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Факторизация дуальных амплитуд в модели с конечным набором осцилляторов и замкнутые петли

В формализме когеретных состояний конечного набора пятимерных осцилляторов выявлена факторизация N -точечной дуальной амплитуды по операторам этих осцилляторов. Получено сходящееся выражение для замкнутой петли.

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Factorization of Dual Amplitudes in the Model with the Finite Set of Oscillators and Loops

In the coherent state formalism with finite set of five-dimensional oscillators, a factorization of the dual N -point amplitude with respect to the operators of these oscillators is proposed. A convergent expression for the planar one-loop diagram is obtained.

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. Introduction

In a recent paper [1] an operator technique for building the dual N-point tree diagram with the aid of N-2 five-dimensional oscillators has been proposed. The factorization of the amplitude, that takes place in this approach is related to the number of the external particles, as the number of the oscillators used is related to the latter, as well. This connection does not appear to be a formal one. A more detailed consideration shows, that the contribution of any particle to the N-point amplitude is determined by the appropriate oscillator. This fact is manifested in the possibility of factorizing automatically the dual amplitude in the oscillator operators. In this additional factorization each multiplier contains only one exponential factor, which corresponds to the external particle.

The above mentioned correspondence is observed still further when one builds the N-particle coherent states which make it possible to factorize the N-point amplitude in both external momenta and oscillators (see §2).

In principle, the oscillators corresponding to each particle can be build up, proceeding from the field operators of this particle. Such a possibility is discussed in ref. 2 .

In the dual models one constructs loops in order to get an imaginary addition to the linear Regge-trajectory, which leads to unitarization of the model. A general expression for planar n-loop diagrams with four external particles has been derived in [3]. Similar diagrams with arbitrary number of external particles were obtained by means of factorization, proposed by Fubini, Gordon, Veneziano/4,5/ (see refs./6,7/ and for a more complete list of references see ref./8/), but they turned out to be exponentially divergent.

In the present paper we build planar one-loop diagrams with an arbitrary number of external particles, using a finite set of five-dimensional oscillators (§ 3). The particular feature of this factorization, namely the number of the creation and annihilation operators is finite and there is a correspondence between oscillators and external particles, leads to the integral representation of these diagrams being convergent.

1. Coherent states of an U(3,2) invariant oscillator.

The five-dimensional U(3,2)-invariant oscillators were introduced in ref. 1 in order to factorize the dual N-point scattering amplitude. The creation and annihilation operators (a^+_μ, b^+ and a_μ, b respectively), corresponding to such an oscillator, satisfy the following commutation relations:

$$
\begin{aligned}
&[a^+_\mu, a_\nu] = g_{\mu\nu} \qquad && g_{\mu\nu} = 0 \quad \mu \neq \nu \\
&[b^+, b] = 1 && g_{00} = -g_{11} = -g_{22} = -g_{33} = 1
\end{aligned}
\tag{1.1}
$$

(the all remaining commutators are equal to zero). In this case the Hamiltonian has the form:

$$H = -g^{\mu\nu} a_\mu^+ a_\nu - b^+ b. \qquad (1.2)$$

It is easy to check, that the eigenvalues of this operator are positive integers n . For every eigenstate of H one can find five positive integers n_i (i = 0,1,2,3,4) such that

$$n = n_0 + n_1 + n_2 + n_3 + n_4' = \sum_{\mu=0}^{3} n_\mu + n_4$$

and which specify completely the state. So, the states can be denoted by $|n_0, n_1, n_2, n_3, n_4\rangle$. The explicit form of these states has been written down in ref. 1 . At the same time one usually considers the so-called coherent states. Their advantage is due to the fact that many relations, concerning the quantum mechanical oscillator take the form analogous to the classical one. Moreover, the coherent states are characterized by continuous complex quantum numbers. Therefore all the discrete sums, that appear when we use the states $|n_0, n_1, n_2, n_3, n_4\rangle$ turn into integrals, which are more convenient to work.

In our case we define the coherent states as follows:

$$|\beta\rangle \equiv |\beta_0, \beta_1, \beta_2, \beta_3, \beta_4\rangle = e^{-\frac{1}{2}\sum_{i=0}^{4}|\beta_i|^2} e^{-g^{\mu\nu}\bar\beta_\mu a_\nu^+ - \bar\beta_4 b^+} |0\rangle$$
$$\langle\beta| \equiv \langle\beta_0, \beta_1, \beta_2, \beta_3, \beta_4| = e^{-\frac{1}{2}\sum_{i=0}^{4}|\beta_i|^2} \langle 0| e^{\sum_{\mu=0}^{3}\beta_\mu a_\mu + \beta_4 b}, \qquad (1.3)$$

where $|0\rangle$ is the vacuum state, i.e. the state with the lowest eigenvalue of H, $n = 0$; β_i are the arbitrary complex numbers and $\bar{\beta}_i$ their complex conjugates. It is obvious, that $|\beta\rangle$ and $\langle\beta|$ are not connected by the operation of hermitian conjugation. However, it is easy to show that the following identity holds:

$$\langle\beta| = (|\beta\rangle)^{+} G \, , \tag{1.4}$$

where

$$G = \prod_{\mu=0}^{3} (-g_{\mu\mu})^{a_\mu^+ a_\mu} (-1)^{b^+ b} \tag{1.5}$$

and the cross denotes the usual hermitian conjugation.

We can calculate also the norm of the states (1.3):

$$\langle\beta|\beta\rangle = 1 \, . \tag{1.6}$$

We note in addition, that in the space of states of our oscillator, the coherent states form a complete set, i.e.

$$\frac{1}{\pi^5} \int |\beta\rangle\langle\beta| \prod_{i=0}^{4} d\mathrm{Re}\,\beta_i \, d\mathrm{Im}\,\beta_i = I \, , \tag{1.7}$$

where I denotes the unit operator in this space and for every β_i the integration is taken over the whole complex plane.

E. (1.7) becomes obvious when we take into account, that:
1) If instead of a^+_μ, b^+_μ and a_μ, b_μ we define $\tilde{\varphi}_i$ and φ_i as

$$\{\tilde{\varphi}_i\} \equiv \{-g^{\mu\nu} a_\nu^+, -b^+\}$$

$$\{\varphi_i\} \equiv \{a_\nu, b\} \tag{1.8}$$

then these new operators will obey the common commutation relations:

$$[\varphi_i, \tilde{\varphi}_k] = \delta_{ik} . \tag{1.9}$$

2) The exponents in the definitions (1.3) can be written as follows:

$$\begin{aligned} -g^{\mu\nu}\bar{\beta}_\mu a_\nu^+ - \bar{\beta}_4 b^+ &= \sum_{i=0}^{4} \bar{\beta}_i \tilde{\varphi}_i \\ \sum_{\mu=0}^{3} \beta_\mu a_\mu + \beta_4 b &= \sum_{i=0}^{4} \beta_i \varphi_i . \end{aligned} \tag{1.10}$$

In these new notations eq.(1.7) takes the form:

$$\frac{1}{\pi^5} \int e^{-\sum_{i=0}^{4}|\beta_i|^2} e^{\sum_{i=0}^{4}\bar{\beta}_i\tilde{\varphi}_i} |0\rangle\langle 0| e^{\sum_{i=0}^{4}\beta_i\varphi_i} \prod_{i=0}^{4} d\,\mathrm{Re}\,\beta_i \, d\,\mathrm{Im}\,\beta_i . \tag{1.11}$$

As is known, owing to the relation (1.9), eq.(1.11) is an identity.

Due to the completeness of the coherent states, an arbitrary state $|F\rangle$ from the space of the U(3,2)-invariant oscillator can be represented in the form:

$$|F\rangle = \int F(\beta)|\beta\rangle \prod_{i=0}^{4} d\mathrm{Re}\beta_i \, d\mathrm{Im}\beta_i \,, \tag{1.12}$$

where $F(\beta)$ is an analytic function of β, for which we can find such a constant $0 < \varrho < 1$, that:

$$|F(\beta)| < e^{\frac{\varrho}{2}\sum_{i=0}^{4}|\beta_i|^2} \qquad if \ |\beta_i| \gg 1 \,. \tag{1.13}$$

In particular, if $F(\beta) = \beta_0^{n_0}\beta_1^{n_1}\beta_2^{n_2}\beta_3^{n_3}\beta_4^{n_4}$ with n_0, n_1, n_2, n_3, n_4 arbitrary non-negative integers, we obtain the states $|n_0, n_1, n_2, n_3, n_4\rangle$.

The following statement has proved to be very useful when one has to apply the coherent states:

Let $f(\beta)$ be a function of five complex variables β_i (i = 0,1,2,3,4). If it is analytic in the neighbourhood of the point $\beta_i = 0$ and if there is a constant $0 < \varrho < 1$ such that

$$|f(\beta)| < e^{\varrho\sum_{i=0}^{4}|\beta_i|^2} \qquad if \ |\beta_i| \gg 1 \tag{1.14}$$

then the equality

$$\frac{1}{\pi^5}\int f(\beta) e^{-\sum_{i=0}^{4}|\beta_i|^2} e^{\sum_{i=0}^{4}\bar{\beta}_i\alpha_i} \prod_{i=0}^{4} d\mathrm{Re}\beta_i \, d\mathrm{Im}\beta_i = f(\alpha) \tag{1.15}$$

is identically fulfilled. (For every β_i the integral is taken over the whole complex plane).

Eq.(1.15) can be verified directly when the Taylor's expansion of $f(\beta)$ around the point $\beta_i = 0$ converges for every complex β_i. If this expansion has a finite radius of convergence R_i, the integral of the left-hand side of (1.15) can be computed inside the circle with radius $M < R_i$. Then (1.15) is proved by means of analytical continuation of the result in M.

Finally we shall note that the coherent states $|\beta\rangle$ are eigenstates of operators φ_i (cl. eq.(1.8)):

$$\varphi_i |\beta\rangle = \bar{\beta}_i |\beta\rangle \tag{1.16}$$

and respectively

$$\langle\beta| \tilde{\varphi}_i = \beta_i \langle\beta| \,. \tag{1.17}$$

In the case of N unconnected oscillators one can generalize all the above mentioned properties of the coherent states. (Similar states were used in ref. 1 in order to factorize the N-point dual amplitude.) This generalization is trivial, as the space of these oscillators is a direct product of all five-dimensional spaces of different oscillators.

2. N-particle coherent states.

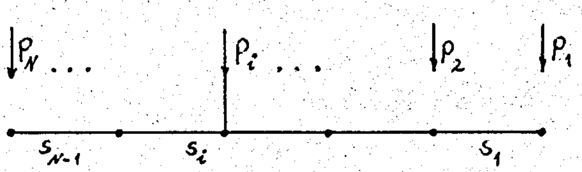
In this section we are going to introduce the N-particle coherent states, which result from the factorization of the dual amplitude proposed in ref. /1/. An important property of this factorization can be formulated in the following manner.

The dual N + 2 - point amplitude is factorized in external momenta with the aid of 5N creation ($a^+_{\mu i}$, b^+_i) and annihilation

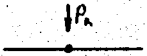
($a_{\mu i}$, b_i) operators ($\mu=0,1,2,3$; $i=1,2,\dots N$). Moreover, it can be additionally factorized in these operators too.

This additional factorization allows us to formulate another (but equivalent to that from ref. 1) procedure of building the N+2 - point amplitude.

First of all, we construct an operator Γ_i , using only the creation and annihilation operators of the i-th oscillator ($a^+_{\mu i}$, b^+_i , $a_{\mu i}$, b_i). This new operator Γ_i corresponds to the diagram



and is build - up by successive multiplication of its elements. The rules of correspondence are as follows:

to every node  there corresponds the operator

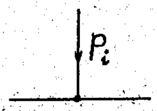
$$V^{L,D}(p_k a_i; b_i) = (1+b_i)^{-\sqrt{2\alpha'}\,\frac{p_k\cdot a_i}{b_i} + \rho^{L,D}_{ik}} \qquad (2.1)$$

where one should write the index L for $k > i$ and D for $k < i$. The constants ρ^{L}_{ki} are defined as follows:

$$\rho^{L}_{i,i+1} = -1-\alpha(0)-2\alpha' p^2$$

$$\rho^{L}_{i,i+2} = \alpha(p^2) \qquad \rho^{L}_{ij}=0 \quad \text{for } j>i+2 \qquad (2.2)$$

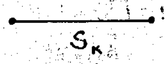
(α' being the slope of the linear Regge-trajectory).
The constants ρ^{D}_{ik} are arbitrary.

to the vertex  there corresponds the operator

$$e^{\sqrt{2\alpha'}\, p_i\cdot a_i^{+} + b_i^{+}} : e^{-H_i} : (1+b_i)^{-\sqrt{2\alpha'}\frac{p_i\cdot a_i}{b_i} + \rho_{ii}} \qquad (2.3)$$

where $\rho_{ii} = 2\alpha' p_i^2 \equiv 2\alpha' p_i^2$

and $H_i = -g^{\mu\nu} a^{+}_{\mu i} a_{\nu i} - b_i^{+} b_i$

to every element  there corresponds the operator $x_k^{H_i}$.

Using these rules of correspondence we find out the operator Γ_i in the form:

$$\Gamma_i = \Big(\prod_{k=i}^{N-1} x_k^{H_i}\Big)\Big(\prod_{n=i+1}^{N}\big(1+\prod_{l=i}^{n-1} x_l b_i\big)^{-\sqrt{2\alpha'}\frac{p_n\cdot a_i}{b_i} + \rho_{in}^{L}}\Big)$$

$$e^{\sqrt{2\alpha'}\, p_i\cdot a_i^{+} + b_i^{+}} : e^{-H_i} : \Big(\prod_{k=1}^{i-1} x_k^{H_i}\Big)\Big(\prod_{n=1}^{i}\big(1+\prod_{l=1}^{n-1} x_l b_i\big)^{-\sqrt{2\alpha'}\frac{p_n\cdot a_i}{b_i} + \rho_{in}^{D}}\Big). \qquad (2.4)$$

If $|0\rangle_i$ denotes the vacuum state of the i^{-th} oscillator, the N+2 - point function can be expressed in the following manner:

$$B_{N+2} = \int_0^1 \prod_{i=1}^{N} \langle 0|\Gamma_i|0\rangle_i \prod_{i=1}^{N-1} x_i^{-\alpha(s_i)-1} dx_i =$$

$$= \int_0^1 \prod_{i=1}^{N-1}\prod_{n=i+1}^{N}\big(1-\prod_{l=i}^{n-1} x_l\big)^{-2\alpha'(p_n\cdot p_i) - \rho^{L}_{ni}} \prod_{i=1}^{N-1} x_i^{-\alpha(s_i)-1} dx_i . \qquad (2.5)$$

It is obvious, that the method of construction of B_{N+2} is equivalent to that, given in ref. /1/.

The additional operators are introduced in order to gain symmetry. So, the explicit form of above mentioned additional factorization is given by the first eq. (2.5).

We denote the N-particle coherent state by $|\chi_N\rangle$ and define it in the following way:

$$|\chi_N\rangle=\int_0^1\prod_{i=1}^{N}\Gamma_i|0\rangle\prod_{i=1}^{N-1}x_i^{-\alpha(s_i)-1}dx_i \quad . \tag{2.6}$$

Inserting Γ_i from eq.(2.4) into (2.6) we get:

$$|\chi_N\rangle=\int_0^1\prod_{i=1}^{N-1}dx_i\,x_i^{-\alpha(s_i)-1}\prod_{i=1}^{N-1}\prod_{n=i+1}^{N}\Big(1-\prod_{l=i}^{n-1}x_l\Big)^{-2\alpha'(p_n\cdot p_i)+\rho_{in}^{L}}\cdot$$
$$e^{\sqrt{2\alpha'}\sum_{i=1}^{N}p_i\,a_i^+\prod_{k=i}^{N-1}x_k+\sum_{i=1}^{N}b_i^+\prod_{k=i}^{N-1}x_k}|0\rangle \tag{2.7}$$

$$\prod_{k=N}^{N-1}x_k\equiv 1 \, .$$

Let us denote the integrand of eq.(2.5) with $\varphi_{N+2}(x)$. Then (2.7) can be rewritten in the form:

$$|\chi_N\rangle=\int_0^1\prod_{i=1}^{N-1}dx_i\,\varphi_{N+2}(x)\,e^{\sqrt{2\alpha'}\sum_{i=1}^{N}p_i a_i^+\prod_{k=i}^{N-1}x_k+\sum_{i=1}^{N}b_i^+\prod_{k=i}^{N-1}x_k} \tag{2.8}$$

The conjugate states are:

$$\langle\chi_M|=\int_0^1\prod_{i=1}^{M-1}dx_i\,\varphi_{M+2}(x)\langle 0|\,e^{\sqrt{2\alpha'}\sum_{i=1}^{M}p_i\,a_i\prod_{k=i}^{M-1}x_k+\sum_{i=1}^{M}b_i\prod_{k=i}^{M-1}x_k} \quad . \tag{2.9}$$

When M = N, the state (2.9) is hermition conjugate to (2.8). If we make in (2.9) the formal substitution:

$$x_M,\,p_M,\,a_M \longrightarrow x_{N+1},\,p_{N+1},\,a_{N+1}$$
$$x_{M-1},\,p_{M-1},\,a_{M-1} \longrightarrow x_{N+2},\,p_{N+2},\,a_{N+2}$$
$$\cdots$$
$$x_1,\,p_1,\,a_1 \longrightarrow x_{N+M},\,p_{N+M},\,a_{N+M}$$

then with these new notations we get:

$$\langle\tilde{\chi}_M| = \int_0^1 \prod_{i=N+1}^{N+M-1} d\lambda_i \, \tilde{\varphi}_{M+2}(x) \langle 0| e^{\sqrt{2\alpha'} \sum_{i=N+1}^{N+M} p_i a_i \prod_{n=N+1}^{i-1} x_n + \sum_{i=N+1}^{N+M} b_i \prod_{k=N+1}^{i-1} x_k} \tag{2.10}$$

The quantities with tilde are the same as in(2.9), but with the account of the above substitution.

In this case B_{N+M+2} can be written in terms of N-particle coherent states as follows:

$$B_{N+M+2} = \langle\tilde{\chi}_M| \int_0^1 dx_N \, x_N^{-\alpha(s_N)-1+H} D_{MN} |\chi_N\rangle \, . \tag{2.11}$$

where

$$D_{MN} = \prod_{i=1}^{N} \prod_{n=N+1}^{N+M} (1 - b_i b_n^+)^{-\frac{a_i a_n^+}{b_i b_n^+} + \beta_{in}^{i}} \tag{2.12}$$

Eq.(2.11) is proved in the Appendix.

When N = M = 1, i.e. in the case of the four-point function, eq.(2.12) coincides with the operator D, defined in ref. 1 . Therefore (2.12) is a generalization of that operator and then eq.(2.11) is completely analogous to the factorization of the four-point function.

The formula (2.11) is remarkable by its full factorization in both external momenta and harmonic oscillators. An arbitrary N-particle coherent state depends only on the external momenta involved in it. (We use the word "coherent" in a very conditional sense, only to stress the fact that these states are superpositions of the actual coherent states). Besides that, only those creation and annihilation operators survive in (2.11), the numbers of which coincide with the numbers of the particles. Therefore we can state, that in this model different U(3,2) - invariant oscillators are connected with the particles and this relationship is reveal in a certain manner in the scattering of these particles.

3. <u>One-loop diagram with N external particles</u>.

It is obvious, that the operator Γ_i (see eq.(2.4)) takes into account the i-th particle contribution to the N-point dual amplitude. The fact, that Γ_i depends on the momenta of other particles should most probably be interpreted as a result of the operations made before the operator Γ_i appears. For instance, in some second-quantized theory one can consider, that Γ_i does not depend on the c - number momenta of external particles, but indeed on their momentum operators. In this case the indices of the momentum operators will only point out that they are related to different particles. The operator Γ_i takes the form (2.4) only after the corresponding average over the second quantized states of external particles has been performed. It is clear from this point of view that the contribution of a given particle to the amplitude is determined rather by the appropriate oscillator, than the momentum.

Another consequence of this consideration is the finite length of the chains Γ_i (if only the number of interacting particles is finite). In fact, the matrix elements between the states with finite number of particles will produce chains Γ_i of the type (2.4) with finite length. Besides, the number of nodes will be defined by the number of the external particles.

Now it is easy to see, that the loops should be built up also with the aid of the operators Γ_i. From the above considerations it follows, that before integrating over the internal loop momentum, the loop with N-external particles should be a product of N closed chains, each having only one external particle. If we cut any of these chains at any point, we should get a linear chain of the type Γ_i. This defines completely the way of constructing closed chains Φ_i

$$\Phi_i(x,\rho) = Sp\left\{x_N^{H_i}\,\Gamma_i(x,\rho)\right\} \qquad (3.1)$$

Then the loop L_N with N external particles is to be determined by the formula:

$$L_N = \int \prod_{i=1}^{N} \Phi_i(x,\rho) \prod_{i=1}^{N} x_i^{\alpha(s_i(\kappa))}\, dx_i\, d^4\kappa \qquad (3.2)$$

κ being the inner four-momentum of the loop. (The chain Γ_i ends in both sides with a node. Then, in order to get a propagator on one of the ends we have ascribed to Γ_i in eq.(3.1) an additional factor $x_N^{H_i}$).

This method of constructing the loops well corresponds to the factorization (2.11). Indeed, from the symmetrized N′-point dual amplitude (N′> N) we can always single out some operator part with N external particles. Because of the commutativity of different oscillators, the trace of this operator chain is represented as the product of the traces of the operators $X_N^{H_i}\Gamma_i(x,\rho)$, i.e.

$$Sp\{x_N^H e^{\rho_N \cdot a_N^+ + b_N^+}: e^{-H_N}: \prod_{i=1}^{N} V(\rho_N \cdot a_i, b_i)\, x_{N-1}^H \cdot$$

$$e^{\rho_{N-1}\cdot a_{N-1}^+ + b_{N-1}^+}: e^{-H_{N-1}}: \prod_{i=1}^{N} V(\rho_{N-1}\cdot a_i, b_i)\cdot \ldots \tag{3.3}$$

$$x_1^H e^{\rho_1 \cdot a_1^+ + b_1^+}: e^{-H_1}: \prod_{i=1}^{N} V(\rho_1 \cdot a_i, b_i)\} \equiv \prod_{i=1}^{N} Sp\{x_N^{H_i}\Gamma_i(x,\rho)\},$$

where $Sp\{x_N^{H_i}\Gamma_i(x,\rho)\}$ is the trace over the states of the i^{-th} oscillator. Using the coherent states introduced in sec.1. we can write down $Sp\{x_N^{H_i}\Gamma_i(x,\rho)\}$ in a more convenient form:

$$Sp\{x_N^{H_i}\Gamma_i(x,\rho)\} = \int \langle \beta_i | x_N^{H_i}\Gamma_i(x,\rho)|\beta_i\rangle (d\,Re\,\beta_i)(d\,Im\,\beta_i), \tag{3.4}$$

where $(d\,Re\,\beta_i) = d(Re\,\beta_i)d(Re\,\beta_i)d(Re\,\beta_2)d(Re\,\beta_3)d(Re\,\beta_4)$ and $(d\,Im\,\beta_i)$ are defined in an analogous way.

After some simple calculations we can get

$$x_N^{H_i}\Gamma_i(x,\rho)|\beta_i\rangle = \frac{1}{\sqrt{\pi}} e^{-\frac{1}{2}\sum_{\mu=0}^{4}\beta_{i\mu}\bar{\beta}_{i\mu}} \prod_{n=1}^{i}\Big(1+\prod_{k=1}^{n-1}x_k\bar{\beta}_{i4}\Big)^{-2\alpha'\frac{\rho_n\cdot\bar{\beta}_i}{\bar{\beta}_{i4}}+\rho_{in}^{D}} \cdot$$

$$\cdot \prod_{n=i+1}^{N}\Big(1-\prod_{\ell=i}^{n-1}x_\ell\Big)^{-2\alpha'(\rho_n\cdot\rho_i)+\rho_{in}^{L}} e^{\rho_i\cdot a_i^+\prod_{k=i}^{N}x_k+\prod_{k=i}^{N}x_k b_i^+}|0\rangle \tag{3.5}$$

The diagonal matrix elements have the form:

$$\langle \beta_i | x_N^{H_i}\Gamma_i(x,\rho)|\beta_i\rangle = \frac{1}{\pi} e^{-\sum_{\mu=0}^{4}\beta_{i\mu}\bar{\beta}_{i\mu}}$$

$$\prod_{n=1}^{i}\Big(1+\prod_{k=1}^{n-1}x_k\beta_{i4}\Big)^{-2\alpha'\frac{\rho_n\cdot\bar{\beta}_i}{\beta_{i4}}+\rho_{in}^{D}} \prod_{n=i+1}^{N}\Big(1-\prod_{\ell=i}^{n-1}x_\ell\Big)^{-2\alpha'\rho_n\cdot\rho_i+\rho_{in}^{L}} e^{-\sum_{\mu=0}^{3}\beta_{i\mu}\rho_{\mu}\prod_{k=i}^{N}x_k-\beta_{i4}\prod_{k=i}^{N}x_k} \tag{3.6}$$

Taking into account (3.6) and (1.15) we find (3.4):

$$
\begin{aligned}
Sp\left\{X_N^{H_i}\Gamma_i(x,p)\right\} = & \prod_{n=1}^{i}\Bigl(1-\prod_{k=1}^{n-1}x_k\prod_{k=i}^{N}x_k\Bigr)^{-2\alpha'(p_n p_i)+\rho_{in}^{D}} \cdot \\
& \prod_{n=i+1}^{N}\Bigl(1-\prod_{\ell=i}^{n-1}x_\ell\Bigr)^{-2\alpha'(p_n p_i)+\rho_{in}^{L}}
\end{aligned}
\tag{3.7}
$$

Finally, substituting (3.7) into (3.2) we get:

$$
L_N = \int \mathcal{N}(x)\prod_{i=1}^{N}x_i^{\alpha(s_i(x))}\,dx_i\,d^4k\,, \tag{3.8}
$$

where

$$
\begin{aligned}
\mathcal{N}(x) = & \prod_{i=1}^{N}\prod_{n=1}^{i}\Bigl(1-\prod_{k=1}^{n-1}x_k\prod_{\ell=i}^{N}x_\ell\Bigr)^{-2\alpha' p_n p_i+\rho_{in}^{D}} \cdot \\
& \prod_{i=1}^{N-1}\prod_{n=i+1}^{N}\Bigl(1-\prod_{\ell=i}^{n-1}x_\ell\Bigr)^{-2\alpha'(p_n p_i)+\rho_{in}^{L}}\,.
\end{aligned}
\tag{3.9}
$$

We have already pointed out that all ρ_{in}^{D} are completely arbitrary. The fact is that they do not participate when obtaining the N-point tree diagrams and, consequently their values can not be determined from the latter. Nevertheless, ρ_{in}^{D} are of importance in eq.(3.9) and should be determined. It turns out, that ρ_{in}^{D} are determined unambiguously, from the requirement of cyclic symmetry of eq.(3.8). Namely, we have to put

$$
\begin{aligned}
\rho_{N,1}^{D} &= \rho_{i,i+1}^{L} \\
\rho_{N-1,1}^{D} &= \rho_{N,2}^{D} = \rho_{i,i+2}^{L}
\end{aligned}
\tag{3.10}
$$

and all the remaining $\rho^{D}_{ij}=0$. Then the expression (3.9) takes the following final form:

$$
\begin{aligned}
\mathcal{N}(x)= & \prod_{i=1}^{N}(1-x_i)^{-1-\alpha[(p_i\cdot p_{i+1})^2]}\prod_{i=1}^{N}(1-x_i x_{i+1})^{-2\alpha'(p_i\cdot p_{i+2})+\alpha(p^2)}\cdot \\
& \prod_{i=2}^{N-2}\Big(1-\prod_{k=1}^{N}x_k\Big)^{-2\alpha'(p_1\cdot p_i)}\prod_{i=3}^{N-1}\Big(1-\prod_{k=i}^{N+1}x_k\Big)^{-2\alpha'(p_2\cdot p_i)}\cdot \\
& \prod_{i=4}^{N}\prod_{n=3}^{i-1}\Big(1-\prod_{k=1}^{n-1}x_k\prod_{l=i}^{N}x_l\Big)^{-2\alpha'(p_n\cdot p_i)}\prod_{i=1}^{N-3}\prod_{n=i+3}^{N}\Big(1-\prod_{l=i}^{n-1}x_l\Big)^{-2\alpha'(p_n\cdot p_i)}.
\end{aligned}
\tag{3.11}
$$

This form of N(x) shows, that the integral representation of the loop converges in some region of parameters $(p_i\cdot p_j)$ and hence for N $>$ 2 the integral (3.2) can be regularized by the well known procedure of analytical continuation.

Let us now consider, for simplicity, the loop with four external particles L_4 . After performing in eq.(3.8) the integration over d^4k , we get

$$
\begin{aligned}
L_4= & -\frac{\pi^2}{\alpha'^2}\int_0^1\prod_{i=1}^{4}dx_i\,\frac{\omega^{-\alpha(0)-1}}{\ln^2\omega}\,e^{-\alpha' s\frac{\ln x_2\ln x_4}{\ln\omega}-\alpha' t\frac{\ln x_1\ln x_3}{\ln\omega}-\alpha' p^2\frac{\ln x_1x_3\ln x_2x_4}{\ln\omega}}\cdot \\
& \times\left[\frac{(1-x_1)(1-x_3)(1-x_2x_3x_4)(1-x_1x_2x_4)}{(1-x_1x_2)(1-x_2x_3)(1-x_3x_4)(1-x_4x_1)}\right]^{-1-\alpha(s)}\left[\frac{(1-x_2)(1-x_4)(1-x_1x_2x_3)(1-x_1x_3x_4)}{(1-x_1x_2)(1-x_2x_3)(1-x_3x_4)(1-x_4x_1)}\right]^{-1-\alpha(t)} \\
& \big[(1-x_1x_2)(1-x_2x_3)(1-x_3x_4)(1-x_4x_1)\big]^{-2-\alpha(p^2)}\big[(1-x_1x_2x_3)(1-x_2x_3x_4)(1-x_1x_2x_4)(1-x_1x_3x_4)\big]^{1+\alpha(2p^2)},
\end{aligned}
\tag{3.12}
$$

where $\omega=x_1x_2x_3x_4$.

First of all we note, that for $\alpha(s)<0$, $\alpha(t)<0$ and $\alpha(0)<0$ the integral (3.12) converges at the points $x_i=1$. The last inequality provides its convergence at the points $x_i=0$ too.

Let us find the asymptotic behavior of (3.12) at $S \to -\infty$. To this end all the power functions in S in the integrand, should be united in one exponential factor

$$e^{\alpha' S f(x)} , \tag{3.13}$$

where

$$f(x) = -\frac{\ln x_2 \ln x_4}{\ln \omega} - \ln \frac{(1-x_1)(1-x_3)(1-x_1x_2x_4)(1-x_2x_3x_4)}{(1-x_1x_2)(1-x_2x_3)(1-x_3x_4)(1-x_4x_1)} > 0 . \tag{3.14}$$

The function $f(x)$ can be expanded in a power series in $(1-x_2)(1-x_4)$, i.e.

$$f(x) = \left(-\frac{1}{\ln x_1x_3} + \frac{x_1}{(1-x_1)^2} + \frac{x_3}{(1-x_3)^2}\right)(1-x_2)(1-x_4) + \cdots \tag{3.15}$$

All the remaining multipliers but the multiplier $\left[(1-x_2)(1-x_4)\right]^{-1-\alpha(t)}$ in the integrand of (3.12) are regular functions of X_2 and X_4 in the neighbourhood of the points $X_2 = 1$ and $X_4 = 1$. Because of this for large S we have

$$L_4 \sim -\frac{\pi}{\alpha'^2} \int_0^1 \prod_{i=1}^{4} dx_i \frac{(x_1x_3)^{-\alpha(0)-1}}{\ln^2 x_1x_3} e^{-\alpha' t \frac{\ln x_1 \ln x_3}{\ln x_1x_3}} \left[(1-x_1)(1-x_3)\right]^{-1+\alpha(\ell t)} \times$$

$$\times (1-x_1x_3)^{-2\alpha(t)+2\alpha(2p^2)} e^{\alpha' s g(x_1x_3)(1-x_2)(1-x_4)} \left[(1-x_2)(1-x_4)\right]^{-1-\alpha(t)} ,$$

where

$$g(x_1,x_3) = -\frac{1}{\ln x_1 x_3} + \frac{x_1}{(1-x_1)^2} + \frac{x_3}{(1-x_3)^2}$$

The simplifications, that occur for large negative s, allow us to take the integral over x_2 and x_4.

$$\int_0^1 dx_2 dx_4\, e^{\alpha' s g(x_1,x_3)(1-x_2)(1-x_4)} [(1-x_2)(1-x_4)]^{-1-\alpha(t)} =$$

$$= [g(x_1,x_3)]^{\alpha(t)} \Gamma(-\alpha(t)) (-\alpha' s)^{\alpha(t)} \ln(-\alpha' s) + \tag{3.16}$$

+ terms of the order of $(-\alpha' s)^{\alpha(t)}$.

Therefore the leading term in the asymptotic of L_4 has the form:

$$L_4 \sim \Gamma(-\alpha(t)) \Delta(t) (-\alpha' s)^{\alpha(t)} \ln(-\alpha' s), \tag{3.17}$$

where

$$\Delta(t) = -\frac{\pi}{\alpha'^2} \int_0^1 dx_1 dx_3 \frac{(x_1 x_3)^{-\alpha(0)-1}}{\ln^2 x_1 x_3} e^{-\alpha' t \frac{\ln x_1 \ln x_3}{\ln x_1 x_3}} .$$

$$\cdot [(1-x_1)(1-x_3)]^{-1+\alpha(2t)} \cdot \left[\frac{x_1}{(1-x_1)^2} + \frac{x_3}{(1-x_3)^2} - \frac{1}{\ln x_1 x_3}\right]^{\alpha(t)} (1-x_1 x_3)^{-2\alpha(t)+2\alpha(2\mu^2)} \tag{3.18}$$

The expressions (3.17) and (3.18) have been obtained earlier (see for instance ref. 8). Eq.(3.12) is in accordance with the general Kikkava, Sakita, Virasoro formula (ref. 3), and coincides with it if one substitutes the arbitrary function $G = 1$. Thus by means of the proposed factorization one gets the simplest expression for the dual planar loop.

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Appendix.

Let us calculate $D_{MN}|\chi_N\rangle$, inserting (2.8) and (2.9) into (2.11). To do this we take into account, that all the creation and annihilation operators in (2.12) commute with one another. Therefore

$$D_{MN}|\chi_N\rangle=\int_0^1\prod_{i=1}^{N-1}dx_i\,\varphi_{N+2}(x)\prod_{i=1}^{N}\prod_{n=N+1}^{N+M}\left(1+\prod_{k=i}^{N-1}x_k\,b_n^+\right)^{-\sqrt{2\alpha'}\frac{\rho_i a_n^+}{b_n^+}+\rho_{in}^L}\cdot$$

$$\cdot e^{\sqrt{2\alpha'}\sum_{i=1}^{N}\rho_i a_i^+\prod_{k=i}^{N-1}x_k+\sum_{i=1}^{N}b_i^+\prod_{k=i}^{N-1}x_k}|0\rangle . \tag{A.1}$$

Multiplying (A.1) by $\int_0^1 x_N^{-\alpha(s_N)-1+H}dx_N$, we get

$$\int_0^1 dx_N\,x_N^{-\alpha(s_N)-1+H}D_{MN}|\chi_N\rangle=$$

$$=\int_0^1\prod_{i=1}^{N}dx_i\,x_N^{-\alpha(s_N)-1}\varphi_{N+2}(x)\prod_{i=1}^{N}\prod_{n=N+1}^{N+M}\left(1+\prod_{k=i}^{N}x_k\,b_n^+\right)^{-\sqrt{2\alpha'}\frac{\rho_i a_n^+}{b_n^+}+\rho_{in}^L}$$

$$e^{\sqrt{2\alpha'}\sum_{i=1}^{N}\rho_i a_i^+\prod_{k=i}^{N}x_k+\sum_{i=1}^{N}b_i^+\prod_{k=i}^{N}x_k}|0\rangle . \tag{A.2}$$

Finally, multiplying (A.2) from the left by $\langle\tilde{\chi}_M|$ (see eq.(2.10) we obtain:

$$\langle\tilde{\chi}_M|\int_0^1 dx_N\, x_N^{-\alpha(s_N)-1+H} D_{MN}|\chi_N\rangle=$$

$$=\int_0^1\prod_{i=1}^{M+N+1} dx_i\; x_N^{-\alpha(s_N)-1}\;\varphi_{N+2}(x)\,\tilde{\varphi}_{M+2}(x)\prod_{i=1}^{N}\prod_{n=N+1}^{N+M}\Big(1-\prod_{k=1}^{N}x_k\prod_{\ell=N+1}^{n-1}x_\ell\Big)^{-2\alpha' p_i\cdot p_n+\rho_{in}^L}. \quad (A.3)$$

All the exponents turn into unity because of the commutativity of their operators.

Now it is easy to show, that the right-hand side of (A.3) coincides with the expression for B_{N+M+2}. We remember, that

$$\varphi_{N+2}(x)=\prod_{i=1}^{N-1}x_i^{-\alpha(s_i)-1}\prod_{i=1}^{N-1}\prod_{n=i+1}^{N}\Big(1-\prod_{\ell=i}^{n-1}x_\ell\Big)^{-2\alpha'(p_n\cdot p_i)+\rho_{in}^L}$$

$$\tilde{\varphi}_{M+2}(x)=\prod_{i=N+1}^{N+M-1}x_i^{-\alpha(s_i)-1}\prod_{i=N+1}^{N+M-1}\prod_{n=i+1}^{N+M}\Big(1-\prod_{\ell=i}^{n-1}x_\ell\Big)^{-2\alpha'(p_n\cdot p_i)+\rho_{in}^L}. \quad (A.4)$$

Inserting (A.4) into (A.3) we get

$$\prod_{i=1}^{N+M-1}x_i\prod_{i=1}^{N-1}\prod_{n=i+1}^{N}\Big(1-\prod_{\ell=i}^{n-1}x_\ell\Big)^{-2\alpha'(p_n\cdot p_i)+\rho_{in}^L}\prod_{i=N+1}^{N+M-1}\prod_{n=i+1}^{N+M}\Big(1-\prod_{\ell=i}^{n-1}x_\ell\Big)^{-2\alpha'(p_n\cdot p_i)+\rho_{in}^L}\times$$

$$\times\prod_{i=1}^{N}\prod_{n=N+1}^{N+M}\Big(1-\prod_{k=1}^{N}x_k\prod_{\ell=N+1}^{n-1}x_\ell\Big)^{-2\alpha'(p_i\cdot p_n)+\rho_{in}^L}. \quad (A.5)$$

As $\prod_{k=i}^{N}x_k\prod_{\ell=N+1}^{n-1}x_\ell=\prod_{k=1}^{n-1}x_k$ one can write down

$$\prod_{i=1}^{N-1}\prod_{n=i+1}^{N}\left(1-\prod_{l=1}^{n-1}x_l\right)^{-2\alpha'(p_n\cdot p_i)+\rho_{in}^{L}}\prod_{i=1}^{N}\prod_{n=N+1}^{N+M}\left(1-\prod_{l=i}^{n-1}x_l\right)^{-2\alpha'(p_n\cdot p_i)+\rho_{in}^{L}}=$$

$$=\prod_{i=1}^{N}\prod_{n=i+1}^{N+M}\left(1-\prod_{l=i}^{n-1}x_l\right)^{-2\alpha'(p_n\cdot p_i)+\rho_{in}^{L}}.$$

Putting this expression into (A.5) we finally get:

$$\langle\tilde{\chi}_M|\int_0^1 dx_N\, x_N^{-\alpha(s_N)-1+H}D_{MN}|\chi_N\rangle=$$

$$=\int_0^1\prod_{i=1}^{M+N-1}dx_i\, x_i^{-\alpha(s_i)-1}\prod_{i=1}^{N+M-1}\prod_{n=i+1}^{N+M}\left(1-\prod_{l=1}^{n-1}x_l\right)^{-2\alpha'(p_n\cdot p_i)+\rho_{in}^{L}}.$$

This last expression coincides with (2.5) for the case of N+M+2 – point function. Thus, the equation (2.11) has been proved.

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