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Introduction

There exists a number of homogeneous spaces group of motion of which may serve for the definition of the Lorentz group. Out of these homogeneous spaces the most familiar is the three-dimensional hyperboloid. It has turned out, however, that in certain respect it is expediently to treat the Lorentz group as a group of motion of the two (complex) dimensional complex sphere $\vec{S}^2 = S_1^2 + S_2^2 + S_3^2$. Namely, it has been pointed out by H. Joos and R. Schrader /1/ and by the authors of /2/ that if the Lorentz group is considered in this spirit, matrix elements of its unitary representation take a rather simple form.

A three-dimensional complex vector $\vec{S}$ is the self-dual part of the Lorentz covariant antisymmetric tensor $S_{\mu\nu}$, i.e. $S_k = S_{0k} + \frac{i}{2}\epsilon_{k\ell m} S_{\ell m} (k,\ell,m=1,2,3)$ Since the real and imaginary part of $\vec{S}$ transform like the electric and magnetic field respectively the invariance of $\vec{S}^2 \approx (\vec{E} + i\vec{B})^2$ under proper Lorentz transformation is evident. And conversely, it can be proved /3/ that the connected part of three-dimensional complex rotation group is isomorphic to the proper Lorentz group.

1. Little Groups on the Complex Sphere of the Zero and Non-Zero Radius

Let us associate to a three-dimensional complex vector $\vec{S} = (S_1, S_2, S_3)$ the matrix $\hat{S} = \begin{bmatrix} S_3 & S_1 - iS_2 \\ S_1 + iS_2 & -S_3 \end{bmatrix}$. Under $g \subset SL(2,C)$ $\hat{S}$ transforms as $T_g \hat{S} \equiv \hat{S}' = g \hat{S} g^{-1}$ and clearly $\vec{S}^2 = S_1^2 + S_2^2 + S_3^2$ is invariant. And conversely, it can be shown that if one excludes the point $\vec{S} = (0,0,0)$ any two complex vectors $\vec{S}$, $\vec{S}'$ of the same length can be translated to each other by means of a suitable $SL(2,C)$ transformation. Consider now the point $\vec{S}_0 = (-iS, S, 0)\ (S \neq 0)$ on the complex sphere of the zero radius Σ_0. The little group of this point i.e. the subgroup satisfying $T_\eta \hat{S}_0 = \hat{S}_0$ constitutes elements of the type $\eta = \begin{bmatrix} 1 & \eta \\ 0 & 1 \end{bmatrix}$. This is the horyspheric subgroup /5,6/ isomorphic to the two-dimensional translation group $T(2)$. An arbitrary other point $\hat{S} = T_g \hat{S}_0$ ($g \in SL(2,C)$) on Σ_0 has the little group $\eta_g = g \eta g^{-1}$. The converse statement is also true i.e. any three-dimensional complex vector having the horyspheric little group η_g is on the sphere of the zero radius. It can be shown in an analogous way that the little group of a vector on the complex sphere of the non-zero radius is the group $H = SO(2) \times SO(1,1)$ /2/. Spherical functions of the Lorentz group with respect to the subgroup H have been studied in /2/. Here we derive the spherical functions with respect to the horyspheric subgroup.

2. Spherical Functions on the Complex Sphere of the Zero Radius

Consider the state $|\,>$ satisfying

$$T_\eta |\,> = |\,>,$$

where T_η is the unitary representation of the horyspheric subgroup. Then spherical functions of the Lorentz group with respect to the subgroup η are defined as

$$f_\Phi(\Sigma_0) = <\Phi|T\ |>^* . \qquad (1)$$

Here T_g is the unitary representation of the Lorentz group and $|\Phi>$ is a basis vector specified below. The quantity Σ_0 indicates that $f_\Phi(\Sigma_0)$ is a function over the factor space g/η, i.e. it is defined over the complex sphere of the zero radius Σ_0.

Explicit form of the spherical functions (1) can be found by solving the eigenvalue equation of the Casimir operators. To this end we introduce the combination

$$\vec{J} = \frac{1}{2}(\vec{M} + i\vec{N}), \quad \vec{K} = \frac{1}{2}(\vec{M} - i\vec{N}),$$

where $\vec{M}$ and $\vec{N}$ are the infinitesimal generators of the spatial and hyperbolic rotations. At first the basis $|\Phi>$ will be labelled by the eigenvalues of J_3 and K_3 i.e. by $m = (\mu + i\nu)/2$, $m^* = (\mu - i\nu)/2$ ($\mu = 0, \pm 1, \pm 2, \ldots, -\infty < \nu < \infty$ continuous).

Introduce the following coordinate system on Σ_0

$$S_1 = -i\cos\Theta\cos\Phi - \sin\Phi, \quad S_2 = -i\cos\Theta\sin\Phi + \cos\Phi \qquad (2)$$

$$S_3 = i\sin\Theta.$$

Here

$$\Theta = \Theta_1 + i\Theta_2, \ \Phi = \Phi_1 + i\Phi_2, \ 0 \leq \Theta_1 < \pi, \ 0 \leq \Phi_1 < 2\pi, \ -\infty < \Theta_2, \Phi_2 < \infty.$$

The spherical functions in unitary spinor basis satisfy the eigenvalue equations of the Casimir operators J^2, K^2 and the generators J_3, K_3. From (2) we obtain

$$\left[tg^2 \Theta \frac{\partial^2}{\partial \Theta^2} - \frac{1}{\cos^2 \Theta} \frac{\partial^2}{\partial \Phi^2} + 2i \frac{\sin \Theta}{\cos^2 \Theta} \frac{\partial^2}{\partial \Theta \partial \Phi} + tg\, \Theta (2 + tg^2 \Theta) \frac{\partial}{\partial \Theta} + \right. \tag{3}$$

$$\left. + \frac{i}{\cos^3 \Theta} \frac{\partial}{\partial \Phi} \right] f^{jj^*}_{mm^*} = j(j+1) f^{jj^*}_{mm^*} ,$$

$$\left[tg^2 \Theta^* \frac{\partial^2}{\partial \Theta^{*2}} - \frac{1}{\cos^2 \Theta^*} \frac{\partial}{\partial \Phi^{*2}} - 2i \frac{\sin \Theta^*}{\cos^2 \Theta^*} \frac{\partial^2}{\partial \Theta^* \partial \Phi^*} + tg\, \Theta^* (2 + tg^2 \Theta^*) \frac{\partial}{\partial \Theta^*} - \right. \tag{4}$$

$$\left. - \frac{i}{\cos^3 \Theta^*} \frac{\partial}{\partial \Phi^*} \right] f^{jj^*}_{mm^*} = j^*(j^*+1) f^{jj^*}_{mm^*} ,$$

$$\frac{1}{i} \frac{\partial}{\partial \Phi} f^{jj^*}_{mm^*} = m f^{jj^*}_{mm^*} , \quad \frac{1}{i} \frac{\partial}{\partial \Phi^*} f^{jj^*}_{mm^*} = m^* f^{jj^*}_{mm^*} . \tag{5}$$

Here j is related to the familiar quantum numbers j_0, σ /4/ as $j = \frac{1}{2}(j_0 - 1 + i\sigma)$ ($j = 0,1,2,\ldots,\ -\infty < \sigma < \infty$ continuous).

The solution of equations (3), (4), (5) can be written in the form

$$f^{jj^*}_{mm^*} = \frac{1}{2\sqrt{2}} \frac{1}{(2\pi)^2} \left(\cos \frac{\Theta}{2}\right)^{j-m} \left(\sin \frac{\Theta}{2}\right)^{j+m} \left(\cos \frac{\Theta^*}{2}\right)^{-j^*-1+m^*} \left(\sin \frac{\Theta^*}{2}\right)^{-j^*-j-m^*} \cdot \tag{6}$$

$$\cdot\, e^{i(m\Phi + m^*\Phi^*)} .$$

These functions are normalized as follows

$$\langle j'j'^*; m'm'^* | jj^*; mm^* \rangle = \left(\frac{i}{2}\right)^2 \int \cos\Theta \cos\Theta^*\, d\Theta\, d\Theta^*\, d\Phi\, d\Phi^* \left(f^{j'j'^*}_{m'm'^*}\right)^* f^{jj^*}_{mm^*} =$$

$$= \delta_{j'_0 j_0}\, \delta(\sigma' - \sigma)\, \delta_{\mu'\mu}\, \delta(\nu' - \nu) .$$

It is worthy of note that $f^{jj^*}_{mm^*}$ is a single valued function. If we cut the $\sin \frac{\Theta}{2}$ plane it is easily seen that as a consequence of integral valuedness of $j_0 \pm \mu$ the discontinuity over the cuts is equal to zero. Or conversely, the requirement of single valuedness leads to the quantization of j_0.

In order to obtain the spherical functions in another basis we have to introduce a suitable coordinate system, e.g. the coordinate system

$$S_1 = e^{a + i\psi} (-\sin\phi + i \cos\theta \cos\phi), \quad S_2 = e^{a+i\psi} (\cos\phi + i \cos\theta \sin\phi)$$

$$S_3 = -i \sin\theta\, e^{a+i\psi} \tag{7}$$

$$-\infty < a < \infty, \quad 0 \le \phi,\ \psi < 2\pi, \quad 0 \le \theta < \pi$$

leads two the spherical functions in angular momentum basis ($|\Phi\rangle = |j_0\sigma; \ell\mu\rangle$);:

$$f^{j_0\sigma}_{\ell\mu} = \sqrt{\frac{2\ell+1}{8\pi^3}}\, e^{a(-1+i\sigma)} D^{\ell}_{\mu j_0}(\phi, \theta, \psi) ,$$

where $D^{\ell}_{\mu j_0}$ is the representation of the real three-dimensional rotation group.

3. Relation to Gelfand's Homogeneous Functions

Consider now the following parametrization of Σ_0 .

$$S_1 = -i(u^2 - v^2),\ S_2 = u^2 + v^2,\ S_3 = 2iuv \quad . \tag{8}$$

It can be easily shown that if u and v transform as spinors of the $SL(2,C)$ group i.e. $\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$ then S transforms as a vector of the three-dimensional complex rotation group. Parametrizations (2), (7) can be considered as special cases of (8) and correspond to the following parametrization of spinors

$$u = \cos\frac{\Theta}{2}\, e^{-i\frac{\Phi}{2}}, \quad v = \sin\frac{\Theta}{2}\, e^{i\frac{\Phi}{2}}$$

and

$$u = -\sin\frac{\theta}{2}\, e^{\frac{a}{2} + i\frac{\psi - \phi}{2}}, \quad v = \cos\frac{\theta}{2}\, e^{\frac{a}{2} + i\frac{\psi + \phi}{2}} .$$

Spherical functions in terms of u , v in the unitary spinor basis read

$$f^{jj^*}_{mm^*} = \frac{1}{2\sqrt{2}}\frac{1}{(2\pi)^2} u^{j-m} v^{j+m} u^{*\,-j^*-1+m^*} v^{*\,-j^*-1-m^*} .$$

If one consideres the linear manifold

$$f(u,v) = \sum_{\mu=-\infty}^{\infty} \int_{-\infty}^{\infty} d\nu\, a_{mm^*} f^{jj^*}_{mm^*}$$

then under the $SL(2,C)$ group the function $f(u,v)$ transforms as

$$T_g f(u,v) = f(g^{-1}(u,v)) = f(\delta u - \beta v, -\gamma u + \alpha v),$$

furthermore, it has the degrees of homogeneity $2j$, $-2j^* - 2$ with respect to u , v and u^*, v^* . Thus, if we fix a basis, say m, m^* the homogeneous functions investigated by Naimark and Gelfand /5,6/ take the form of the spherical functions (6) defined over the two-dimensional complex sphere of the zero radius.

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