{9/2}}\int e^{-i\sum_m^3 \vec{p}_m\vec{x}_m}\langle c|\psi_1(c,\vec{x}_1)\psi_2(c,\vec{x}_2)\psi_3(c,\vec{x}_3)|n\rangle(d\vec{x}) \tag{3.12d}$$

$$\delta(\vec{P}-\vec{K}_n)\,\chi^{+}_{cn}(\vec{\tilde{p}},\vec{p})=\frac{2\sqrt{2E_n}}{(2\pi)^{9/2}}\int e^{i\sum_m^3 \vec{p}_m\vec{x}_m}\langle n|\psi^{+}_1(c,\vec{x}_1)\psi^{+}_2(c,\vec{x}_2)\psi^{+}_3(c,\vec{x}_3)|0\rangle\,(d\vec{x}).$$

Using Eq. (3.5) the three-particle Green function can be written as iteration series

$$G=G_{r0}+G_{r0}KG_{r0}+G_{r0}KG_{r0}KG_{r0}+\cdots\,. \tag{3.13}$$

Now integrating over relative energies as is shown in (3.10) and picking out the retarded part from each term of (3.13) we get

$$\tilde{\tilde{G}}_1^{r}=\tilde{\tilde{G}}_{10}^{r}+\left[\overline{\overline{G_{r0}K\,G_{r0}}}\right]^{r}+\left[\overline{\overline{G_{r0}K\,G_{r0}K\,G_{r0}}}\right]^{r}+\cdots\,, \tag{3.13*}$$

It is easy to check that

$$\tilde{\tilde{G}}_0^{\tau}(P,\vec{\tilde{p}}_i,\vec{\tilde{p}}_i,\vec{\tilde{q}}_i,\vec{\tilde{q}}_i)=\frac{i\pi^2\delta(\vec{\tilde{p}}_i-\vec{\tilde{q}}_i)\,\delta(\vec{\tilde{p}}_i-\vec{\tilde{q}}_i)}{2\omega_1(\vec{p}_1)\omega_2(\vec{p}_2)\omega_3(\vec{p}_3)\left[P_0-\omega_1(\vec{p}_1)-\omega_2(\vec{p}_2)-\omega_3(\vec{p}_3)+i\varepsilon\right]} \quad (3.14)$$

Treating $\tilde{\tilde{G}}^{\tau}(P,\vec{\tilde{p}},\vec{\tilde{p}},\vec{\tilde{q}},\vec{\tilde{q}})$ as an operator in the three-particle Hilbert space of functions of three-dimensional relative momenta $f(\vec{\tilde{p}},\vec{\tilde{p}})$

$$f(\vec{\tilde{p}},\vec{\tilde{p}})\xrightarrow{\tilde{\tilde{G}}^{\tau}}\int\tilde{\tilde{G}}^{\tau}(P,\vec{\tilde{p}},\vec{\tilde{p}},\vec{\tilde{q}},\vec{\tilde{q}})\,d\vec{\tilde{q}}\,d\vec{\tilde{q}}\cdot f(\vec{\tilde{q}},\vec{\tilde{q}})$$

we obtain for the inverse operator $\left(\tilde{\tilde{G}}^{\tau}\right)^{-1}$

$$\left(\tilde{\tilde{G}}^{\tau}\right)^{-1}=\left(\tilde{\tilde{G}}_0^{\tau}\right)^{-1}-\left(\tilde{\tilde{G}}_0^{\tau}\right)^{-1}\left[\widetilde{\widetilde{G_0KG_0}}\right]^{\tau}\left(\tilde{\tilde{G}}_0^{\tau}\right)^{-1}+\dots \quad (3.15)$$

Introducing the quasipotential of three-body system K^{τ}

$$\frac{i}{\pi^2}K^{\tau}=\left(\tilde{\tilde{G}}_0^{\tau}\right)^{-1}\left[\widetilde{\widetilde{G_0KG_0}}\right]^{\tau}\left(\tilde{\tilde{G}}_0^{\tau}\right)^{-1}+\left(\tilde{\tilde{G}}_0^{\tau}\right)^{-1}\left[\widetilde{\widetilde{G_0KG_0KG_0}}\right]^{\tau}\left(\tilde{\tilde{G}}_0^{\tau}\right)^{-1}+\dots \quad (3.16)$$

the following equation can be derived for the R.G.F of the three particles

$$\tilde{\tilde{G}}^{\tau}=\tilde{\tilde{G}}_0^{\tau}+\frac{i}{\pi^2}\tilde{\tilde{G}}_0^{\tau}K^{\tau}\tilde{\tilde{G}}^{\tau}. \quad (3.17)$$

From the definition of the quasipotential (3.16) it is obvious that it can be represented as follows

$$K^{\tau}=V_1+V_2+V_3+V_T \quad (3.18)$$

where (see Appendix)

$$V_i(P, \vec{p}, \vec{p}_i, \vec{q}, \vec{q}_i) = \delta(\vec{p}_i - \vec{q}_i)\,\omega_i(\vec{p})\,K_i^{(2)\tau}(P_0 - \omega_i, \vec{P}_i, \vec{p}_i, \vec{q}_i) \qquad (3.18a)$$

is the sum of all terms of (3.16) containing the function $\delta(\vec{p}_i - \vec{q}_i)$ and corresponds to the pair interaction. V_T is analogical to the potential of the three-body forces.

Here $K_i^{(2)\tau}$ denotes the quasipotential of the i-th two-body subsystem which is defined by (2.16) and occurs in (2.17), (2.19) and (2.22).

4. Bound States in the Three-Body System

Let us now write the term corresponding to the contribution to (3.12) of a one-particle state $|M, \vec{K}, \alpha\rangle$ characterized by the mass M, four-momentum $\vec{K}$ and discrete quantum numbers α

$$I_2(E, \vec{P}, \vec{p}, \vec{p}, \vec{q}, \vec{q}) \approx \frac{i\pi^2}{2E}\,\delta\left(E - \sqrt{\vec{P}^2 + M^2}\right) \chi_\alpha(\vec{p}, \vec{p}) \otimes \chi_\alpha(\vec{q}, \vec{q}), \qquad (4.1)$$

where

$$\chi_\alpha(\vec{p}, \vec{p})\,\delta(\vec{P} - \vec{K}) = \frac{2\sqrt{2E_M}}{(2\pi)^{9/2}} \int e^{-i\sum_m \vec{p}_m \vec{x}_m} (d\vec{x}) \langle 0|\, \psi_1(0,\vec{x}_1)\psi_2(0,\vec{x}_2)\psi_3(0,\vec{x}_3)|\alpha\rangle. \qquad (4.2)$$

From (4.1) and (3.12b) it follows that the pole contribution to the R.G.F. has the form

$$\tilde{\tilde{G}}_1^{\,2}(P,\vec{\tilde{p}},\vec{p},\vec{\tilde{q}},\vec{q}) \approx \frac{i\pi^2}{2E_M}\,\frac{\sum_\alpha \chi_\alpha(\vec{\tilde{p}},\vec{p})\otimes\chi_\alpha^+(\vec{\tilde{q}},\vec{q})}{P_0-\sqrt{\vec{P}^2+M^2}+i\varepsilon} \, . \tag{4.3}$$

Picking out the pole $P_0=\sqrt{\vec{P}^2+M^2}$ from Eq.(3.17) a homogeneous equation for $\chi_\alpha(\vec{\tilde{p}},\vec{p})$ can be obtained

$$\left[\omega_1(\vec{p}_1)+\omega_2(\vec{p}_2)+\omega_3(\vec{p}_3)-P_0\right]\chi_\alpha(\vec{\tilde{p}},\vec{p})=\frac{1}{\omega_1(\vec{p}_1)\omega_2(\vec{p}_2)\omega_3(\vec{p}_3)}\int K^2(P,\vec{\tilde{p}},\vec{p},\vec{\tilde{q}},\vec{q})\chi_\alpha(\vec{\tilde{q}},\vec{q})\,d\vec{\tilde{q}}\,d\vec{q} \tag{4.4}$$

$$P_0=\sqrt{\vec{P}^2+M^2}\equiv E_M \, .$$

Here one should especially take notice of the properties (3.18), (3.18a) of the quasipotentials defined by (2.16) and (3.16). This again stresses the deep analogy of the equal-time Bethe-Salpeter amplitudes (2.10) and (3.12) with the non-relativistic wave functions of two and three particles respectively.

Writing the identity

$$\tilde{\tilde{G}}_1^{\,2}\left[(\tilde{\tilde{G}}_0^{\,2})^{-1}-\frac{i}{\pi^2}K^2\right]\tilde{\tilde{G}}_1^{\,2}=\tilde{\tilde{G}}_1^{\,2} \tag{4.5}$$

and picking out the residue at the bound-state pole from both sides of (4.5), it is easy to find the following normalization condition

$$\int 2\omega_1(\vec{p}_1)\omega_2(\vec{p}_2)\omega_3(\vec{p}_3)\chi_\alpha^+(\vec{\tilde{p}},\vec{p})\chi_\alpha(\vec{\tilde{p}},\vec{p})\,d\vec{\tilde{p}}\,d\vec{p}+\left(\chi_\alpha^+\frac{\partial K^2}{\partial P_0}\chi_\alpha\right)\Big|_{P_0=E_M}=2E_M \, . \tag{4.6}$$

Note, that if the quasipotential in (4.4) correctly reproduces the spectral representation (3.12b) for the R.G.F. of three particles by means of Eq.(3.17), then Eq.(4.4) has only positive energy spectrum.

5. <u>Scattering Amplitudes in the Three-Body Systems.</u>

In this section we will show that the R.G.F. Eq.(3.12b) contains the complete information about all possible transitions in the three-particle system and the equations required can be constructed by direct analogy with the non-relativistic potential theory.

Thus, we start with the definition of the two-particle operators $\tilde{\tilde{G}}_i^{\tau}$ and T_i in the three-particle Hilbert space

$$\tilde{\tilde{G}}_i^{\tau} = \tilde{\tilde{G}}_0^{\tau} + \frac{i}{\pi^2}\tilde{\tilde{G}}_0^{\tau} V_i \tilde{\tilde{G}}_i^{\tau} \qquad (5.1)$$

$$T_i = V_i + \frac{i}{\pi^2} V_i \tilde{\tilde{G}}_0^{\tau} T_i \;. \qquad (5.2)$$

Taking into account (3.18a) one gets

$$\tilde{\tilde{G}}_i^{\tau}(P, \vec{\tilde{P}}_i, \vec{P}_i, \vec{q}_i, \vec{\tilde{q}}_i) = \frac{\pi}{\omega_i(\vec{p}_i)}\, \delta(\vec{\tilde{p}}_i - \vec{q}_i)\, \tilde{G}_i^{(2)\tau}(P_0 - \omega_i, \vec{P}_i, \vec{\tilde{p}}_i, \vec{q}_i) \qquad (5.3)$$

$$T_i(P, \vec{\tilde{P}}_i, \vec{P}_i, \vec{\tilde{q}}_i, \vec{q}_i) = \omega_i(\vec{p}_i)\, \delta(\vec{\tilde{p}}_i - \vec{\tilde{q}}_i)\, T_i^{(2)\tau}(P_0 - \omega_i, \vec{P}_i, \vec{\tilde{p}}_i, \vec{q}_i) \;. \qquad (5.4)$$

The upper index "2" indicates that the operator belongs to the two-particle Hilbert space considered in section 2. The suffix numerates the two-particle subsystem.

Using (5.1) and (5.2) or (5.3) and (5.4) it is easy to obtain

$$\tilde{\tilde{G}}_i^{\tau} = \tilde{\tilde{G}}_0^{\tau} + \frac{i}{\pi^2}\tilde{\tilde{G}}_0^{\tau} T_i \tilde{\tilde{G}}_0^{\tau} \;. \qquad (5.5.)$$

To construct the S- matrix elements we introduce now the operators $M_{\alpha\beta}$ which will be used as the so-called transition operators in /13,14/. To this end we recall some results obtained in the quantum field theory /2,3/. In what follows we agree to denote by (α) $\alpha = 0,1,2,3$ either the state in which the α-th particle is free and the two others form the bound-state (if $\alpha \neq 0$) or the state of three free particles (if $\alpha = 0$). Then the part of the 6-time Green function corresponding to the sum of all connected graphs

$$G_1' = G_1 - G_{10} - \sum_{k=1}^{3} \left(G_{1k} - G_{10} \right) \tag{5.6*}$$

can be written as

$$G_1'(P, \tilde{p}_\beta, \bar{p}_\beta, \tilde{q}_\alpha, \bar{q}_\alpha) = \frac{i}{p_\beta^2 - m_\beta^2 + i\varepsilon} \left\{ \frac{i\pi}{2E_\beta^{(1)}} \frac{\Psi_3(\bar{p}_\beta)}{(P_\beta^0 - E_\beta^{(1)} + i\varepsilon)} T_{\beta\alpha}(P, \tilde{p}_\beta, \tilde{q}_\alpha) \frac{\Psi_\alpha^+(\bar{q}_\alpha)}{(Q_\alpha^0 - E_\alpha^{(1)} + i\varepsilon)} \frac{i\pi}{2E_\alpha^{(1)}} + \right.$$
$$\left. + \operatorname{Reg}\left(P_\beta^0 - E_\beta^{(1)}; \ Q_\alpha^0 - E_\alpha^{(1)} \right) \right\} \frac{i}{q_\alpha^2 - m_\alpha^2 + i\varepsilon} \tag{5.7}$$

$$G_1'(P, \tilde{p}, \bar{p}, \tilde{q}_\alpha, \bar{q}_\alpha) = \frac{(i)^3}{(p_1^2 - m_1^2 + i\varepsilon)(p_2^2 - m_2^2 + i\varepsilon)(p_3^2 - m_3^2 + i\varepsilon)} \left\{ T_{0\alpha}(P, \tilde{p}, \bar{p}, \tilde{q}_\alpha) \times \right.$$
$$\left. \times \frac{i\pi}{2E_\alpha^{(1)}} \frac{\Psi_\alpha^+(\bar{q}_\alpha)}{(Q_\alpha^0 - E_\alpha^{(1)} + i\varepsilon)} + \operatorname{Reg}\left(Q_\alpha^0 - E_\alpha^{(1)} \right) \right\} \frac{i}{q_\alpha^2 - m_\alpha^2 + i\varepsilon} \tag{5.8}$$

$$E_\beta^{(1)} = \sqrt{\vec{P}_\beta^2 + M_\beta^{(1)2}} \ ; \quad E_\alpha^{(1)} = \sqrt{\vec{Q}_\alpha^2 + M_\alpha^{(1)2}} \ .$$

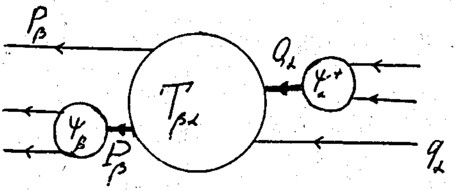
Here the poles of the channels $\alpha \neq 0$, $\beta \neq 0$ and $\alpha \neq 0, \beta = 0$ are explicitly picked out. $\Psi_\alpha(\bar{p}_\alpha)$, $\Psi_\alpha^+(\bar{p}_\alpha)$ are the 2-time Bethe-Salpeter amplitudes. The wave-functions (2,19) are connected with them in the following way

$$\chi_\alpha^{(1)}(\vec{p}_\alpha) = \int_{-\infty}^{\infty} d\bar{p}_\alpha^0 \, \Psi_\alpha(\bar{p}_\alpha) \qquad \chi_\alpha^{(1)+}(\vec{p}_\alpha) = \int_{-\infty}^{\infty} d\bar{p}_\alpha^0 \, \Psi_\alpha^+(\bar{p}_\alpha) .$$

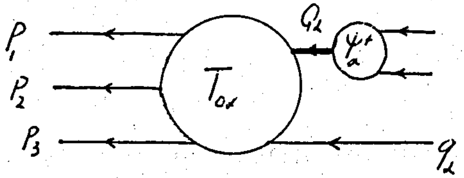
---

* $G_k = S_k \cdot G_k^{(2)}$ where $G_k^{(2)}$ is the 4-time Green function of the k-th two-particle subsystem.

The terms (5.7) and (5.8) can be pictured graphically



(5.7a)



(5.8a)

The considerations of the cases $\alpha = 0$ are quite similar.

The S- matrices are given by the residues of G' (5.6) at the poles on the various mass shells

$$S_{\beta\leftarrow\alpha} = \delta_{\alpha\beta}\delta_{ij}\,\delta(\vec{q}_\alpha - \vec{p}_\alpha)\,\delta(\vec{Q}_\alpha - \vec{P}_\alpha) + \pi^3\delta(P-Q)\times$$

(5.9)

$$\times\frac{1}{\sqrt{\omega_\beta(\vec{p}_\beta)}\sqrt{E_\beta^{(i)}}\sqrt{\omega_\alpha(\vec{q}_\alpha)}\sqrt{E_\alpha^{(\psi)}}}\,T_{\beta\alpha}(P,\tilde{p}_\beta,\tilde{q}_\alpha)\Bigg|_{\substack{P_\beta^2 = M_\beta^{(i)2} \quad p_\beta^2 = m_\beta^2 \\ Q_\alpha^2 = M_\alpha^{(\psi)2} \quad q_\alpha^2 = m_\alpha^2}}$$

if $\alpha \neq 0$ and $\beta \neq 0$;

$$S_{0\,\alpha} = \pi^3\delta(P-Q)\frac{T_{0\alpha}(P,\tilde{p},\tilde{p},\tilde{q}_\alpha)\Big|_{p_e^2 = m_e^2;\ q_\alpha^2 = m_\alpha^2;\ Q_\alpha^2 = M_\alpha^{(\psi)2}}}{\sqrt{\omega_1(\vec{p}_1)}\sqrt{\omega_2(\vec{p}_2)}\sqrt{\omega_3(\vec{p}_3)}\sqrt{\omega_\alpha(\vec{q}_\alpha)}\sqrt{E_\alpha^{(\psi)}}} \qquad (5.10)$$

if $\alpha \neq 0$;

and

$$S'_{c\leftarrow c} = \delta(\vec{p}_1-\vec{q}_1)\,\delta(\vec{p}_2-\vec{q}_2)\,\delta(\vec{p}_3-\vec{q}_3) + \tag{5.11}$$
$$+\pi^3\delta(P-Q)\Big[\sum_{m=1}^{3}\delta(\vec{p}_m-\vec{q}_m)f_m + T_{00}(P,\tilde{p},\bar{p},\tilde{q},\bar{q})\Big|_{p_i^2=q_i^2=m_i^2}\Big].$$

Let us now introduce the operators $M_{\alpha\beta}(P,\tilde{p}_\alpha,\bar{p}_\alpha,\tilde{q}_\beta,\bar{q}_\beta)$

$$\tilde{\tilde{G}}'^{\,z} = \tilde{\tilde{G}}^{\,oz}_{\alpha} + \frac{i}{\pi^2}\tilde{\tilde{G}}^{\,z}_{\alpha} M_{\alpha\beta}\tilde{\tilde{G}}^{\,z}_{\beta} \tag{5.12}$$

here $\tilde{\tilde{G}}^{\,oz}_{\alpha} = \tilde{\tilde{G}}^{\,oz}_{r_0}$ if $\alpha = 0.$

Integrating Eq.(3.10) over the relative energies and picking out the retarded parts from both sides of the equality (5.6) one gets

$$\sum_{i\neq\beta}\frac{i}{\pi^2}\tilde{\tilde{G}}^{\,z}_{r_0}T_i\tilde{\tilde{G}}^{\,z}_{r_0} + \tilde{\tilde{G}}^{\,oz} = \frac{i}{\pi^2}\tilde{\tilde{G}}^{\,z}_{\beta}M_{\beta\alpha}\tilde{\tilde{G}}^{\,oz}_{\alpha}. \tag{5.13}$$

Using (5.7) and (5.8) it can be shown that $\tilde{\tilde{G}}^{\,z}(P,\vec{\tilde{p}},\vec{\bar{p}},\vec{\tilde{q}},\vec{\bar{q}})$ has the following pole structure

$$\tilde{\tilde{G}}'^{\,z} \approx \frac{i\pi^2\quad\chi_\beta(\vec{p}_\beta)}{(P_0-E^{(i)}_\beta-\omega_\beta(\vec{p}_\beta)+i\varepsilon)2E^{(i)}_\beta\omega_\beta}\,[T_{\beta\alpha}]\,\frac{i\pi^2\chi^+_\alpha(\vec{p}_\alpha)}{(Q_0-E^{(i)}_\alpha-\omega_\alpha(\vec{q}_\alpha)+i\varepsilon)2E^{(i)}_\alpha\omega_\alpha} \tag{5.7'}$$

$$\tilde{\tilde{G}}^{\,oz} \approx \frac{i\pi^2}{(P_0-\omega_1(\vec{p}_1)-\omega_2(\vec{p}_2)-\omega_3(\vec{p}_3)+i\varepsilon)2\omega_1\omega_2\omega_3}\,[T_{0\alpha}]\,\frac{i\pi^2\chi^+_\alpha(\vec{p}_\alpha)}{(Q_0-E^{(i)}_\alpha-\omega_\alpha(\vec{q}_\alpha)+i\varepsilon)2E^{(i)}_\alpha\omega_\alpha}. \tag{5.8'}$$

These are the poles corresponding to the initial (α) and final states (β).

Here $[T_{\beta\alpha}]$ are the amplitudes $T_{\beta\alpha}$ on the appropriate mass shell. Then, according to (5.13) we find

1) $\alpha \neq 0$; $\beta \neq 0$

$$\frac{i}{\pi^2}\int \chi_\beta^{+(2)}(\vec{p}_\beta)\, d\vec{p}_\beta\, M_{\beta\alpha}(P,\vec{P}_\beta,\vec{p}_\beta,\vec{q}_\alpha,\vec{q}_\alpha)\, d\vec{q}_\alpha\, \chi_\alpha^{(2)}(\vec{q}_\alpha)\Big|_{P_0=E_\beta^{(1)}+\omega_\beta(\vec{p}_\beta)=E_\alpha^{(1)}+\omega_\alpha(\vec{q}_\alpha)} = \qquad (5.14a)$$

$$= \left[T_{\beta\alpha}\right]$$

2) $\alpha \neq 0$; $\beta = 0$

$$\frac{i}{\pi^2}\int M_{0\alpha}(P,\vec{P},\vec{P},\vec{q}_\alpha,\vec{q}_\alpha)\, d\vec{q}_\alpha\, \chi_\alpha^{(2)}(\vec{q}_\alpha)\Big|_{P_0=\omega_1(\vec{p}_1)+\omega_2(\vec{p}_2)+\omega_3(\vec{p}_3)=E_\alpha^{(1)}+\omega_\alpha(\vec{q}_\alpha)} = \left[T_{0\alpha}\right] \qquad (5.14b)$$

3) $\alpha = 0$; $\beta = 0$

$$\frac{i}{\pi^2} M_{00}(P,\vec{P},\vec{P},\vec{q},\vec{q})\Big|_{P_0=\sum_e \omega_e(\vec{p}_e)=\sum_e \omega_e(\vec{q}_e)} = \left[T_{00}\right] + \qquad (5.14c)$$

$$+\frac{i}{\pi^2}\sum \delta(\vec{p}_\kappa-\vec{q}_\kappa)\,\omega_\kappa(\vec{p}_\kappa)\, T_\kappa^{(2)2}(P_0-\omega_\kappa,\vec{P}_\kappa,\vec{p}_\kappa,\vec{q}_\kappa)\Big|_{P_0-\omega_\kappa=\omega_i(\vec{p}_i)+\omega_e(\vec{p}_e)=\omega_i(\vec{q}_i)+\omega_e(\vec{q}_e)}.$$

Using (5.9), (5.10) and (5.11) we express finally the various S-matrices in terms of $M_{\alpha\beta}$

1) $\alpha \neq 0$; $\beta \neq 0$

$$S_{\beta\leftarrow\alpha} = \delta_{\alpha\beta}\,\delta_{ij}\,\delta(\vec{P}_\alpha-\vec{Q}_\alpha)\,\delta(\vec{p}_\alpha-\vec{q}_\alpha) + i\pi\,\delta(P-Q)\times \qquad (5.15a)$$

$$\times\left[\omega_\beta(\vec{p}_\beta)E_\beta^{(1)}\omega_\alpha(\vec{q}_\alpha)E_\alpha^{(1)}\right]^{-\frac{1}{2}}\left(\chi_\beta^+ M_{\beta\alpha}\chi_\alpha\right)\Big|_{P_0=\omega_\beta(\vec{p}_\beta)+E_\beta^{(1)}=\omega_\alpha(\vec{q}_\alpha)+E_\alpha^{(1)}}$$

2) $\beta = 0$ $\quad \alpha \neq 0$

$$S_{0\leftarrow\alpha} = i\pi\,\delta(P-Q)\frac{M_{0\alpha}\chi_\alpha\Big|_{P_0=\sum_e\omega_e(\vec{p}_e)=\omega_\alpha(\vec{q}_\alpha)+E_\alpha^{(1)}}}{\sqrt{\omega_1(\vec{p}_1)}\sqrt{\omega_2(\vec{p}_2)}\sqrt{\omega_3(\vec{p}_3)}\sqrt{\omega_\alpha(\vec{q}_\alpha)}\sqrt{E_\alpha^{(1)}}} \qquad (5.15b)$$

3) $\alpha = 0$; $\beta = 0$

$$S_{0\leftarrow 0} = \delta(\vec{p}_1-\vec{q}_1)\,\delta(\vec{p}_2-\vec{q}_2)\,\delta(\vec{p}_3-\vec{q}_3) + i\pi\,\delta(P-Q)\times \qquad (5.15c)$$

$$\times\left[\prod_e \omega_e(\vec{p}_e)\,\omega_e(\vec{q}_e)\right]^{-\frac{1}{2}} M_{00}\Big|_{P_0=\sum_e\omega_e(\vec{p}_e)=\sum_e\omega_e(\vec{q}_e)}$$

P_0 is put throughout equal to the total energies of the initial and final states. Now it is clear that $M_{\alpha\beta}$ have the same meaning as the transition operators $U^{(+)}_{\alpha\beta}$ /7,13,14,16/ of the non-relativistic three-body theory. The operators $U^{(-)}_{\alpha\beta}$ could be introduced in a similar way.

6. Three-Particle Equations.

In this section we will derive some useful relations for $M_{\alpha\beta}$ and $\tilde{\tilde{C}}_1^{\tau}$ following from the main equation (3.17). To this end we will consider the pair interaction approximation $K_p^{\tau} = V_1 + V_2 + V_3$. In such a case the kernels of the various three-particle equation are expressed only in terms of the two-body quantities defined in section 2. The three-body forces can be included in the following way. If $\tilde{\tilde{C}}_{1p}^{\tau}$ satisfies Eq.(3.17) in the approximation $K^{\tau} = K_p^{\tau}$

$$\tilde{\tilde{C}}_{1p}^{\tau} = \tilde{\tilde{C}}_{10}^{\tau} + \frac{i}{\pi^2} \tilde{\tilde{C}}_{10}^{\tau} K_p^{\tau} \tilde{\tilde{C}}_{1p}^{\tau} . \tag{6.1}$$

Then the exact equation can be written in the form

$$\tilde{\tilde{C}}_1^{\tau} = \tilde{\tilde{C}}_{1p}^{\tau} + \frac{i}{\pi^2} \tilde{\tilde{C}}_{1p}^{\tau} V_T \tilde{\tilde{C}}_1^{\tau} . \tag{6.2}$$

In any case the solution of the approximate equation (6.1) is interesting of itself in some sense. From (6.1) it follows that

$$\tilde{\tilde{C}}^{\tau} = \tilde{\tilde{C}}_{10}^{\tau} + g_1 + g_2 + g_3 , \tag{6.3}$$

where

$$g_i = \frac{i}{\pi^2} \tilde{\tilde{C}}_{10}^{\tau} V_i \tilde{\tilde{C}}_{1p}^{\tau} . \tag{6.4}$$

Using (5.1) we rewrite (6.1) in the form

$$\tilde{\tilde{G}}_{p}^{z} = \tilde{\tilde{G}}_{i}^{z} + \frac{i}{\pi^2} \tilde{\tilde{G}}_{i}^{z} \sum_{k \neq i} V_k \tilde{\tilde{G}}_{p}^{z} . \tag{6.5}$$

Consequently,

$$g_i = \frac{i}{\pi^2} \tilde{\tilde{G}}_{0}^{z} V_i \tilde{\tilde{G}}_{i}^{z} + \left(\frac{i}{\pi^2}\right)^2 \tilde{\tilde{G}}_{0}^{z} V_i \tilde{\tilde{G}}_{i}^{z} \sum_{k \neq i} V_k \tilde{\tilde{G}}_{p}^{z} . \tag{6.6}$$

Comparing (5.1) with (5.5) we have

$$\tilde{\tilde{G}}_{0}^{z} V_i \tilde{\tilde{G}}_{i}^{z} = \tilde{\tilde{G}}_{0}^{z} T_i \tilde{\tilde{G}}_{0}^{z} . \tag{6.7}$$

Taking into account the latter for (6.6) we finally derive the system of three integral equations with three unknown functions g_i

$$g_i = \frac{i}{\pi^2} \tilde{\tilde{G}}_{0}^{z} T_i \tilde{\tilde{G}}_{0}^{z} + \frac{i}{\pi^2} \sum_{k \neq i} \tilde{\tilde{G}}_{0}^{z} T_i g_k . \tag{6.8}$$

These equations can be rewritten in a matrix form.

$$\begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} = \frac{i}{\pi^2} \tilde{\tilde{G}}_{0}^{z} \begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix} \tilde{\tilde{G}}_{0}^{z} + \frac{i}{\pi^2} \begin{pmatrix} 0 & T_1 & T_1 \\ T_2 & 0 & T_2 \\ T_3 & T_3 & 0 \end{pmatrix} \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} . \tag{6.9}$$

The relations (6.4) and (6.9) represent together the equation for R.G.F. of three particles expressed in terms of the two-particle off-shell amplitudes.

Using the definition (5.12) for the transition operators and Eq.(6.5) we get the following equations for $M_{\alpha\beta}$

$$M_{\alpha\beta}=\sum_{i\neq\alpha} V_i+\frac{i}{\pi^2}\sum_{i\neq\beta} M_{\alpha i}\,\tilde{\tilde{G}}_i^{\,r} V_i \qquad (6.10)$$

or

$$M_{\alpha\beta}=\sum_{i\neq\alpha} V_i+\frac{i}{\pi^2}\sum_{i\neq\beta} M_{\alpha i}\,\tilde{\tilde{G}}_0^{\,r} T_i \quad . \qquad (6.11)$$

We can also introduce the new operators $A_{\alpha\beta}$ /16/ in a symmetric way

$$\tilde{\tilde{G}}^{\,r}=\delta_{\alpha\beta}\,\tilde{\tilde{G}}_\alpha^{\,r}+\frac{i}{\pi^2}\,\tilde{\tilde{G}}_\alpha^{\,r} A_{\alpha\beta}\,\tilde{\tilde{G}}_\beta^{\,r} \quad . \qquad (6.12)$$

It is easy to show that

$$\chi_\alpha^+ A_{\alpha\beta}\,\chi_\beta=\chi_\alpha^+ M_{\alpha\beta}\,\chi_\beta \qquad (6.13)$$

provided P_0 is equal to the total energies of the initial and final states. In this sense the operators $A_{\alpha\beta}$ are equivalent to $M_{\alpha\beta}$. The advantage is that the equation for $A_{\alpha\beta}$ can be expressed only in terms of the two-particle off-shell amplitudes

$$A_{\alpha\beta}=\frac{\pi^2}{i}\left(\tilde{\tilde{G}}_0^{\,r}\right)^{-1}\left(1-\delta_{\alpha\beta}\right)+\frac{i}{\pi^2}\sum_{m\neq\beta} A_{\alpha m}\,\tilde{\tilde{G}}_0^{\,r} T_m \quad . \qquad (6.14)$$

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APPENDIX.

Let us prove the relation (3.18a)

$$V_i(P_0,\vec{P}_i,\vec{p}_i,\vec{q}_i,\vec{q}_i)=\delta(\vec{p}_i-\vec{q}_i)\,\omega_i(\vec{p}_i)\,K_i^{(2)\tau}(P_0-\omega_i,\vec{P}_i,\vec{p}_i,\vec{q}_i)\ .$$

Rewriting the series (3.16) in a compact form

$$\frac{i}{\pi^2}K^{\tau}=\left(\tilde{\tilde{G}}_0^{\tau}\right)^{-1}-\left(\tilde{\tilde{G}}^{\tau}\right)^{-1} \qquad (A.1)$$

we see that

$$\frac{i}{\pi^2}V_i=\left(\tilde{\tilde{G}}_0^{\tau}\right)^{-1}-\left(\tilde{\tilde{G}}_i^{\tau}\right)^{-1}\ , \qquad (A.2)$$

where G_i is the 6-time Green function in the approximation of non-interacting i-th particle. This is evident since by definition V_i is the sum of all terms of (3.16) which are diagonal with respect to the i-th particle three-dimensional momentum. Then, according to (3.12d) we have

$$\delta(\vec{P}-\vec{K}_n)\chi_{on}(\vec{P}_i,\vec{p}_i)=\frac{2\sqrt{2E_n}}{(2\pi)^{9/2}}\int(d\vec{x})\,e^{-i\vec{p}_i\vec{x}_i}\langle 0|\psi_i(0,\vec{x}_i)|\vec{K}_i\rangle$$
$$e^{-i\vec{p}_e\vec{x}_e-i\vec{p}_k\vec{x}_k}\langle 0|\psi_e(0,\vec{x}_e)\,\psi_k(0,\vec{x}_k)|\vec{K}_n-\vec{K}_i,n(2)\rangle= \qquad (A.3)$$
$$=\delta(\vec{p}_i-\vec{K}_i)\,\delta(\vec{P}_i-\vec{K}_n+\vec{K}_i)\frac{2\sqrt{2E_n}\,(2\pi)^{9/2}}{(2\pi)^{9/2}\sqrt{2\omega_i(\vec{p}_i)}\,2\sqrt{E_{n(2)}}}\chi_{on}^{(2)i}(\vec{P}_i)$$

hence

$$\chi_{on}(\vec{P}_i,\vec{p}_i)=\delta(\vec{p}_i-\vec{K}_i)\frac{\sqrt{2E_n}}{\sqrt{2\omega_i(\vec{p}_i)}\sqrt{E_{n(2)}}}\chi_{on}^{(2)i}(\vec{P}_i)\ . \qquad (A.4)$$

Similarly one gets for χ_{on}^{+}

$$\chi^{+}_{on}(\vec{\tilde{p}}_i,\vec{\tilde{p}}_i)=\delta(\vec{p}_i-\vec{k}_i)\frac{\sqrt{2E_n}}{\sqrt{2\omega_i(\vec{p}_i)}\sqrt{E_{n(2)}}}\chi^{+(2)i}_{on}(\vec{p}_i) \quad . \qquad (A.5)$$

Here $\chi^{(2)i}_{on}, \chi^{+(2)i}_{on}$ are the two-particle wave functions of the i-th subsystem which are defined by (2.10). Inserting (A.3) and (A.4) into (3.12c) we find

$$I_2(E,\vec{P},\vec{\tilde{p}}_i,\vec{\tilde{p}}_i,\vec{\tilde{q}}_i,\vec{\tilde{q}}_i)=\sum_n \frac{i\pi^2}{2\omega_i(\vec{p}_i)E_{n(2)}}\int d\vec{k}_i\,\delta(E-E_{n(2)}-\omega_i(\vec{k})) \qquad (A.6)$$
$$\delta(\vec{p}_i-\vec{k}_i)\delta(\vec{q}_i-\vec{k}_i)\delta(\vec{P}-\vec{K}_{n(2)}-\vec{k}_i)\chi^{(2)i}_{on}(\vec{\tilde{p}}_i)\otimes\chi^{+(2)i}_{on}(\vec{\tilde{q}}_i) \; .$$

Then, integrating Eq.(A.6) over $\vec{k}_i$ one obtains

$$I_2(E,\vec{P},\vec{\tilde{p}}_i,\vec{\tilde{p}}_i,\vec{\tilde{q}}_i,\vec{\tilde{q}}_i)=\frac{\pi}{\omega_i(\vec{p}_i)}\frac{i\pi}{2E_{n(2)}}\sum_{n(2)}\delta(E-\omega_i(\vec{p}_i)-E_{n(2)})\times \qquad (A.7)$$
$$\times\delta(\vec{P}_i-\vec{K}_{n(2)})\chi^{(2)i}_{on}(\vec{\tilde{p}}_i)\otimes\chi^{+(2)i}_{on}(\vec{\tilde{q}}_i) \; .$$

Therefore I_2 can be written as follows

$$I_2(E,\vec{P},\vec{\tilde{p}}_i,\vec{\tilde{p}}_i,\vec{\tilde{q}}_i,\vec{\tilde{q}}_i)=\delta(\vec{\tilde{p}}_i-\vec{\tilde{q}}_i)\frac{\pi}{\omega_i(\vec{p}_i)}I_2^{(2)i}(E-\omega_i,\vec{P}_i,\vec{\tilde{p}}_i,\vec{\tilde{q}}_i) \qquad (A.8)$$

$I_2^{(2)i}$ is the spectral density of R.G.F. of the i-th two-particle subsystem which is defined by (2.9a). Finally, inserting (A.8) into (3.12b) and comparing it with (2.9) we obtain

$$\tilde{\tilde{G}}_i^{2}(P,\vec{\tilde{p}}_i,\vec{\tilde{p}}_i,\vec{\tilde{q}}_i,\vec{\tilde{q}}_i)=\frac{\pi}{\omega_i(\vec{p}_i)}\tilde{\tilde{G}}_i^{(2)2}(P_0-\omega_i,\vec{P}_i,\vec{\tilde{p}}_i,\vec{\tilde{q}}_i)\delta(\vec{\tilde{p}}_i-\vec{\tilde{q}}_i) \; . \qquad (A,9)$$

A similar relation holds for $\tilde{\tilde{G}}_0^{2}$

$$\tilde{\tilde{G}}_0^{2}(P,\vec{\tilde{p}}_i,\vec{\tilde{p}}_i,\vec{\tilde{q}}_i,\vec{\tilde{q}}_i)=\frac{\pi}{\omega_i(\vec{p}_i)}\tilde{\tilde{G}}_0^{(2)2}(P_0-\omega_i,\vec{P}_i,\vec{\tilde{p}}_i,\vec{\tilde{q}}_i)\delta(\vec{\tilde{p}}_i-\vec{\tilde{q}}_i) \; . \qquad (A,10)$$

Inserting now (A,9) and (A,10) into (A,2) it is easy to get (3,18a).

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