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GLOBAL AND LOCAL STRUCTURE OF RELATIVISTIC QUANTUM SYSTEMS

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Introduction.

Algebraic approach in relativistic quantum theory still contains a significant number of unsolved problems. There are among them the problems of paramount importance such as the definition of field starting from local observable algebras ("F-problem" for the sake of brevity) and the construction of purely algebraic scattering formalism ("S-problem"). We suppose one of main reasons of this to be the following: the initial form of algebraic approach developed by Haag, Araki and the other absorbed a very poor physical information, thus giving no possibility either to solve or even to formulate the problems like afore-mentioned ones, for which it was necessary to operate with objects not reducing to local observables (such as field or scattering state, etc.). For the solution of problems like these the algebraic approach should be provided with additional information; i.e. an algebraic quantum theory allowing the introduction of a field or S-matrix, etc., should represent a net of local algebras (abstract or concrete), which satisfy a set of fundamental axioms (Haag-Kastler's or Haag-Araki's) as well as some additional conditions.

It seems very natural to expect that such additional conditions will include, first of all, certain constraints on global structure of theory. In fact, all axioms are conditions on local algebras, whereas it is most probable that global algebra of a physical system cannot be arbitrary too, especialy if the system in question enjoys some special properties, like the existence of field or S-matrix, etc. This argument can be illustrated, for instance, by the situation in statistical mechanics, where conditions on global algebra exist always and are of essential importance. Nevertheless no general and physically grounded conditions fixing the global structure were proposed for relativistic quantum theory. (In W^* - approach one assumed often that global W^* - algebra R is a factor or is irreducible. However the first assumption was not justified by anything except mathematical convenience and the second, as is well- known, is too restrictive). Besides this, it is not excluded at all that other additional conditions, those of local nature, will also be necessary to pick out of general formalism of local observables more concrete physical theories.

With these arguments in mind, we set ourselves as an object to develop such a modification of Haag-Araki-Kastler theory, where physically well-grounded additional conditions to fundamental axioms were found and investigated. Due to these conditions it could be used as a basis for constructing algebraic structures corresponding, for instance, to field or S-matrix theories, physically more rich and interesting than the general theory of local observables. In the first two parts of our work, published in Russian in /1,2/ and for convenience stated briefly in §1, we have analyzed the global structure of observable algebras in "concrete" W^* - approach. We have found out that all physical systems possessing superselection rules (i.e. very wide class of systems including, in particular, field systems) are characterized by a very definite global structure, which was called the "$\mathcal{H} = \mathcal{H}_\rho$ structure" by us and presupposes, in particular, that global W^*-algebra R is a direct sum of discrete factors. The latter are physically interpreted as global observable algebras of coherent superselection sectors. From purely mathematical viewpoint our proposition to use as a global algebra R the direct sum of discrete factors (instead of older variants, $R = \mathcal{B}(\mathcal{H})$, or R being an arbitrary factor) is of course, not so great innovation. However the main element which was here important to us is that in our scheme the global structure is not introduced ad hoc, but is prescribed directly by the analysis of physical phenomena.

Further, in § 2 we provide the deduced global structure $R = \bigoplus_{\alpha \in C} R_\alpha$ with a net $\{R(O)\}_{O \subset M}$ of local algebras R(O) satisfying Haag-Araki axioms. Then we study in detail an arising superposition of the global (sector) and local (net) structures and generalize to the resulting "crossed" structure all classical theorems and results of the Haag-Araki theory. In particular, quasilocal C^* - algebra $\mathfrak{A}$ of our scheme appears to be simple, what is important for further developments. Also a number of new results is obtained. They include, for instance, theorem 2.4 containing a strictly axiomatic proof of global nature of superselection rules. The properties of translational automorphisms of algebra $\mathfrak{A}$ are investigated. Using asymptotical abelianness of $\mathfrak{A}$ with respect to translation group, we describe the structure of the set of all translation-invariant vectors as well as properties of vacuum coherent sector and vacuum state. This completely clarifies the problem of relations between many possible formulations of "vacuum postulate". As a result, we obtain for any theory of $\mathcal{H} = \mathcal{H}_\rho$ class a well-developed scheme of axiomatic theory of local observables, hoping by this to provide a basis for all possible applications of the theory. In this section the structure $\mathcal{H} = \mathcal{H}_\rho$ is our only additional condition.

In § 3 we proceed to problems connected with introducing of a field and we find our global structure to be quite suitable for these purposes. We represent the coherent superselection sectors as representations π_α of the quasilocal C^*-algebra $\mathfrak{A}$ and study equivalence properties of those representations as well as their restrictions to different subalgebras $\mathfrak{A}_1 \subset R$ (keeping in mind that starting objects for the construction of fields must be intertwining operators of some or other representations connected with coherent sectors /3-6/. We completely describe the equivalence properties of representations π_α (i.e. the global equivalence properties of coherent sectors) and then proceed to local equivalence properties which are of greater importance for the F-problem. At first we consider the equivalence properties of restrictions $\pi_\alpha^{(o)} \doteq \pi_\alpha |_{R(o)}$ with O, by definition, a bounded region. Here it turns out that local unitary equivalence of coherent sectors $\mathcal{H}_\alpha$ and $\mathcal{H}_\beta$ ($\pi_\alpha^{(o)} \simeq \pi_\beta^{(o)}$) requires, besides the axioms, certain additional conditions relating to dimensions of irreducible subspaces of algebras R and R' in sectors $\mathcal{H}_\alpha$ and $\mathcal{H}_\beta$. This means that our starting condition $\mathcal{H} = \mathcal{H}_\rho$ is joined with another additional condition, also of global nature. This new condition is rather general and unrestrictive.

Finally, we investigate in § 4 properties of our scheme with respect to unbounded regions. Following /3-6/ we characterize these properties by two families of C^*_ω-algebras: $\mathfrak{A}^c(O) \doteq R(O)' \cap \mathfrak{A}$ and $\mathfrak{A}(O') \doteq \vee R(\hat{O})$ (C^*-algebra generated by local algebras $R(\hat{O})$, $\hat{O} \subset O'$, $\hat{O} \in \mathcal{B}(M)$. Equivalence properties of corresponding representations $\pi^c_\alpha \doteq \pi_\alpha |_{\mathfrak{A}^c(O)}$ and $\pi^{(O')}_\alpha \doteq \pi_\alpha |_{\mathfrak{A}(O')}$ are of most importance for the F-problem because intertwining operators of representations $\pi^{(O')}_\alpha$ and $\pi^{(O')}_\beta$ or π^c_α and π^c_β (if such exist) possess localization properties and can directly be used for constructing a field group and field operators. We find, however, that the behaviour of the theory in unbounded regions, as described by the families $\{\pi^c_\alpha\}_{\alpha\in\sigma}$ and $\{\pi^{(O')}_\alpha\}_{\alpha\in\sigma}$ strongly differs from that in bounded regions described by the family $\{\pi^{(O)}_\alpha\}_{\alpha\in\sigma}$. The reason is that weak closures of algebras $\mathfrak{A}^c(O)$ and $\mathfrak{A}(O')$ do not belong to $\mathfrak{A}$ in general case; that's why the equivalence properties of corresponding representations are not governed by the fact of simplicity of $\mathfrak{A}$ and worsen considerably.

We study at first the representations π^c_α and weak duality condition /4, 5/ closely connected with them. Under the very general global conditions we prove the fulfillment of weak duality in coherent sectors and then, adding one more global condition, which means physically the absence of continuous superselection rules, we prove weak duality in the whole space $\mathcal{H} = \bigoplus_{\alpha\in\sigma} \mathcal{H}_\alpha$ and pair-wise disjointness of all π^c_α. This means that these representations are useless for the F-problem.

We proceed then to the representations $\pi_\alpha^{(O')}$ and draw the conclusion (however not having the formal proof for the time being) that local observable theory even if provided with any global constraints, allows the arbitrariness in equivalence properties of these representations. It means (if the primariness of the $\pi_\alpha^{(O')}$ is taken into account) that all local observable theories with superselection rules can be divided into two classes, the first of them having $\pi_\alpha^{(O')} \approx \pi_\beta^{(O')}$ and the second $\pi_\alpha^{(O')} \; \delta \; \pi_\beta^{(O')}$ for all $\alpha, \beta \in C$. All field theories fall in the first class and so the problem of formulating the necessary and sufficient conditions of belonging to this class becomes of a real interest. The role of these conditions is to single out the class of field theories within the larger class of relativistic quantum theories of local observables. We have proved that one form of such criterion consists in the presence in every coherent sector $\mathcal{H}_\alpha$ of a total set $S\ell(O)$ of vectors <u>representing</u> states strictly localized in the region O: $\overline{L\{S\ell(O)\}} = \mathcal{H}_\alpha$ for all $O \in B(M)$. The existence of other similar criterions is not impossible at all, but the physical meaning of them is always the same, i.e. to put restrictions on the behaviour of states at the infinity in space-like directions. Loosely speaking, this means that field theories can be picked out in the set of all local observable theories by means of some "asymptotical condition", of which one possible form was found by us.

As a result, we obtain a full picture of equivalence properties of coherent sectors for all regions. This picture becomes quite clear in the light of the following inclusions

$$R(\hat{O}) \subset \mathfrak{A}(O') \subset \mathfrak{A}^c(O) \subset \mathfrak{A} \qquad (0.1)$$

(where $\hat{O}$ is a bounded region lying in O). Indeed, if $\mathfrak{A}_{1,2}$ are C^*-algebras such that $\mathfrak{A}_1 \subset \mathfrak{A}_2$ and $\pi_{i,k}$ are some representations of $\mathfrak{A}_2$, then equivalence properties of the restrictions $\pi_i|_{\mathfrak{A}_1}$ and $\pi_k|_{\mathfrak{A}_2}$ may be stronger in general case than the equivalence properties of π_i and π_k themselves. In virtue of this fact, in the chain (0.1) the representations $\pi_\alpha^{(o)}$ must enjoy the strongest equivalence properties, while the representations π_α the weakest ones. This is in complete accordance with our results, which give $\pi_\alpha^{(o)} \simeq \pi_\beta^{(o)}$ and $\pi_\alpha \,\delta\, \pi_\beta$. The asymptotical representations π_α^c and $\pi_\alpha^{(o')}$ are intermediate between local and global ones, and so their equivalence properties become well-determined only under additional restrictions.

The methods and aims of our work lie close to the works by Borchers /3,4/ and especially by Doplicher, Haag and Roberts /5, 6/ dedicated to the F-problem. Detailed comparison with the Doplicher, Haag and Roberts' results is made in the Conclusion. As a main general distinction of both schemes we can point out that at the moment we have gone not so far in what concerns the direct construction of a field (which was the main task of Doplicher et al.), but in return we have developed a more general and elaborated formalism, which is suitable, as we hope, for wider range of applications. Also we regarded more critically to the introduction of additional conditions, controlling their independence from each other and from the axioms and their necessity for desired results. In correspondence with this principle, our scheme starts with the global condition $\mathcal{H} = \mathcal{H}_p$, which is necessary for the formulation of the superselection theory, and ends (at the present stage) with the local condition $L\{S\ell(O)\}_\alpha = \mathcal{H}_\alpha$ which is necessary for the existence of fields.

NOTATIONS. Most part of our notations follows the books by J.Dixmier /7, 8/, the references on which are denoted as DW and DC respectively, so that, for instance, [DW-5] means page 5 of /7/. R^1 (C^1) is the set of all real (resp. complex) numbers. The four-dimensional Minkowski space will be denoted by M, open bounded regions in M by O, and B(M) is the notation for the set of all such O in M. O' denotes the set of all points in M, which are space-like to all points of O. If $\mathfrak{M}$ is any subset of some linear space, $L\{\mathfrak{M}\}$ denotes the linear hull of $\mathfrak{M}$. In topological spaces the closure of any subset $\mathfrak{M}$ in the topology τ will be denoted by the line provided with corresponding index: $\overline{\mathfrak{M}}^{\tau}$. The closure in the norm topology of any Banach space will be denoted by the line without the index. In Hilbert space $\mathcal{H}$ we denote as $\mathcal{B}(\mathcal{H})$ the algebra of all linear bounded operators on $\mathcal{H}$, and $\mathcal{C}(\mathcal{H})$ the algebra of all multiples of the identity operator $\mathbf{I}$. The subspaces in $\mathcal{H}$ generated by the action of a $*$-set of operators $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ on a subset $\mathfrak{M} \subset \mathcal{H}$ will be denoted as $\mathcal{H}^{\mathcal{A}}_{\mathfrak{M}}$ i.e. $\mathcal{H}^{\mathcal{A}}_{\mathfrak{M}} = L\{\mathcal{A}\mathfrak{M}\}$. Projection on $\mathcal{H}^{\mathcal{A}}_{\mathfrak{M}}$ is denoted as $P^{\mathcal{A}}_{\mathfrak{M}}$. S^1 denotes the unit sphere in $\mathcal{H}$. The weak, strong and uniform operator topologies in $\mathcal{B}(\mathcal{H})$ are denoted as by the indices w, s, u respectively.

vector states$_W$ in the representation $\pi \in \mathrm{Rep}\,\mathfrak{A}$ such that $\pi(\mathfrak{A}) = R$. One possible variant of such connection will be described in § 3 below.

Definition 1.1.

State of physical system described by W^*-algebra of observables $R \subset \mathcal{B}(\mathcal{H})$ is linear functional ω on the algebra R, positive ($\forall_{A \in R}\ \omega(A^*A) \geq 0$) and normed to the unity (ω (I) = 1). The set of all states will be denoted as R_1^{*+}

Extremal points of the set R_1^{*+} will be called pure states, all other points mixed states (mixtures).

State defined for all $A \in R$ by means of the correspondence $A \to (A\Phi, \Phi)$ with $\Phi \in \mathcal{H}$ and $\|\Phi\| = 1$ will be called vector state and denoted as ω_Φ. The set of all vector states on R will be denoted as V(R) and the set of all pure vector states as PV(R).

Vector $\Phi \in \mathcal{H}$ such that $\omega_{\|\Phi\|^{-1}\Phi} \in$ PV(R) will be called pure vector, and the set of all such vectors will be denoted as $\mathcal{P}$.

The most convenient object of stadying possessing in addition clear physical interpretation, appears to be the set of all vectors in $\mathcal{H}$ representing the same vector state on R.

Definition 1.2.

For every vector state $\omega_\Phi \in$ V(R), the set of all unit vectors $\Psi \in \mathcal{H}$ such tbat $\omega_\Psi = \omega_\Phi$ is called the H-image of ω_Φ ard denoted as $\mathfrak{M}^1_\Phi$. I.e.,

$$\mathfrak{M}^1_\Phi \doteq \{ \Psi \in \mathcal{H} \mid \|\Psi\| = 1, \forall_{A \in R}\ (A\Psi, \Psi) = (A\Phi, \Phi) \quad (1.1)$$

If and only if $R = \mathcal{B}(\mathcal{H})$, H-images of all vector states from V(R) are unit rays, $R \neq \mathcal{B}(\mathcal{H})$ implies the presence of non-onedimensional H-images. From physical viewpoint introducing of the H-image notion seems to be quite natural. The object (related to $\mathcal{H}$) directly manifesting itself in the experiment and thus in the exact sense representing physical state, is not the vector $\Phi \in \mathcal{H}$ itself, but the set of all expectation values $(A\Phi, \Phi)$ for all observables A. By definition, for all vectors in the same H-image this set of expectation values is the same.

Now let us list a few main properties of H-images.

1) for every vector state ω_Φ the closure of the linear hull of its H-image $\mathfrak{M}^1_\Phi$ coincides with cyclic subspace $\mathcal{H}^{R'}_\Phi$:

$$\overline{L\{\mathfrak{M}^1_\Phi\}} = \mathcal{H}^{R'}_\Phi \tag{1.2}$$

2) the state $\omega_\Phi \in V(R)$ is pure if and only if

$$\mathfrak{M}^1_\Phi = \mathcal{H}^{R'}_\Phi \cap S^1. \tag{1.3}$$

Class of W^*-algebras possessing pure vector states is closely related to the class of type I W^*-algebras, as the simple property shows:

the following three sets are in one-to-one correspondence between each other: 1) the set PV(R) of all pure vector states on the W^*-algebra R; 2) the set of all minimal projections in R; 3) the set of all minimal projections in R'.

If R is a factor, this property means that

$$PV(R) \neq \emptyset \Leftrightarrow R \text{ is of the type I}. \tag{1.4}$$

If $3 \neq \mathcal{B}(\mathcal{H})$ this relation can be destroyed in general case, but any algebra R with $PV(R) \neq \emptyset$ still cannot belong to the types II and III.

For the comparison of algebraic theory with the old Hilbert space language (in which all pure states are considered to be unit rays in $\mathcal{H}$) the following questions are of interest: 1) under what conditons the H-image of a given vector state is unit ray? 2) under what conditions all pure vector states are represented by unit rays? Below the precise answers to both questions are given.

Proposition I.I.

The H-image of vector state ω_φ is unit ray, i.e.

$$\mathfrak{M}^1_\varphi = \{ e^{i\psi}\varphi \}_{\psi \in R^1}$$

if and only if

1) $\omega_\varphi \in PV(R)$

2) $\mathcal{H}^R_\varphi \supset \mathcal{H}^{R'}_\varphi$

The condition (2) can be replaced by the equivalent one:

2) $\mathcal{H}^R_\varphi = \mathcal{H}^{R'}_\varphi$.

Proposition 1.2

H-images of all pure vector states are one-dimensional if and only if the algebra R is of the following form:

$$R = R_1 \oplus R_2 \quad \text{with } R_1 = \bigoplus_{\gamma\in\Gamma} \mathcal{B}(\mathcal{H}_\gamma) \quad \text{and } PV(R_2) = \emptyset .$$

After these preliminaries let us introduce the algebraic structure, which will be at very centre of all the further account.

Definition 1.3.

We shall say that W^*-algebra R possesses sufficient amount of pure vector states or, equivalently, belongs to the " $\mathcal{H}=\mathcal{H}_p$ class", if the linear hull of the set of vectors $\Psi \in \mathcal{H}$ representing pure states is dense in $\mathcal{H}$:

$$\mathcal{H}_p \doteq \overline{L\{\Psi \in \mathcal{H} \mid \omega_\Psi \in PV(R)\}} = \mathcal{H} . \quad (1.5)$$

The class of von Neumann algebras satisfying this definition is completely characterized by the following theorem.

Theorem 1.3 *)

The following conditions are equivalent:

1) W^*-algebra R belongs to the $\mathcal{H} = \mathcal{H}_p$ class,

2) R is W^*-algebra of the type **I** and its centre 3 contains only the operators with purely point spectrum,

3) R is direct sum of type **I** factors.

Next, we establish that W^*-algebras of the $\mathcal{H} = \mathcal{H}_p$ class can be physically interpreted as observable algebras of physical systems with superselection rules; conversely, every physical system with superselection rules corresponds to observable algebra belonging to the $\mathcal{H} = \mathcal{H}_p$ class.

We draw these conclusions from the analysis of concrete physical systems possessing superselection rules. On the grounds of such analysis we formulate a comprehensive algebraic definition of arbitrary superselection rule.

Definition 1.4

Let quantum system be given with observable algebra $R \subset \mathcal{B}(\mathcal{H})$. We shall say that a superselection rule is acting in this system, if there exists the decomposition $\mathcal{H} = \bigoplus_{\alpha \in \sigma} \mathcal{H}_\alpha$ satisfying the conditions I - III below, and each of these conditions implies two others.

*) Examples of statistical systems with observable algebras of the types II and III being explicitly known, it is clear that in the statistical mechanics (in contrast to relativistic quantum theory) the structure proposed by us cannot be too universal.

I. Every vector not lying in some $\mathcal{H}_\alpha$ corresponds to mixed state.

II. Transitions between different $\mathcal{H}_\alpha$ by means of observable operators are forbidden

$$\forall_{A \in R} \ \forall_{\alpha_1 \neq \alpha_2} \ (A \Phi_{\alpha_1}, \Phi_{\alpha_2}) = 0 \qquad (\text{II}, \alpha)$$

and in addition all the H-images of vector states ω_{Φ_α} with $\Phi_\alpha \in \mathcal{H}_\alpha$ lie entirely in $\mathcal{H}_\alpha$:

$$\forall_{\Phi_\alpha \in \mathcal{H}_\alpha} \ \mathfrak{M}^1_{\Phi_\alpha} \subset \mathcal{H}_\alpha \qquad (\text{II}, \beta)$$

III. There exists an operator $T \eta \mathfrak{Z}$ (i.e. affiliated to $\mathfrak{Z}$ and unbounded in general case) such that all $\mathcal{H}_\alpha$, $\alpha \in \sigma$ are its eigen-spaces.

In this situation the subspaces $\mathcal{H}_\alpha$ are called superselection sectors and the operator T superselection operator (corresponding to a given superselection rule).

Theorem I.4

Quantum system possesses superselection rules if and only if its observable algebra R belongs to the $\mathcal{H} = \mathcal{H}_\rho$ class. Each operator $T \eta \mathfrak{Z}$ determines uniquely a certain superselection rule, i.e. the structure $\mathcal{H} = \bigoplus_{\alpha \in \sigma} \mathcal{H}_\alpha$ satisfying the definition I.4.

Let us note main features peculiar to this treatment of superselection rules.

1) In the definition I.4 only the condition (II, β) and the requirement of equivalence of the conditions I - III were not marked in previous literature. As a consequence, all the new features of our scheme go back to these two distinctions. Namely, the condition (II, β) implies that

superselection operators have to be affiliated not only to the commutant R' but to the centre $\mathfrak{Z}$ of R; the equivalence of the properties I – III implies the necessity for observable algebras of systems with superselection rules to belong to the $\mathcal{H} = \mathcal{H}_p$ class. Properties that follow represent the secondary consequences of the same initial distinctions.

2) All superselection rules commute automatically between each other.

3) Let quantum system be given possessing the decomposition of the following form:

$$\mathcal{H} = \int^{\oplus} d\mu(\xi)\,\mathcal{H}(\xi)\ ;\ R = \int^{\oplus} d\mu(\xi) R(\xi), \quad (I.6)$$

where R (ξ) are factors almost everywhere in the measure μ and so the algebra of diagonalizable operators is some abelian W^*-algebra generated by an operator $T \in \mathfrak{Z}$ with continuous spectrum. Such decompositions are impossible in $\mathcal{H} = \mathcal{H}_p$ and in our scheme they are refused to be interpreted as superselection structures. This follows automatically from the scheme, but we also represented the independent arguments in favour of this (in /2/,§2), which are close to those developed by J.Antoine /9/.

Absence of the decompositions (I.6) does not mean at all that continuous superselection rules are excluded in our scheme. What is excluded is only one a priori possible kind of them, namely, the superselection rules, for which the superselection operator possesses continuous spectrum. But still is perfectly allowed another kind, the superselection rules, for which the set $\mathfrak{S}$ of all coherent superselection sectors is uncountable. We verified that all known examples of the continuouos superselection rules (such as Bargmann's superselection rule or the superselection rule in the BCS model found by Emch and Guenin /10/) belong to the second kind. The assertion about complete impossibility of the first mentioned kind of continuous superselection rules may be considered as a prediotion made by our scheme.

Thus we define the discrete and the continuous superselection rules as those characterized respectively by the countable and the uncountable set σ of coherent sectors; but in both cases by an operator with point speotrum only. As the following simple proposition shows, these cases correspond to two subclasses of the $\mathcal{H} = \mathcal{H}_p$ class, which differ by an essential structural property.

Proposition I.5

Algebra R of the $\mathcal{H} = \mathcal{H}_p$, class describes the quantum system with discrete superselection rules if and only if the centre $\mathfrak{Z}$ is a countably decomposable algebra. Otherwise ($\mathfrak{Z}$ is not countably decomposable) R describes the quantum system with continuous superselection rules.

4) As is well-known, existence of decomposition of the theory into coherent superselection sectors is very desirable for any superseleotion soheme. This property is also reached automatically in our treatment.

Definition I.5

Let the Hilbert spaoe $\mathcal{H} = \mathcal{H}_p$ be decomposed into a certain direct sum of superselection seotors, $\mathcal{H} = \bigoplus_{\alpha \in \sigma} \mathcal{H}_\alpha$. The subspace $\mathcal{H}_\alpha$ in this decomposition will be oalled coherent superselection sector, if $\mathcal{H}_\alpha$ is the superseleotion sector for any of superselection rules presented and within $\mathcal{H}_\alpha$ there are no more superselection sectors.

Proposition I.6

Any theory possessing superselection rules (in the sense of our definition I.4 and theorem I.4) allows the decomposition into coherent superselection sectors:

$$\mathcal{H} = \bigoplus_{\alpha \in \sigma} \mathcal{H}_\alpha \quad ; \quad R = \bigoplus_{\alpha \in \sigma} R_\alpha \qquad (I.7)$$

and the observable algebras R_α of the coherent sectors are factors of the type I.

The problem arising immediately with the decomposition (I.7) is to study the internal structure of coherent sector. In this point our scheme represents the generalization of the usual treatment, in which only irreducible subspaces of R ($R_\alpha = \mathcal{B}(\mathcal{H}_\alpha)$) were considered as coherent sectors. (The reason lies again in the condition II, β, due to which the eigen-spaces of our superselection operators are irreducible for $\mathfrak{Z}'$, but not for R in general case). First of all, this generalization implies that in the case $R_\alpha \neq \mathcal{B}(\mathcal{H}_\alpha)$ the vectors representing mixed states are possible in coherent sectors.

Proposition I.7

Let $\mathcal{H}_\alpha$ be coherent superselection sector. Then all vectors Ψ which are of the form $\Psi = \Phi_1 + \Phi_2$ with Φ_1 and Φ_2 pure and belonging to different irreducible subspaces of R, as well as to different irreducible subspaces of R′, represent mixed states:

$$\omega_\Psi \notin PV(R) \Leftarrow \Psi = \Phi_1 + \Phi_2 \, , \; \Phi_k \in \mathcal{P}_\alpha \, , \; \mathcal{H}^R_{\Phi_1} \cap \mathcal{H}^R_{\Phi_2} = \mathcal{H}^{R'}_{\Phi_1} \cap \mathcal{H}^{R'}_{\Phi_2} = 0 \qquad (I.8)$$

Every non-pure vector in $\mathcal{H}_\alpha$ is the linear combination of the vectors of the form (I.8) and the set of all such vectors is void if and only if $R_\alpha = \mathcal{B}(\mathcal{H}_\alpha)$.

Due to this proposition, the superposition principle in its usual form

$$\Phi_1, \Phi_2 \in \mathcal{P} \implies \Phi_1 + \Phi_2 \in \mathcal{P}_1$$

is not fulfilled in $\mathcal{H}_\alpha$. That's why we introduce a certain generalization of this principle, valid in all coherent sectors.

Definition I.6

We shall say that in the subspace $\mathcal{H}_1 \subset \mathcal{H}$ generalized superposition principle is fulfilled, if for every vectors $\Phi_1, \Phi_2 \in \mathcal{H}_1 \cap \mathcal{P}$ there can be found vectors $\Psi_1, \Psi_2 \in \mathcal{H}_1 \cap \mathcal{P}$ corresponding to states ω_{Φ_1} and ω_{Φ_2} respectively and such that $\Psi_1 + \Psi_2 \in \mathcal{H}_1 \cap \mathcal{P}$. I.e.

$$\forall_{\Phi_1, \Phi_2 \in \mathcal{H}_1 \cap \mathcal{P}} \ \exists_{\Psi_1, \Psi_2 \in \mathcal{H}_1 \cap \mathcal{P}} \quad \Psi_k \in \mathfrak{M}^1_{\Phi_k}, \ k = 1, 2; \ \Psi_1 + \Psi_2 \in \mathcal{H}_1 \cap \mathcal{P}. \qquad (I.9)$$

Proposition I.8

In every coherent sector $\mathcal{H}_\alpha$ the generalized superposition principle (I.9) holds.

Further, for every coherent sector $\mathcal{H}_\alpha$ we prove the representations:

$$\forall_{\Phi_\alpha \in \mathcal{P}_\alpha} \ \mathcal{H}_\alpha = \mathcal{H}^{3'}_{\Phi_\alpha} \simeq \mathcal{H}^{R}_{\Phi_\alpha} \otimes \mathcal{H}^{R'}_{\Phi_\alpha} \simeq \bigoplus_{\{\Phi_k\}} \mathcal{H}^{R}_{\Phi_k}, \ R_\alpha \simeq \mathcal{B}(\mathcal{H}^{R}_{\Phi_\alpha}) \otimes \mathcal{C}(\mathcal{H}^{R'}_{\Phi_\alpha}) \qquad (I.10)$$

where $\{\Phi_k\}_{k \in \mathcal{K}'_\alpha}$ is any orthonormal basis in $\mathcal{H}^{R'}_{\Phi_\alpha}$.

Hence it follows that every coherent sector is completely determined by corresponding values of the following algebraic invariants:

$$\varkappa_\alpha \doteq \dim \mathcal{H}^{R}_{\Phi_\alpha} ; \ \varkappa'_\alpha \doteq \dim \mathcal{H}^{R'}_{\Phi_\alpha}. \qquad (I,II)$$

It is easy to show that

$$\varkappa_\alpha = \text{card}\ \mathcal{K}_\alpha\ ;\ \varkappa'_\alpha = \text{card}\ \mathcal{K}'_\alpha\ , \qquad (\text{I},12)$$

where card $\mathcal{K}_\alpha$ (card $\mathcal{K}'_\alpha$) is the power of the complete orthogonal system of projections in R (resp. R′). The case $\varkappa'_\alpha = I$ corresponds to the old definition of coherent sector ($R_\alpha = \mathcal{B}(\mathcal{H}_\alpha)$) and will be called abelian coherent sector /5/. Generally speaking, both parameters $\varkappa_\alpha$ and $\varkappa'_\alpha$ are allowed to take the values of any cardinal number. However, it will be established in § 2 that local structure of the theory implies the infinity of the physical algebras R_α. This means that R_α are factors of the type I_∞ or, equivalently, $\varkappa_\alpha \geq \aleph_0$ (the countable set cardinal). Further, it will be shown in § 3 that a special class is formed by the coherent sectors $\mathcal{H}_\alpha$ with $\varkappa'_\alpha \leq \aleph_0$. The following proposition gives complete characterization of such sectors.

Proposition I.9

Let $\mathcal{H}_\alpha$ be coherent superselection sector. The following conditions are equivalent:

1) $\varkappa'_\alpha \leq \aleph_0$,

2) R_α possesses cyclic vectors,

3) R'_α is a countably decomposable algebra,

4) irreducible subspaces of R′(or, equivalently, the H-images of pure vector states from $\mathcal{H}_\alpha$) are separable.

Of course, the completely analogous proposition holds for R_α . These propositions give us, in particular, the necessary and sufficient conditions of existence of cyclic and separating vectors for sector algebras R_α and R'_α .

Later on we shall need such conditions also for the "full" algebras R, R', $\mathfrak{Z}$. It is easy to verify that these conditions are the following ones:

$$\begin{aligned} &\exists_{\Psi\in\mathcal{H}} \overline{R\Psi} = \mathcal{H} && \Longleftrightarrow \forall_{\alpha\in\sigma} \varkappa'_\alpha \leq \aleph_0, \text{card } \sigma \leq \aleph_0 \\ &\exists_{\Psi\in\mathcal{H}} \overline{R'\Psi} = \mathcal{H} && \Longleftrightarrow \forall_{\alpha\in\sigma} \varkappa_\alpha \leq \aleph_0 \text{, card } \sigma \leq \aleph_0 \qquad (I,13) \\ &\exists_{\Psi\in\mathcal{H}} \overline{\mathfrak{Z}'\Psi} = \mathcal{H} && \Longleftrightarrow \text{card } \sigma \leq \aleph_0 . \end{aligned}$$

The conditions of separability of the spaces $\mathcal{H}_\alpha$ and $\mathcal{H}$ will also be useful:

$$\mathcal{H}_\alpha \text{ is separable } \Longleftrightarrow \varkappa_\alpha \leq \aleph_0 ,\ \varkappa'_\alpha \leq \aleph_0$$

$$\mathcal{H} \text{ is separable } \Longleftrightarrow \varkappa_\alpha \leq \aleph_0 ,\ \varkappa'_\alpha \leq \aleph_0 ,\ \text{card } \sigma \leq \aleph_0 .$$

Finally, the whole theory of the $\mathcal{H}=\mathcal{H}_\rho$ class is completely determined by the following set of algebraic invariants:

$$\{ \Sigma \doteq \text{card } \sigma ;\ \forall_{\alpha\in\sigma} \varkappa_\alpha ;\ \forall_{\alpha\in\sigma} \varkappa'_\alpha \} .$$

According to proposition I,5, the cases $\Sigma \leq \aleph_0$ and $\Sigma > \aleph_0$ correspond to theories with discrete and continuous superselection rules respectively. The formulas (I.13) show that one of the principal distinctions of these theories is that in the continuous case algebras R, R', $\mathfrak{Z}$, $\mathfrak{Z}'$ cannot possess either cyclic or separating vectors.

2. THEORY OF LOCAL OBSERVABLES FOR SYSTEMS WITH SUPERSELECTION RULES

Now let us proceed to our main problem, which consists in investigation of correlation between global and local properties of general systems with superselection rules. As the first step to this we shall adopt the starting positions of the Haag-Araki concrete algebraic formalism. This means to suppose that associated with each open bounded region O in the Minkowski space M there is a W^*-algebra $R(O)$ acting on a Hilbert space $\mathcal{H}$ and the set $\{R(O)\}\ O \in B(M)$ of all these algebras satisfies the following (Haag-Araki) postulates.

I. Isotony

$$O_I \subset O_2 \Rightarrow R(O_I) \subset R(O_2)$$

If the set B(M) is considered as partially ordered by the inclusion relation $O_I \subset O_2$, then it follows from the postulate I that the sequence $\{R(O)\}\ O \in B(M)$ represents a net. Indeed, B(M) is in this case a filtrating partially ordered set, i.e. for each pair O_I, $O_2 \in B(M)$ there exists $O_3 \in B(M)$ (for instance, $O_3 = O_I \cup O_2$) such that O_I, $O_2 \subset O_3$, and from here I ensures that $R(O_I) \cup R(O_2) \subset R(O_3)$. According to the usual definition, this means that $\{R(O)\}_{O \in B(M)}$ is a net.

Sometimes we shall require the fulfillment of the following stronger form of I.

I - α . Continuous isotony /11/ : Let $\{O_k\}_{k=1}^{\infty}$ be a decreasing sequence of regions $O_k \in B(M)$, $O_1 \supset O_2 \supset O_3 \supset \ldots$ and $O = \text{int} \bigcap_{k=1}^{\infty} O_k$. Then $R(O) = \bigcap_{k=1}^{\infty} R(O_k)$.

II. Additivity

$$R(O_1 \cup O_2) = \overline{R(O_1) \vee R(O_2)}^{w}.$$

On the basis of I and II we can also associate a W^*-algebra $R(\tilde{O})$ with each unbounded region $\tilde{O} \subset M$, putting by definition

$$R(\tilde{O}) \doteq \overline{\bigvee_{O \subset \tilde{O}} R(O)}^{w}, \qquad (2.I)$$

where the union is taken over all bounded regions 0 contained in $\widetilde{O}$.

II-1 Weak additivity

$$\forall_{O \in B(M)} \overline{\bigvee_{a \in M} R(O+a)}'' = R(M).$$

III. Causality (Locality): $O_1 \subset O_2' \Rightarrow R(O_1) \subset R(O_2)'$.

IV. Primitive causality: Every time-slice $\mathcal{T}_\varepsilon \doteq \{x \in M \mid |x^0| < \varepsilon\}$ satisfies the condition $R(\mathcal{T}_\varepsilon) = R(M)$.

V. Translational covariance: An unitary strongly continuous representation U of the translation group M of Minkowski space is acting in $\mathcal{H}$

$$M \ni a \longrightarrow U(a) = \int e^{ipa} dE(p)$$

such that

$$\forall_{O \in B(M)} \forall_{a \in M} U(a) R(O) U(-a) = R(O_a) \qquad (2.2)$$

O_a being the image of O under the translation $a \in M$.

V is usually treated as a part of the physically more important postulate

V-1.Relativistic covariance: There exists a unitary representation in $\mathcal{H}$ of the Poincaré group with the properties analogous to (2.2).

However, existence of the Lorentz transformations is not used by us anywhere and so we consider the translation group separately.

VI. Spectrum condition: Support of the spectral measure E(p) of the representation U is contained in the forward light cone, i.e.

$$\text{supp } E(p) \subset \overline{V_+} \doteq \overline{\{p \in M \mid p^2 \geq 0, p^0 \geq 0\}}.$$

The succession of axioms adopted here somewhat differs from the usual one, but it seemed to us more natural. The first group of the axioms (I - IV) concerns only with the structure $\mathfrak{A} \doteq \overline{\bigvee_{O \in \mathcal{B}(M)} R(O)}$ i.e. C^*-algebra with the net, while the second group (V, VI) concerns with the structure $\{\mathfrak{A}, U\}$, C^*-algebra with the net and the group of automorphisms.

W^*-algebras R(O), satisfying I - VI, will be called algebras of local observables (local algebras). The union $\mathcal{A} \doteq \bigcup_{O \in \mathcal{B}(M)} R(O)$ is called $*$ - algebra of all local observables, its uniform closure $\mathfrak{A} \doteq \overline{\mathcal{A}}$ is called algebra of quasilocal observables (quasilocal C^*-algebra), its weak closure $R(M) \doteq \overline{\mathcal{A}}^{w}$ is called algebra of global observables (global W^*-algebra).

Main premises of the Haag-Araki approach include also that global algebra R(M) coincides with the observable algebra R of described system. Taken together with the § 1 results. this gives us the following fundamental

Property 0

For every physical system with superselection rules, global observable algebra R(M) is a W^*-algebra of the $\mathcal{H} = \mathcal{H}_\rho$ class.

This fact represents the initial formulation of the interrelation between the local structure described by the axioms I - VI and global structure generated by superselection rules. Its immediate consequence is that the algebra $R(M) \equiv R$ represents itself in the form $R = \bigoplus_{\alpha \in \sigma} R_\alpha$, R_α being discrete factors, and the Hilbert space $\mathcal{H}$ is decomposed into a direct sum $\bigoplus_{\alpha \in \sigma} \mathcal{H}_\alpha$ of coherent superselection sectors $\mathcal{H}_\alpha$ with the projections P_α belonging to the centre $\mathfrak{Z}$ of R. Next task is to investigate what the property O implies for the algebras R(O), $\mathcal{A}$, $\mathfrak{A}$. Let us begin with

<u>Definition 2,I</u>

Inductions of algebras R(O), $\mathcal{A}$, $\mathfrak{A}$, R by the projections $P_\alpha \in R' = \mathfrak{A}' = \mathcal{A}' \subset R(O)'$ will be called sector algebras and denoted $X_\alpha = X_{P_\alpha}$, X being any of these algebras. The algebras X will be called sometimes "full algebras" as distinct from sector ones.

<u>Proposition 2.I.</u>

The net $\{R(O)\}_{O \in B(M)}$ of local sector algebras satisfies the axioms I, I-α II, II-α, III - VI, if the full net $\{R(O)\}_{O \in B(M)}$ does. Besides this,

$$\mathcal{A}_\alpha = \bigcup_{O \in B(M)} R(O)_\alpha \quad , \quad \mathfrak{A}_\alpha = \overline{\mathcal{A}_\alpha} \quad , \quad R_\alpha = \overline{\mathcal{A}_\alpha}^{w} .$$

<u>Proof</u>. Validity of the axioms I - IV for $\{R(O)_\alpha\}_{O \in B(M)}$ can be established trivially, using the properties of the induction operation /DW-18/.

To obtain I-α, we have to make sure that $R(O) = \bigcap_{k=1}^{\infty} R(O_k)$

$$\text{with } O = \text{int} \bigcap_{k=1}^{\infty} O_k \quad \text{implies } R(O)_\alpha = \bigcap_{k=1}^{\infty} R(O_k) .$$

It is more convenient to deduce the equivalent property: $R(O)_\alpha' = \{\bigcup_k^{\infty} R(O_k)_\alpha'\}''$. It follows from $R(O_1)' \subset R(O_2)' \subset$ ... that $\bigcup_{k=1}^{\infty} R(O_k)'$ is a $*$ - algebra, which is w-dense in $R(O)' = \{\bigcup_{k=1}^{\infty} R(O_k)'\}''$ due to the axiom I-α for R(O). Projection P_α lying in $R(O)'$, we have from /DW-18/ that the induction $[\bigcup_{k=1}^{\infty} R(O_k)]_{P_\alpha}$ is w-dense in $R(O)_\alpha'$ i.e.

$$\{[\bigcup_{k=1}^{\infty} R(O_k)']_{P_\alpha}\}'' = R(O)_\alpha'$$

whence desired property follows immediately.

The proof of II-α is analogous.

V and VI follow from the Araki-Borchers theorem (see proposition 2,19 below) stating that translation operators U(a) belong to R. Due to this theorem, $P_\alpha \in U(M)'$ and the induction $U(M)'' \to U(M)''_{P_\alpha}$ defines in the sector $\mathcal{H}_\alpha$ s-continuous unitary representation U_α of translation group, satisfying V, VI. Properties of the representations U and U_α will be considered in more detail at the end of this section in connection with the discussion of vacuum sector.

Finally, in the last assertions of propositions 2.1 those for $\mathcal{A}_\alpha$ and $\mathcal{R}_\alpha$ are obvious (due to (DW-18)). So we have to prove only that $\mathfrak{A}_\alpha = \overline{\mathcal{A}_\alpha}$.

At first let us remark that the induction $R \to R_\alpha$ being a $*$ - homomorphism, image $\mathfrak{A}_\alpha$ of C^*-algebra $\mathfrak{A} \subset R$ is also a C^*-algebra, whence it follows that

$$\mathfrak{A}_\alpha \supset \overline{\mathcal{A}_\alpha} .$$

Let us obtain an inverse inclusion. For each $A_\alpha \in \mathfrak{A}_\alpha$ there is $A \in \mathfrak{A}$ such that the restriction $A|_{\mathcal{H}_\alpha}$ of A to $\mathcal{H}_\alpha$ is equal to A_α. If $\{B^n\}_{n=1}^{\infty}$ is a sequence of local observables $B^n \in \mathcal{A}$ u-converging to A, then the restrictions $B^n|_{\mathcal{H}_\alpha} \equiv B^n_\alpha$ belong to $\mathcal{A}_\alpha$ and form the sequence $\{B^n_\alpha\}_{n=1}^{\infty}$ u-converging to A_α in virtue of $\|A_\alpha - B^n_\alpha\|_{\mathcal{H}_\alpha} \leq \|A - B^n\|$. This means that $A_\alpha \in \overline{\mathcal{A}_\alpha}$ hence the result follows.

As a consequence, there arise in our scheme two kinds of local observable theories: the "full" theory in $\mathcal{H} = \mathcal{H}_P$ and the sectorial or " coherent" theories in each $\mathcal{H}_\alpha$. Henceforth we shall study both these kinds of theories in parallels. First of all we see that a number of well-known results obtained in the Haag-Araki theory holds automatically in our scheme. These are the results, which can be proved using the axioms I - VI only, without any assumptions about the structure of global algebra R. Such results are valid in our formalism for full algebras as well as for sector ones. The most important of them are the following: the theorem by Borchers on ideals in quasilocal algebra $\mathfrak{A}$; the Reeh-Schlieder theorem on analytical for the energy vectors; the theorem by Borchers about belonging of translation operators to algebra R.

Each of these theorems implies in our formalism a number of important consequences and so appears to be a kernel of a certain complex of properties. Now we shall consider these three complexes in consequtive order.

Proposition 2.2 (Borchers /12/).

Let the axioms I - III, V, VI be satisfied. Then the set $J \subset \mathfrak{A}$ is a closed two-sided ideal in $\mathfrak{A}$ if and only if $J \cap \mathfrak{Z}$ is a non-trivial ideal in $\mathfrak{Z}$.

This leads immediately to important results.

Theorem 2.3

Let quantum theory in $\mathcal{H} = \mathcal{H}_p$ be given and the axioms I-III, V, VI be satisfied. Then quasilocal algebra $\mathfrak{A}$ as well as quasilocal sector algebras $\mathfrak{A}_\alpha$ are simple.

Proof. Taking into account the propositions 2.1 and 2.2 and the theorem I we see that sector algebras $\mathfrak{A}_\alpha$ cannot contain closed two-sides ideals, because the centre of R is being trivial. As a consequence, the C^*-algebra $\mathfrak{A}_\alpha$ cannot contain any two-sided ideals, i.e. $\mathfrak{A}_\alpha$ is simple for any $\alpha \in \sigma$.

Let us assume now that there is a two-sided ideal $J \neq \{0\}$ in $\mathfrak{A}$. Then there exists always some $\alpha_0 \in \sigma$ such that $J_{\alpha_0} \neq \{0\}$, where J_{α_0} is the image of J under the induction $R \longrightarrow R_{\alpha_0}$ defined by projection $P_{\alpha_0} \in \mathfrak{Z}$. It is easy to see that J_{α_0} should be a two-sided ideal in $\mathfrak{A}_{\alpha_0}$

in contradiction with the simplicity of the latter. Thus $\mathfrak{A}$ is simple.

Let us note here that global algebra R of the $\mathcal{H} = \mathcal{H}_p$ class (with non-trivial σ) can never be simple because the inductions R $\longrightarrow$ R_α are $*$ -homomorphisms with non-zero kernels. Further, due to the fact that $R(O) \subset \mathfrak{A}$ but in general case $R(\tilde{O}) \not\subset \mathfrak{A}$ for $\tilde{O} \notin B(M)$, the simplicity of $\mathfrak{A}$ (and non-simplicity of R) induces differences between properties of observable algebras associated with the bounded and unbounded regions. These differences will play the most essential part in §§ 3, 4 where we study field-like properties of our theory. Here we collect another consequences of the simplicity of $\mathfrak{A}$, which are also of importance, but are not related to field-like properties.

Theorem 2.4 (Global nature of superselection rules).

Let quantum theory of the $\mathcal{H} = \mathcal{H}_p$ class be given and all the axioms I-VI (except possibly IV) be satisfied. Then centre $\mathfrak{Z}$ of global algebra R does not contain either local or quasilocal observables:

$$\mathfrak{A} \cap \mathfrak{Z} = \mathbb{C}(\mathcal{H}) .$$

Proof. Let us take an arbitrary operator $S \in \mathfrak{A} \cap \mathfrak{Z}$, $S \neq 0$. Due to the theorem 1.3, item 2 and the proposition I.6, $S = \sum_{\alpha \in \sigma} S_\alpha P_\alpha$ and due to $S \neq 0$ there is $\alpha_0 \in \sigma$ such that $S_{\alpha_0} \neq 0$. Let us introduce restriction Π_{α_0} of the induction $\hat{\Pi}_{\alpha_0}$: R $\longrightarrow$ R_{α_0} to the algebra $\mathfrak{A}$. Π_{α_0} is a $*$ - representation of $\mathfrak{A}$ in $\mathcal{B}(\mathcal{H}_\alpha)$. Considering this representation on the element $T = (I - S_{\alpha_0}^{-1} S) \in \mathfrak{A}$ we obtain

$$\Pi_{\alpha_0}(T) = T P_{\alpha_0} = P_{\alpha_0} - S_{\alpha_0}^{-1} S P_{\alpha_0} = 0$$

i.e. $T \in \ker \Pi_{\alpha_0}$. However, $\mathfrak{A}$ is simple C^*-algebra and so $\ker \Pi_{\alpha_0} = 0$ what implies $T = 0$. Whence it follows that $S = S_{\alpha_0} I$, $S_{\alpha_0} \in C^1$ and $\mathfrak{A} \cap \mathfrak{Z} = \mathcal{C}(\mathcal{H})$.

In terms of our theory of superselection rules the result of the theorem 2.4 means that all superselection operators, and first of all, the projections P_α on coherent sectors, are purely global observables. For a long time this fact was assumed for physical reasons but had no general axiomatic proof

Corollary 2.5

Let quantum theory in $\mathcal{H} = \mathcal{H}_p$ satisfy the axioms I-α-VI and the region D be an "diamond", $D = D''$. Then

1) local algebras R(D) are factors,

2) $\mathfrak{Z}(D') \subset \mathfrak{Z}$,

3) sector observable algebras $R(D)_\alpha$ and $R(D')_\alpha$ are factors.

Proof. Statements of the point (3) follow from (I) and (2) respectively. To deduce the latter let us use the result by Kraus /II/, according to which the axioms I-α and IV imply $R(D \cup D') = R$ Whence it follows with the aid of the axiom II that $R(D)' \cap R(D')' = R'$. From here the axiom III gives immediately $\mathfrak{Z}(D) \subset \mathfrak{Z}$ and $\mathfrak{Z}(D') \subset \mathfrak{Z}$. The statement (2) is proved, and so is (1) if we take into account that

$$\mathfrak{Z}(D) \subset \mathfrak{A} \quad \text{and} \quad \mathfrak{A} \cap \mathfrak{Z} = \mathcal{C}(\mathcal{H}).$$

Corollary 2.6

Quantum theory of the $\mathcal{H} = \mathcal{H}_p$ class satisfying the axioms I-VI, satisfies also the postulate of extended locality /13/ i.e. for any two diamonds D_1 and D_2 space-like to each other

$$R(D_1) \cap R(D_2) = \mathcal{C}(\mathcal{H})$$

Proof. Due to the axiom III and corollary 2.5

$$R(D_1) \cap R(D_2) \subset R(D_1) \cap R(D_1)' = \mathfrak{Z}(D_1) = \mathcal{C}(\mathcal{H}).$$

Besides the extended locality for diamonds, the theory in $\mathcal{H} = \mathcal{H}_\rho$ satisfies the strict locality condition /II, 14, 15/.

Proposition 2.7

Let $\mathcal{H} = \mathcal{H}_\rho$ and the axioms I-VI be satisfied. If region $O \in B(M)$ is space-like to diamond D, $A \in R(O)$, $B \in R(D)$ and $A \neq 0$, $B \neq 0$, then $AB \neq 0$.

Proof consists in direct application of the following general

Lemma 2.8

Let R_1 be a factor in Hilbert space $\mathfrak{H}$ and $R_2 \subset \mathcal{B}(\mathfrak{H})$ be a W^*-algebra commuting with R_1: $R_2 \subset R_1'$. Then for all pairs $0 \neq A_k \in R_k$, k = 1,2 we have $A_1 A_2 \neq 0$.

Proof of the lemma. Let us assume that $A_1 A_2 = 0$. Since $A_1 \neq 0$ there exists $\Phi \in \mathfrak{H}$ such that $A_1 \Phi \neq 0$. Further, R_1 being a factor, we have $\mathfrak{Z}' = \mathcal{B}(\mathfrak{H})$. This gives us

$$\mathfrak{H} = \mathcal{H}^{\mathfrak{Z}'}_{A_1 \Phi} = \mathcal{H}^{R_1}_{A_1 \mathfrak{M}}, \quad \text{rge} \quad \mathfrak{M}' = \mathcal{H}^{R_1'}_{\Phi}$$

By the other side, taking into account that $A_2 \in R_2 \subset R_1'$ we obtain

$$A_2 \mathfrak{H} = A \mathcal{H}^{R_1'}_{A_1 \mathfrak{M}} = \mathcal{H}^{R_1}_{A_2 A_1 \mathfrak{M}} = 0$$

and this is in contradiction with $A_2 = 0$.

Proposition 2.9 (Borchers /12/)

Let all the axioms I-VI (except possibly IV) be satisfied and regions O_1 and O_2 be such that $O_1 \subset O$ and $O_1' \cap \theta \neq \emptyset$. Then every projection $P \in R(O_1)$ is infinite with respect to the algebra R(O).

Corollary 2.10

In the theory of the $\mathcal{H} = \mathcal{H}_\rho$ class for any region $O \subset M$ (bounded or not) full local algebras R(O) and sector local algebras R(O) as well as commutants $R(O)'$, $R(O)'_\alpha$ are infinite W^*-algebras.

The proof of this fact in the original works /16-18/ was based on the assumption that R(O) (or $R(O)_\alpha$) possess cyclic and separating vectors. According to § 1 results, in our theory this assumption can be not valid in general case.

Corollary 2.11

All coherent sectors $\mathcal{H}_\alpha$ are infinite-dimensional, i.e. in terms of § 1

$$\forall_{\alpha \in J} \; \aleph_\alpha \geqslant \aleph_0 .$$

Corollary 2.12

Let $\tilde{O}$ be an arbitrary region (possibly unbounded one) with $\tilde{O}' \neq \emptyset$. Then $R(\tilde{O}) \neq R$.

Proof. $\tilde{O}' \neq \emptyset$ implies the existence of a non-void open region $O_1 \subset \tilde{O}'$. For this region the assumption $R(\tilde{O}) = R$ leads to

$$R(O_1) \subset R(\tilde{O}') \subset R(\tilde{O})' = R' \subset R(O_1)'$$

i.e. $R(O_1)$ is abelian what contradicts to the corollary 2.10.

Development of the corollary 2.12 leads to the following

Proposition 2.13 (Wightman's inequality /16/)

Let $\mathcal{H} = \mathcal{H}_\rho$, the axioms I, II-$\alpha$, III, V be satisfied and $O_1 \subset O \subset M$. If the euclidean distance $d[\partial O_1, \partial O]$ between the boundaries ∂O_1 and ∂O of the regions O_1 and O is strictly positive then

$$\forall_{\alpha \in \sigma} \; R(O_1)_\alpha \neq R(O)_\alpha \quad \text{and} \quad R(O_1) \neq R(O)$$

Proof. Let us assume that $R(O_1)_\alpha = R(O)_\alpha$ for some $\alpha \in \sigma$. It follows from $d[\partial O_1, \partial O] > 0$ and proposition 2.1 that there exists a neighbourhood N of zero in M such that

$$\forall_{b \in N} \; R(O_1 + b)_\alpha = U_\alpha(b) R(O_1)_\alpha U_\alpha(-b) \subset R(O)_\alpha = R(O_1)_\alpha .$$

Since any vector $a \in M$ can be represented as sum of vectors from N, so

$$\forall_{a \in M} \; R(O_1 + a)_\alpha = R(O_1)_\alpha .$$

Hence $R_\alpha = \overline{\bigvee_{a \in M} R(O_1 + a)_\alpha}^{w} \subset R(O)_\alpha$ i.e. $R_\alpha = R(O_1)_\alpha$ in contradiction with the corollary 2.12.

Further information about the structure of $R(O)$, $\mathfrak{A}$, R in the theory of the $\mathcal{H} = \mathcal{H}_p$ class can be extracted from the analysis of the set of all analytic for the energy vectors.

Definition 2.2

Let P^0 be the generator of time translations (the energy operator). Vector $\psi \in \mathcal{H}$ is called an analytic for the energy vector if $\forall_{n=0,1,2,\dots} \; \psi \in \text{dom}\,(P^0)^n$ and the series $\sum_{n=0}^{\infty} \|(P^0)^n \psi\| \frac{z^n}{n!}$ has non-zero radius of convergence. The set of all analytic for the energy vectors will be denoted as $\mathfrak{M}$

Proposition 2.14

Let $\mathcal{H} = \mathcal{H}_p$ and the axiom V be satisfied. Then

a) the set $\mathfrak{M}$ is linear and dense in $\mathcal{H}$,

b) there is in every coherent sector $\mathcal{H}_\alpha$ a linear dense set $\mathfrak{M}_\alpha$ of vectors analytic for the energy and there are pure vectors in $\mathfrak{M}_\alpha$ so that $\mathfrak{M}_\alpha \cap \mathfrak{P} \neq \emptyset$

c) the set $\mathfrak{M} \cap \mathcal{H}_i$ is linear and dense in every subspace $\mathcal{H}_i \subset \mathcal{H}$ such that the projection $P_i \in R'$.

d) $\Psi \in \mathfrak{N}$ implies $R'\Psi \subset \mathfrak{N}$. If, besides this, Ψ is pure, i.e. $\Psi \in \mathfrak{N} \cap \mathfrak{P}$ then $\mathfrak{M}^1_\Psi \subset \mathfrak{N}$ and $\mathcal{H}^{R'}_\Psi \subset \mathfrak{N}$.

Proof. The operator P^0 being self-adjoint, (a) follows from the well-known Nelson's criterion of self-adjointness. Other assertions can be deduced in a straightforward way from the definition 2.2 and properties of cyclic subspaces $\mathcal{H}^R_\Psi$ and $\mathcal{H}^{R'}_\Psi$ given, for example, in /1/.

Proposition 2.15 (Reeh-Schlieder /19/, Borchers /3/).

Let the axioms II- α , V, VI be satisfied. Then for any vector Ψ analytic for the energy and for any region $\tilde{O} \subset M$ (unbounded in general case, but spatially incomplete)

$$\mathcal{H}^{R(\tilde{O})}_\Psi = \mathcal{H}^R_\Psi .$$

If in addition Ψ is separating for $\mathfrak{Z}$, then it is separating for all $R(\tilde{O})$ and all $R(\tilde{O})' \cap R$.

Proof. The first part of the statement is virtually proved by Borchers in /3/ and he proved also that when Ψ is separating for $\mathfrak{Z}$, Ψ is separating for all $R(O)$ and all $R(O)' \cap R$ with $O \in B(M)$. The extension of these results to unbounded regions can be made straightforward. Let $\tilde{O}$ be unbounded region with $\tilde{O}' \neq \emptyset$. Then there exist $O_1 \subset \tilde{O}'$, $O_1 \in B(M)$ and by the axiom III $R(O_1) \subset R(\tilde{O})'$ as well as $\overline{R(O_1) \vee R'}^w \subset R(\tilde{O})'$. This means that all vectors separating for $R(O_1)' \cap R$ are also separating for $R(\tilde{O})$. Finally, taking any $C_2 \subset \tilde{O}$, $O_2 \in B(M)$ we see that all vectors separating for $R(C_2)' \cap R$ are also separating for $R(\tilde{O})' \cap R$ and the proposition is proved.

Corollary 2.16

Every vector Ψ analytic for the energy and lying in the coherent sector $\mathcal{H}_\alpha$ is cyclic for $\overline{\overline{R(\tilde{O})_\alpha \setminus R'_\alpha}}^w$ and separating for $R(\tilde{O})_\alpha$, $\tilde{O} \subset M$ being any region with $\tilde{O}' \neq \emptyset$.

Proposition 2.17

Let $\mathcal{H} = \mathcal{H}_\rho$ and all the axioms I-VI be satisfied, except possibly IV. Then for any spatially incomplete region $\tilde{O} \subset M$ there are the following necessary and sufficient conditions of the existence of cyclic vectors for sector $R(O)_\alpha$ and full $R(O)$ local algebras respectively:

$$\exists_{\Psi_\alpha \in \mathcal{H}_\alpha} \mathcal{H}_{\Psi_\alpha}^{R(\tilde{O})_\alpha} = \mathcal{H}_\alpha \Longleftrightarrow \varkappa'_\alpha \leq \aleph_0 \qquad (2.3-\alpha)$$

$$\exists_{\Psi \in \mathcal{H}} \mathcal{H}_{\Psi}^{R(\tilde{O})} = \mathcal{H} \Longleftrightarrow \forall_{\alpha \in \sigma} \varkappa'_\alpha \leq \aleph_0, \text{ and } \sigma \leq \aleph_0 \qquad (2.3-\beta)$$

Proof. According to proposition 1.9 the following relation takes place:

$$\exists_{\Phi_\alpha \in \mathcal{H}_\alpha} \mathcal{H}_{\Phi_\alpha}^{R} = \mathcal{H}_\alpha \Longleftrightarrow \varkappa'_\alpha \leq \aleph_0 \qquad (2.4).$$

From here the formulas $(2.3-\alpha,\beta)$ will be deduced as follows.

a) If Ψ_α is cyclic for $R(\tilde{O})_\alpha$ then Ψ_α is cyclic for R_α and consequently $\varkappa'_\alpha \leq \aleph_0$.

b) Let now be $\varkappa'_\alpha \leq \aleph_0$ then there exists in virtue of (2.4) a vector Φ_α cyclic for R_α . In this case the vector $\Psi_\alpha = e^{-P^0}\Phi_\alpha$ is also cyclic for R_α and is in addition analytic for the energy. Hence it follows that due to proposition 2.15

$$\mathcal{H}_{\Psi_\alpha}^{R(\tilde{O})} = \mathcal{H}_{\Psi_\alpha}^{R} = \mathcal{H}_\alpha$$

i.e. Ψ'_α is cyclic for $R(\tilde{O})_\alpha$.

c) The condition (2.3-β) follows from (2.3-α) if we take into account that R-cyclic vector exists if and only if cardσ $\leq \aleph_0$ and R_α -cyclic vectors exist for all $\alpha \in \sigma$.

According to (1.10) every coherent sector can be represented in the form

$$\mathcal{H}_\alpha = \bigoplus_{\{\Phi_{\kappa,\alpha}\}} \mathcal{H}^R_{\Phi_{\kappa,\alpha}}, \qquad (2.5)$$

where $\{\Phi_{\kappa,\alpha}\}_{\kappa \in K'_\alpha}$ is an orthonormal basis in $\mathcal{H}^{R'}_{\Phi_\alpha}$ and $\Phi_\alpha \in \mathcal{S} \cap \mathcal{H}_\alpha$. This leads to the following corollary convenient for application.

Corollary 2.18

Under the assumptions of the preceding proposition, any vector $\Psi_\alpha \in \mathcal{H}_\alpha$ analytic for the energy is cyclic for $R(O)_\alpha$, if it possesses non-zero projections on each subspace $\mathcal{H}^R_{\Phi_{\kappa,\alpha}}$ from some decomposition of the form (2.5). The set of all such vectors will be denoted as $\mathcal{M}_{\alpha,\infty}$. The point (b) in the proof of the proposition 2.17 shows that this set is non-void.

The last cycle of properties we want to describe in this section concerns with the structure of the representation U of the translation group M in $\mathcal{H} = \mathcal{H}_\rho$.

Proposition 2.19

Let $\mathcal{H} = \mathcal{H}_\rho$ and all the axioms I-VI be satisfied, except possibly IV. Then

a) all translation operators U(a) belong to R,

b) spectrum of the representation U as well as spectrum of the restriction U_α of U to any coherent sector $\mathcal{H}_\alpha$ is unbounded.

c) translation operators $U(a)$ cannot represent superselection operators:

$$U(M)'' \cap \mathfrak{Z} = \mathbb{C}(\mathcal{H})$$

d) the set of all translationally invariant observables contains the centre $\mathfrak{Z}$ but does not coincide with it:

$$U(M)' \cap R \supsetneq \mathfrak{Z} \qquad (2.6)$$

e) quasilocal algebra $\mathfrak{A}$ does not contain non-trivial translationally invariant observables

$$U(M)' \cap \mathfrak{A} = \mathbb{C}(\mathcal{H})$$

what implies, in particular, that all spectral projections of the energy-momentum operator are purely global observables.

<u>Proof</u>.

a) is the well-known Borchers' theorem /20/,

b) According to the same work by Borchers /20/, the spectrum of translation group representation is unbounded if there exists a region $O \in B(M)$ and a neighbourhood N of the zero in M such that $\overline{\bigvee_{a \in N} R(O+a)}^{w} \neq R$. Remembering the corollary 2.12 we conclude that this inequality takes place for full as well as sectorial theories and for any region $O \in B(M)$ and any bounded neighbourhood of the zero in M.

c) If there exists a translation $a \neq 0$ such that $U(a) \in \mathfrak{Z}$ then all spectral projections of the representation U belong to $\mathfrak{Z}$, and this gives $U(M)'' \subset \mathfrak{Z}$. In this case the restriction U_α to any sector $\mathcal{H}_\alpha$ is a trivial representation with bounded spectrum, what is impossible according to the preceding point.

d) First part of this assertion is the well-known Araki theorem /21/ and follows in our case from (a). The inequality in (2.6) follows from (c) and (a) put together.

e) Let us prove firstly the corresponding result for the coherent sectors:

$$U(M)'_\alpha \cap \mathfrak{A}_\alpha = \mathcal{C}(\mathcal{H}_\alpha). \qquad (2.7)$$

Taking into account that all $\Psi_\alpha \in \mathcal{H}_\alpha$ are separating for $\mathfrak{Z}_\alpha = \mathcal{C}(\mathcal{H}_\alpha)$ and repeating Borchers' arguments in the proof of theorem I in /3/, it is easy to show that for any $A_\alpha \in \mathfrak{A}_\alpha$ there exists a sequence $\{a_n\}_{n=1}^{\infty}$ of space-like vectors $a_n \in M$ such that

$$U_\alpha(a_n) A_\alpha U_\alpha(-a_n) \xrightarrow{w} z_\alpha(A_\alpha) I_\alpha,$$

where $z_\alpha(A_\alpha)$ is complex number depending on A_α in general case. Hence, if $B_\alpha \in \mathfrak{A}_\alpha \cap U(M)'_\alpha$ then $z_\alpha(B_\alpha) I_\alpha = \text{w-lim}\, U_\alpha(a_n) B_\alpha U_\alpha(-a_n) = B_\alpha$ what means exactly (2.7). Now let us take an arbitrary $A \in U(M)' \cap \mathfrak{A}$. In virtue of (2.7), $A = \sum_\alpha z_\alpha P_\alpha$ and for the algebra of the $\mathcal{H} = \mathcal{H}_p$ class this is equivalent to $A \in \mathfrak{Z}$. In other words, $A \in U(M)' \cap \mathfrak{A}$ implies $A \in \mathfrak{A} \cap \mathfrak{Z}$ and this is equal to $\mathcal{C}(\mathcal{H})$ due to theorem 2.4.

Now let us consider the set $\mathcal{V}$ of all translationally invariant vectors in $\mathcal{H}$:

$$\mathcal{V} \doteq \{ \Omega \in \mathcal{H} \mid \forall_{a \in M}\ U(a)\Omega = \Omega \}$$

$\mathcal{V}$ being a subspace, let us denote the corresponding projection as $P_{\mathcal{V}}$. It is clear that $P_{\mathcal{V}} \in U(M)'' \subset R$. The structure of the subspace $\mathcal{V}$ appears to be governed by the property of asymptotical abelianness of $\mathfrak{A}$.

Proposition 2.20

Let $\mathcal{H} = \mathcal{H}_p$ and the axioms I-III, V, VI be satisfied. Then

a) Quasilocal algebra $\mathfrak{A}$ is asymptotically abelian with respect to the representation U of the translation group, in the Störmer's sense /22/: for every quasilocal observable $A \in \mathfrak{A}$ there exists a sequence $\{a_n(A)\}_{n=1}^{\infty}$ of translations $a_n(A) \in$ M depending on A in general case and such that

$$\forall_{B \in \mathfrak{A}} \lim_{n \to \infty} \| [U(a_n(A)) A U(-a_n(A)), B] \| = 0. \quad (2.8)$$

b) All translationally invariant vectors Ω, belonging to coherent sector $\mathcal{H}_\beta$ are pure:

$$\forall_{\beta \in \sigma} \mathcal{V}_\beta \doteq \mathcal{V} \cap \mathcal{H}_\beta \subset \mathcal{P}. \quad (2.9)$$

c) For every non-zero $\Omega_\beta \in \mathcal{V}_\beta$ the projection E_β on the subspace $\mathcal{H}^R_{\Omega_\beta} \cap \mathcal{V}$ consisting of all translationally invariant vectors in the cyclic subspace $\mathcal{H}^R_\Omega$ is one-dimensional

$$E_\beta \doteq P^R_{\Omega_\beta} P_{\mathcal{V}} = P[\Omega_\beta], \quad (2.10)$$

where $P[\Omega_\beta]$ is the projection on subspace spanned by the vector Ω_β.

Proof. Formula (2.6) can be deduced with the aid of the fact that A and B from $\mathfrak{A}$ can be uniformly approximated with arbitrary exactness by elements from local algebras $R(O_A)$ and $R(O_B)$ respectively. Since O_A and O_B are chosen, the sequence $\{a_n(A)\}_{n=1}^{\infty}$ should be chosen in such a way that the region $O + A_n(A)$ becomes space-like to O_B in the limit $n \to \infty$. Further, properties (2.9) and (2.10) follow from general theory of asymptotically abelian C^*-algebras (see, for instance, the theorem /5.2/ of Störmer's work).

Now let us denote as $\sigma_{\mathcal{V}}$ the set of all coherent sectors $\mathcal{H}_\beta$ containing at least one translationally invariant vector:

$$\sigma_{\mathcal{V}} \doteq \{ \beta \in \sigma \mid \mathcal{H}_\beta \cap \mathcal{V} \neq 0 \} . \qquad (2.11)$$

Projection on the set $\mathcal{V}_\beta$ defined in the formula (2.9) will be denoted as $P_{\mathcal{V}_\beta}$. It is easy to see that

$$P_{\mathcal{V}_\beta} = P^{U(M)'}_{\Omega_\beta} \qquad \text{for any } \Omega_\beta \in \mathcal{V}_\beta . \qquad (2.12)$$

Theorem 2.21 (Uniqueness of the vacuum state in the coherent sector).

The set of all normed translationally invariant vectors belonging to a given sector $\mathcal{H}_\beta$, $\beta \in \sigma_{\mathcal{V}}$ coincides with the H-image of unique pure vector state $\omega_{\Omega_\beta} \in PV(R)$:

$$\forall_{\beta \in \sigma_{\mathcal{V}}} \quad \mathcal{V}_\beta \cap S^1 = \mathcal{M}^1_{\Omega_\beta} \qquad (2.13)$$

Ω_β being any normed vector from $\mathcal{V}_\beta$. The formula (2.13) is equivalent to

$$P_{\mathcal{V}_\beta} = P^{U(M)'}_{\Omega_\beta} = P^{R'}_{\Omega_\beta} . \qquad (2.14)$$

The set $\bigcup_{\beta \in \sigma_{\mathcal{V}}} \mathcal{V}_\beta$ of all pure translationally invariant vectors is dense in the set $\mathcal{V}$ so that

$$\mathcal{V} = \bigoplus_{\beta \in \sigma_{\mathcal{V}}} \mathcal{V}_\beta . \qquad (2.15)$$

Proof. The formula (2.13) will be proved on the basis of the points b) and c) in proposition 2.20. According to c), if the subspace $\mathcal{H}^R_{\Phi_\beta}$ with $\Phi_\beta \in \mathcal{H}_\beta$ contains a normed translationally invariant ray Ω_β, then such a ray is unique in $\mathcal{H}^R_{\Phi_\beta}$, and in addition such $\mathcal{H}^R_{\Phi_\beta}$ is irreducible due to b). Further, owing to the formula (2.12), the following simple relation takes place:

$$\Omega_\beta \in V_\beta \Rightarrow \mathcal{H}^{R'}_{\Omega_\beta} \subset V_\beta . \qquad (2.16)$$

Now let us assume that there can be found in the sector two different translationally invariant states, say, ω_{Ω_1} and ω_{Ω_2}. Then irreducible subspaces $\mathcal{H}^{R'}_{\Omega_1}$ and $\mathcal{H}^{R'}_{\Omega_2}$ of R′ do not intersect with each other and, on the contrary, they intersect non-trivially with every irreducible subspace $\mathcal{H}^R_\Phi$ of R (see the corollary to the proposition 13 of /1/). Hence it follows together with (2.16) that, for instance, $\mathcal{H}^R_{\Omega_1}$ contains two different normed translationally invariant rays. This contradicts c) in proposition 2.20 and so (2.13) is proved. The equivalence of (2.13) and (2.14) is obvious in the light of the properties (I.2) and (I.3) of H-image.

Finally, the formula (2.15) follows from the definition of σ_ν, the mutual orthogonality of coherent sectors and the relationship

$$P_\nu = \Big(\sum_{\alpha\in\sigma} P_\alpha\Big) P_\nu = \sum_{\beta\in\sigma_\nu} P_\beta P_\nu = \sum_{\beta\in\sigma_\nu} P_{\nu_\beta} .$$

<u>Corollary 2.22</u>

The following enhancement of the point b) in proposition 2.20 takes place: every irreducible subspace $\mathcal{H}^R_\Phi$ in the sector $\mathcal{H}_\beta$, $\beta\in\sigma_\nu$ contains one, and only one normed translationally invariant vector (a vacuum vector, in the usual terminology).

Proof follows straightforward from the theorem 2.21 and corollary to proposition 13 of /I/.

Results of the statements 2.20, 2.21 and 2.22 provide us with complete description of the "vacuum structure" of an arbitrary theory of the $\mathcal{H} = \mathcal{H}_p$ class and make it possible to analyze and compare different possible forms of the postulate of the existence and uniqueness of vacuum.

1) In general case (no restrictions on vacuum structure) theory possesses arbitrary set σ_v of vacuum coherent sectors $\mathcal{H}_\beta$, each of them containing the unique and pure vacuum state with the H-image of the arbitrary dimension. Besides this, there are also mixed vacuum states ω_Ω, $\Omega = \sum_{\gamma \in \sigma_v} P_\gamma$, $P_\gamma \in \mathcal{V}_\gamma$.

2) The weakest possible restriction on vacuum structure is the condition of the uniqueness of vacuum sector. According to theorem 2.21, this condition is completely equivalent to (a priori) much stronger one: there exists a unique vacuum state (still with the arbitrary H-image).

3) The strongest (and also the most wide-spread) form of the "vacuum postulate" is the requirement of the existence of a unique vacuum vector. This simplest vacuum structure can be described by the following elementary

Corollary 2.23

Let $\mathcal{H} = \mathcal{H}_p$. Then the following conditions are equivalent:

1) there exists in $\mathcal{H}$ a unique vacuum vector,

2) the vacuum sector is unique and abelian

3) the vacuum sector is unique and contains a cyclic (for $R_\mathcal{J}$, of course) vacuum vector.

Proof can be performed easily by any reader.

3. FIELD-LIKE PROPERTIES OF QUANTUM THEORIES IN $\mathcal{H} = \mathcal{H}_\rho$: BOUNDED REGIONS

Now we have demonstrated that our scheme in $\mathcal{H} = \mathcal{H}_\rho$ possesses practically all properties which can be required from a well-developed axiomatic theory of local observables. After this we intend to show that due to its specific global structure the scheme possesses also a wide complex of other properties which are characteristic for field theories. However, the real existence of a field appears to be ensured only under still another necessary constraints, besides our starting conditon $\mathcal{H} = \mathcal{H}_\rho$.

It is known for a long time /3-5/ that field properties of algebraic theory are connected with the existence of operators mapping states and observables from one coherent sector into another. A natural way to constructing such operators is to make of our coherent sectors representations of some C^*-algebra and then to establish equivalence properties of these representations[x)]. In other words, as a preliminary we have to reformulate our theory in form of Haag-Kastler's abstract algebraic approach. Doing this we must take into account that fundamental algebra of abstract algebraic theory, for which all "concrete" or "coherent" physical theories are its representations, is by its physical meaning the algebra of quasilocal but not global observables. This circumstance was firstly pointed out by Haag and Kastler /23/ on the basis of quantum measurement theory argu-

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x) Of course, it is possible in our scheme to study the relations between coherent sectors by means of the theory of W^*-algebras, without introducing representations of C^*-algebra. However such a way is less effective and hinders the comparison with known results.

ments. Practically, if we have an abstract C^*-algebra $\mathfrak{A}$ and some physical representation π of $\mathfrak{A}$, the image $\pi(\mathfrak{A}) \subset \mathcal{B}(\mathcal{H})$ is not a W^*-algebra in general case, so it does not include global observables and the latter are added only by the weak closure operation: $\pi(\mathfrak{A}) \to \overline{\pi(\mathfrak{A})}^{w}$. Due to this, performing an abstract reformulation of our theory we should consider not the global, but the quasilocal algebra $\mathfrak{A}$ as a fundamental C^*-algebra, representations of which are coherent sectors. Then the images of these representations should coincide obviously with quasilocal sector algebras $\mathfrak{A}_\alpha$ and R will play the part of an enveloping W^*-algebra of C^*-algebra $\mathfrak{A}$.

Thus we shall consider the sectorial structures as representations of quasilocal algebra

$$\pi_\alpha : \quad \mathfrak{A} \to \mathcal{B}(\mathcal{H}_\alpha)$$

and we can define canonical extensions of these representations to representations of global algebra R:

$$\hat{\pi}_\alpha : \quad R \to \mathcal{B}(\mathcal{H}_\alpha)$$

as well as local restrictions: $\hat{\pi}_\alpha^{(O)} \doteq \hat{\pi}_\alpha\big|_{R(O)}$ for bounded regions $O \in B(M)$ and $\pi_\alpha^{(\tilde{O})} \doteq \pi_\alpha\big|_{\mathfrak{A}(O)}$ for unbounded regions O. All representations introduced we shall set in a completely explicit form by defining the mappings $\mathfrak{A} \to \mathcal{B}(\mathcal{H}_\alpha)$ as follows

$$\forall_{A \in \mathfrak{A}} \quad \Pi_\alpha : \quad A \rightarrow A_{P_\alpha} \tag{3.1}$$

what implies

$$\forall_{A \in R} \quad \hat{\Pi}_\alpha : \quad A \rightarrow A_{P_\alpha} . \tag{3.2}$$

In other words, we realize the representations $\hat{\Pi}_\alpha$ as the inductions (because of $P_\alpha \in R'$) of W^*-algebra R. In this point our scheme allows a certain arbitrariness, which belongs to its specific distinctions from the usual Haag-Kastler formalism. In fact, by giving a coherent sector we determine only the space $\mathcal{H}_\alpha$ of the representation Π_α and the image $\Pi_\alpha(\mathfrak{A}) = \mathfrak{A}_\alpha$ of the latter; but after this the element-wise action of the $*$ - homomorphism

$$A \in \mathfrak{A} \longrightarrow \Pi_\alpha(A) \in \mathfrak{A}_\alpha \tag{3.3}$$

still can be very diverse and not at all coinciding with (3.1). Nevertheless, this arbitrariness has no essential influence on equivalence properties of Π_α, $\Pi_\alpha^{(0)}$, $\Pi_\alpha^{(\tilde{c})}$ (we are only interested in). As can be seen, for example, from well-known criterions of quasuequivalence and unitary equivalence of representations

$$\Pi_\alpha \approx \Pi_\beta \iff \mathfrak{Z}\text{-supp}\, P_\alpha = \mathfrak{Z}\text{-supp}\, P_\beta\, ; \quad \Pi_\alpha \simeq \Pi_\beta \iff V(\Pi_\alpha) = V(\Pi_\beta)$$

($V(\Pi) \subset \mathfrak{A}^{*+}$ being the set of all vector functionals in the representation Π) the relations $\approx$ and $\simeq$ are completely determined by spaces $\mathcal{H}_\alpha, \mathcal{H}_\beta$ and algebras $\Pi_\alpha(\mathfrak{A})$, $\Pi_\beta(\mathfrak{A})$ and the same is valid, of course, for any restriction $\Pi_\alpha|_{\mathfrak{A}_1}$, $\mathfrak{A}_1 \subset \mathfrak{A}$. Only weak equivalence, $\Pi_\alpha \sim \Pi_\beta$ depends on the element-wise correspondence (3.3), according to the criterion $\ker \Pi_\alpha = \ker \Pi_\beta$. However in this case it is obvious that such a choice of the correspondence (3.3) will be physically preferable, which ensures the fulfillment of weak equivalence and we shall show that this is just the case for the correspondence (3.I). We can now summarize that the choice of the correspondence (3.3) makes no difference for equivalence relations $\approx$ and $\simeq$, and for the relation $\sim$ the choice of this correspondence in the form (3.I) is preferable. So it is this choice that will be accepted by us from now on.

Now when the reformulation of our scheme in form of the family of C^*-algebra representations is completely stated, let us give

<u>Definition 3.I</u>

Coherent superselection sectors $\mathcal{H}_\alpha$ and $\mathcal{H}_\beta$ characterized by representations π_α and π_β of quasilocal algebra $\mathfrak{A}$ will be called:

1) physically equivalent, if $\pi_\alpha \sim \pi_\beta$

2,3) locally (asymptotically) physically equivalent, if $\pi_\alpha^{(0)} \sim \pi_\beta^{(0)}$ (resp., $\pi_\alpha^{(0')} \sim \pi_\beta^{(0')}$) for all $0 \in B(M)$,

4) quasiequivalent, if $\pi_\alpha \approx \pi_\beta$

5,6) locally (asymptotically) quasiequivalent, if $\pi_\alpha^{(0)} \approx \pi_\beta^{(0)}$ (resp., $\pi_\alpha^{(0')} \approx \pi_\beta^{(0')}$) for all $0 \in B(M)$,

7) unitarily equivalent, if $\pi_\alpha \simeq \pi_\beta$

8,9) locally (asymptotically) unitarily equivalent, if $\pi_\alpha^{(0)} \simeq \pi_\beta^{(0)}$ (resp. $\pi_\alpha^{(0')} \simeq \pi_\beta^{(0')}$) for all $0 \in B(M)$.

It is clear that any equivalence property of representations π_α, π_β implies the same property of their restrictions $\pi_\gamma \big|_{\mathfrak{A}_1}$ for any $\mathfrak{A}_1 \subset \mathfrak{A}$ and analogously any equivalence relation between $\pi_\gamma^{(0')}$ implies the same relation between $\Pi_\gamma^{(0)}$ because always $O \subset O_1'$ for some $O_1 \in B(M)$. As a consequence, the following implications take place: I $\Rightarrow$ 3 $\Rightarrow$ 2, 4 $\Rightarrow$ 6 $\Rightarrow$ 5, and 7 $\Rightarrow$ 9 $\Rightarrow$ 8. By the other side, the usual relations between equivalence properties ($\simeq \Rightarrow \approx \Rightarrow \sim$) give us 7 $\Rightarrow$ 4 $\Rightarrow$ I, 8 $\Rightarrow$ 5 $\Rightarrow$ 2 and 9 $\Rightarrow$ 6 $\Rightarrow$ 3.

In this section we give complete description of equivalence properties of the representations Π_α and $\Pi_\alpha^{(0)}$.

Proposition 3.I

Let quantum theory of the $\mathcal{H}=\mathcal{H}_p$ class be given, satisfying the axioms. Then all coherent sectors are:

1) physically equivalent

2) locally quasiequivalent

3) disjoint

Proof. 1) Physical (i.e. weak) equivalence of the sectors $\mathcal{H}_\alpha$ and $\mathcal{H}_\beta$ means that some arbitrary isomorphism of $\mathfrak{A}_\alpha$ and $\mathfrak{A}_\beta$ exists. The existence of the isomorphism follows directly from the simplicity of $\mathfrak{A}$. $\mathfrak{A}$ being simple, all the representations Π_γ, $\gamma \in \sigma$ are faithful and so the isomorphisms exist: $\Pi_\gamma : \mathfrak{A} \to \mathfrak{A}_\gamma$, $\gamma \in \sigma$ as well as the inverse isomorphisms Π_γ^{-1}. It is obvious that the composition $\Pi_{\alpha\beta} \doteq \Pi_\alpha \circ \Pi_\beta^{-1}$ of the mappings Π_α, Π_β exists and represents the desired isomorphism of $\mathfrak{A}_\alpha$ and $\mathfrak{A}_\beta$. It is also clear that $\Pi_{\alpha\beta}$ cannot be extended to isomorphism of corresponding weak closures $R_\alpha = \overline{\mathfrak{A}_\alpha}^w$ and $R_\beta = \overline{\mathfrak{A}_\beta}^w$ because the extensions $\hat{\Pi}_\gamma : R \to R_\gamma$ are not faithful representations (due to $\ker \Pi_\gamma \ni P_\alpha$ for all $\alpha \in \sigma$, $\alpha \neq \gamma$).

2) However, local restrictions $\Pi_\alpha^{(0)}$ of representations are faithful representations of R(O)'s and generate

an isomorphism $T_{\alpha\beta}^{(0)} = \Pi_{\alpha}^{(0)} \circ (\Pi_{\beta}^{(0)})^{-1}$ which is at the same time the isomorphism of algebras $\Pi_{\alpha}^{(0)}(R(0))$ and $\Pi_{\beta}^{(0)}(R(0))$ as well as their weak closures ($\Pi_{\gamma}^{(0)}(R(0)) = R(0)_{\gamma}$ being W^*-algebras). Further, from the definition of $\Pi_{\alpha\beta}^{(0)}$ we have:

$$\forall_{A \in R(0)} \quad \Pi_{\alpha}^{(0)}(A) = \Pi_{\alpha\beta}^{(0)}\left(\Pi_{\beta}^{(0)}(A)\right) \quad (3.4)$$

so that $\Pi_{\alpha}^{(0)}$ and $\Pi_{\beta}^{(0)}$ are quasiequivalent.

3) Now we shall establish the disjointness of Π_{α} and Π_{β} by proving orthogonality of corresponding central supports. Π_{α} and Π_{β} being subrepresentations of the identical representation of $\mathfrak{A}$, their central supports coincide, by definition, with those of the projections P_{α} and P_{β} in W^*-algebra $\overline{\mathfrak{A}}^{w}$ $= R'$. Due to P_{α}, $P_{\beta} \in \mathfrak{Z}$ these central supports coincide with P_{α} and P_{β} and are orthogonal.

The property of local quasiequivalence, proved in this proposition, is closely connected with a number of other structural properties of local algebras. In order to describe these connections we shall prove

<u>Proposition 3.2</u>

Let quantum theory of the $\mathcal{H} = \mathcal{H}_{p}$ class be given and let $\mathcal{M}_{\alpha}$ be the set of all analytic for the energy vectors from arbitrary sector $\mathcal{H}_{\alpha}$. Then the following conditions are equivalent

1) all sectors are locally quasiequivalent,
2) inductions $R(0) \longrightarrow R(0)_{P_{\alpha}}$ are isomorphisms,
3) all $\Psi_{\alpha} \in \mathcal{M}_{\alpha}$ are separating for $R(0)$,

4) there exists $\Psi_\alpha \in \mathcal{H}_\alpha$ separating for R(0),

5) all $\Psi_\alpha \in \mathcal{H}_\alpha$ are separating for $\mathfrak{Z}(0) \doteq R(0) \cap R(0)'$

6) there exists $\Psi_\alpha \in \mathcal{H}_\alpha$ separating for $\mathfrak{Z}(0)$.

Proof. There are the following obvious implications between the conditions (I - 6); $2 \Rightarrow 1$, $3 \Rightarrow 4$, $5 \Rightarrow 6$, $3 \Rightarrow 5$, $4 \Rightarrow 6$ It is easy to verify that it is sufficient now to deduce the relations $I \Rightarrow 2$, $2 \Rightarrow 3$ and $6 \Rightarrow 2$.

$I \Rightarrow 2$. We whall show that the central support of $\Pi_\alpha^{(0)}$ as a subrepresentation of the identical representation of R(0), equal to the central support of P_α in algebra R(0)′, is equal to I. According to general formula, $\mathfrak{Z}(0)$-supp P_α = $= P^{\mathfrak{Z}(0)'}_{\mathcal{H}_\alpha}$ and using /DC-5.3.I/ we obtain

$$\Pi_\alpha^{(0)} \approx \Pi_\beta^{(0)} \iff P^{\mathfrak{Z}(0)'}_{\mathcal{H}_\alpha} = P^{\mathfrak{Z}(0)'}_{\mathcal{H}_\beta}.$$

Let here α be fixed and β run over all the set $\mathfrak{S}$. Then taking into account $I \in R(0)$ we have

$$\mathcal{H}_\beta \subset \mathcal{H}^{\mathfrak{Z}(0)'}_{\mathcal{H}_\beta} = \mathcal{H}^{\mathfrak{Z}(0)'}_{\mathcal{H}_\alpha}$$

whence it follows

$$\mathcal{H} = \bigoplus_{\beta \in \mathfrak{S}} \mathcal{H}_\beta \subset \mathcal{H}^{\mathfrak{Z}(0)'}_{\mathcal{H}_\alpha}.$$

This means that $P^{\mathfrak{Z}(0)'}_{\mathcal{H}_\alpha} = I$ and the property 2) is fulfilled.

$2 \Rightarrow 3$. By the Reeh-Sohlieder theorem (proposition 2.15) all $\Psi_\alpha \in \mathcal{H}_\alpha$ are separating for $R(0)_\alpha$. From here we shall deduce with the aid of the condition 2) that all such vectors are separating for R(0). For all $\Phi_\alpha \in \mathcal{H}_\alpha$ and $0 \neq A \in R(0)$ we have $A_{P_\alpha}\Phi_\alpha = A\Phi_\alpha$. If $R(0) \to R(0)_\alpha$ is an isomorphism, then $A = 0$ implies $A_{P_\alpha} = 0$. Thus if Ψ_α is separating for $R(0)_\alpha$ i.e.

$$A_{P_\alpha}\Psi_0 = 0 \Rightarrow A_{P_\alpha} = 0$$

then

$$\forall_{A\in R(0)} \; A\Psi_\alpha = 0 \Rightarrow A_{P_\alpha}\Psi_\alpha = 0 \Rightarrow A_{P_\alpha} = 0 \Rightarrow A = 0$$

q.e.d.

6 ⇒ 2. Let us use the standart expression for central support of representation $\pi_\alpha^{(0)}$: $\mathfrak{Z}(0)$=supp $P_\alpha = P^{R(0)'}_{\mathcal{H}_\alpha}$. We have $P_\alpha = P^{R(0)}_{\mathcal{H}_\alpha}$ so $P^{R(0)'}_{\mathcal{H}_\alpha} = P^{\mathfrak{Z}(0)'}_{\mathcal{H}_\alpha}$ and

$$\mathfrak{Z}(0)\text{-supp } P_\alpha = I \iff \overline{L\{\mathfrak{Z}(0)'\mathcal{H}_\alpha\}} = \mathcal{H}.$$

Now it is clear that if there is a cyclic vector for $\mathfrak{Z}(0)'$ in $\mathcal{H}_\alpha$ (condition 6), then $\mathfrak{Z}(0)$-supp P_α =I, whence the condition 2) follows.

As it follows from the conditions 3,4 due to proposition 3.2 everyone from the conditions (1 - 6) implies the following property: R(0) is countably decomposable. Vice versa, the commutant R(0)′ being an infinite W^*-algebra, countable decomposability of R(0) implies the existence of vector separating for it, according to the criterion/(DW-233)/. However we cannot guarantee now that the set of R(0)-separating vectors includes all analytic for the energy vectors from all $\mathcal{H}_\alpha$ or, at least, one such vector from each $\mathcal{H}_\alpha$. As a result, the countable decomposability of R(0) is the necessary but not sufficient condition of the properties (I - 6).

Returning to proposition 3.1 and definition 3.1, we see that our theory possesses the equivalence properties 1, 2, 3, 5 and can never possess the properties 4,7. Now to exhaust the problem of describing equivalence properties related to representations $\Pi_\alpha^{(0)}$ and $\Pi_\beta^{(0)}$ we have only to consider the property 8, local unitary equivalence of coherent sectors. This task is completely fulfilled by the following

Theorem 3.3.

Let quantum theory of the $\mathcal{H}=\mathcal{H}_p$ class be given and let $\mathcal{H}_\alpha$ and $\mathcal{H}_\beta$ be coherent sectors. Then:

I. Let $\varkappa'_\alpha \leq \aleph_0$. Then $\Pi_\alpha^{(0)} \simeq \Pi_\beta^{(0)}$ if and only if $\varkappa'_\beta \leq \aleph_0$ and $\varkappa_\alpha = \varkappa_\beta$.

II. Let $\varkappa'_\alpha > \aleph_0$. Then $\Pi_\alpha^{(0)} \simeq \Pi_\beta^{(0)}$ if and only if $\varkappa'_\alpha = \varkappa'_\beta$ and $\dim \mathcal{H}_\alpha = \dim \mathcal{H}_\beta$ *)

Proof. Throughout all the proof we shall take into account that according to proposition 3.I.

$$\underset{\alpha,\beta\in\sigma}{\forall} \quad \underset{O\in B(M)}{\forall} \quad \Pi_\alpha^{(0)} \approx \Pi_\beta^{(0)}. \qquad (3.5)$$

The proof consists in the deduction of necessary and sufficient conditions under which the isomorphism corresponding to (3.5) is spatial.

I - sufficiency. According to proposition 2.17, $\varkappa'_\gamma \leq \aleph_0$ is the necessary and sufficient condition, under which local algebras $R(O)_\gamma$ of arbitrary sector $\mathcal{H}_\gamma$ possess cyclic vectors. By the other side, there exists always separating vectors for $R(O)_\gamma$, $\gamma\in\sigma$. As a result, conditions $\varkappa'_\alpha \leq \aleph_0$ and $\varkappa'_\beta \leq \aleph_0$ imply that algebras $R(O)_\alpha$ and $R(O)_\beta$ both possess cyclic and separating vectors. Due to the well-known criterion /DW-233/, in this case any isomorphism of the algebras $R(O)_\alpha$ and $R(O)_\beta$ is spatial.

*) The last condition also can easily be written in terms of parameters $\varkappa_\alpha, \varkappa_\beta$ and $\varkappa'_\alpha, \varkappa'_\beta$ with the aid of the formula: $\varkappa_\alpha, \varkappa'_\alpha \geq \aleph_0 \implies \dim\mathcal{H}_\alpha = \max\{\varkappa_\alpha, \varkappa'_\alpha\}$. However such a form is less convenient because it depends on the relationship between $\varkappa_\alpha$ and $\varkappa'_\alpha$, $\varkappa_\beta$ and $\varkappa'_\beta$.

I - necessity. Let us prove firstly that the conditions $\varkappa'_\alpha \le \aleph_0$ and $\Pi_\alpha^{(0)} \simeq \Pi_\beta^{(0)}$ imply $\varkappa'_\beta \le \aleph_0$. To this end we shall show that if $R(0)_\alpha$ has cyclic vectors (what is the case according to proposition 2.17) and $\Pi_\alpha^{(0)} \simeq \Pi_\beta^{(0)}$ then $R(0)_\beta$ has cyclic vectors too. Let $V_{\beta\alpha}(0)$ be isometric operator from $\mathcal{H}_\alpha$ on $\mathcal{H}_\beta$ implementing unitary equivalence of algebras $R(0)_\alpha$ and $R(0)_\beta$

$$V_{\beta\alpha}(0)\,\mathcal{H}_\alpha = \mathcal{H}_\beta\,;\quad V_{\beta\alpha}(0)\,R(0)_\alpha\,V_{\beta\alpha}(0)^{-1} = R(0)_\beta\,. \tag{3.6}$$

Due to /DC-5.I.2/, the operator $V_{\beta\alpha}(0)$ as intertwining operator of subrepresentations of algebra R(0), has the following property:

$$V_{\beta\alpha}(0)\,P_\alpha \in R(0)'. \tag{3.7}$$

From here we obtain the useful relationship:

$$\forall_{\varphi_\alpha \in \mathcal{H}_\alpha} \quad V_{\beta\alpha}(0)\,\mathcal{H}^{R(0)}_{\varphi_\alpha} = \mathcal{H}^{R(0)}_{V_{\beta\alpha}(0)\varphi_\alpha}\,. \tag{3.8}$$

In fact, using (3.7), property $P_\alpha \in R(0)'$ and continuity of $V_{\beta\alpha}(0)$ one obtains (3.8):

$$V_{\beta\alpha}(0)\,\overline{R(0)\varphi_\alpha} = \overline{V_{\beta\alpha}(0)\,R(0)\varphi_\alpha} = \overline{R(0)\,V_{\beta\alpha}(0)\,\varphi_\alpha}\,,$$

Now we see from (3.8) and (3.6) that $\mathcal{H}^{R(0)}_{\varphi_\alpha} = \mathcal{H}_\alpha$ implies $\mathcal{H}^{R(0)}_{\varphi_\beta} = \mathcal{H}_\beta$ with $\varphi_\beta = V_{\beta\alpha}(0)\,\varphi_\alpha$, i.e. the mapping $V_{\beta\alpha}(0)$ transforms cyclic vectors of $R(0)_\alpha$ into cyclic vectors of $R(0)_\beta$. The using of proposition 2.17 for sector $\mathcal{H}_\beta$ gives then $\varkappa'_\beta \le \aleph_0$.

We remark further that $\Pi_\alpha^{(0)} \simeq \Pi_\beta^{(0)}$ implies obviously $\dim \mathcal{H}_\alpha = \dim \mathcal{H}_\beta$. According to § 1, $\forall_{\delta \in \sigma}$ $\dim \mathcal{H}_\delta = \varkappa_\delta \cdot \varkappa'_\delta$

and according to corollary 2.II it follows from the axioms I-III, V, VI that $\varkappa_\alpha \ge \aleph_0$, $\varkappa_\beta \ge \aleph_0$. If in addition $\varkappa'_\alpha, \varkappa'_\beta \le \aleph_0$ then the equality $\dim \mathcal{H}_\alpha = \dim \mathcal{H}_\beta$ provides us with the last condition we need: $\varkappa_\alpha = \varkappa_\beta$.

II - sufficiency. We shall prove that under the condition $\mathcal{X}_\alpha' = \mathcal{X}_\beta'$ the algebras $R(0)_\alpha$ and $R(0)_\beta$ satisfy the following criterion /DW-321/:

Let R_1 and R_2 be von Neumann algebras. Let us suppose that there exists in R_1' (resp., R_2') an infinite family $\{E_i\}_{i\in J}$ (resp., $\{F_i\}_{i\in J}$) of projections, which are

(D) mutually equivalent, orthogonal, with the sum equal to I and such that R'_{1E_i} (resp., R'_{2F_i}) are countably decomposable. Then every isomorphism Π of algebra R_1 on R_2 is spatial.

R_α being type I factor for all $\alpha\in\sigma$, the space $\mathcal{H}_\alpha$ can be represented in the form $\mathcal{H}_\alpha = \oplus_\kappa \mathcal{H}_{\kappa,\alpha}$ where $\mathcal{H}_{\kappa,\alpha}$ are the spaces on which mutually unitarily equivalent irreducible representations $\Pi_{\kappa,\alpha}$ of the algebra R are acting. Due to $P_{\kappa,\alpha}\in R'$ and proposition 2.14, every $\mathcal{H}_{\kappa,\alpha}$ contains a linear dense set of analytical for the energy vectors. For all such vectors $\Psi_{\kappa,\alpha}$ using irreducibility of $\mathcal{H}_{\kappa,\alpha}$ with respect to R, Reeh-Schlieder theorem and the property $P_{\kappa,\alpha}\in R(0)'$ we obtain

$$\overline{R(0)_{P_{\kappa,\alpha}}\Psi_{\kappa,\alpha}} = \overline{R(0)_\alpha\Psi_{\kappa,\alpha}} = \overline{R_\alpha\Psi_{\kappa,\alpha}} = \mathcal{H}_{\kappa,\alpha}.$$

This means that all $\Psi_{\kappa,\alpha}$ are cyclic vectors for $R(0)_{P_{\kappa,\alpha}}$ and separating for $R(0)'_{P_{\kappa,\alpha}}$. As a result, $R(0)'_{P_{\kappa,\alpha}}$ is countably decomposable in virtue of /DW-6/.

Further, unitary equivalence of $\Pi_{\kappa_1,\alpha}$ and $\Pi_{\kappa_2,\alpha}$ implies according to /DC-5.I.3/ that corresponding projections $P_{\kappa_1,\alpha}$ and $P_{\kappa_2,\alpha}$ are equivalent with respect to algebra R_α'

$$\forall_{\alpha\in\sigma}\ \forall_{\kappa_1,\kappa_2\in\mathcal{K}_\alpha}\quad P_{\kappa_1,\alpha}\sim P_{\kappa_2,\alpha}\quad (\text{mod } R')$$

since for all W^{*}-algebras R_1, R_2 and projections $P, Q \in R_2$ $R_1 \subset R_2$, $P \sim Q (\mathrm{mod}\ R_1) \Longrightarrow P \sim Q(\mathrm{mod}\ R_2)$.

So we have completely shown that for any sectors $\mathcal{H}_\alpha$ and $\mathcal{H}_\beta$ the families of projections $\{P_{\kappa,\alpha}\}_{\kappa \in \mathcal{K}_\alpha} \subset R(0)'_\alpha$ and $\{P_{\epsilon,\beta}\}_{\epsilon \in \mathcal{K}_\beta} \subset R(0)'_\beta$ possess all properties required by the criterion (D) with the exception, possibly, of the condition card $\mathcal{K}_\alpha$ = card $\mathcal{K}_\beta$. But if we have in addition $\varkappa'_\alpha = \varkappa'_\beta$ then this condition is satisfied too, since it takes place exactly $\varkappa'_\alpha$ = card $\mathcal{K}_\alpha$, $\varkappa'_\beta$ = card $\mathcal{K}_\beta$. As a result, we have proved that for any $\varkappa'_\alpha \geqslant \aleph_0$, $\varkappa'_\alpha = \varkappa'_\beta$ plus, of course dim $\mathcal{H}_\alpha$ = dim $\mathcal{H}_\beta$ give $\Pi^{(0)}_\alpha \simeq \Pi^{(0)}_\beta$.

<u>II - necessity.</u> Let $\mathcal{H}_\alpha, \mathcal{H}_\beta$ be coherent sectors such that $\varkappa'_\alpha, \varkappa'_\beta > \aleph_0$ and $\Pi^{(0)}_\alpha \simeq \Pi^{(0)}_\beta$ by means of partially isometric operator $V_{\beta\alpha}(0)$. We shall deduce that in this case $\varkappa'_\alpha = \varkappa'_\beta$.

Let us use again the system of projections $\{P_{\kappa,\alpha}\}_{\kappa \in \mathcal{K}_\alpha} \subset$ $\subset R(0)'_\alpha$ and consider how $V_{\beta\alpha}(0)$ is acting on the subspaces $\mathcal{H}_{\kappa,\alpha}$. Since there exists a non-void set of analytical for the energy vectors $\mathcal{M}_{\kappa,\alpha} \subset \mathcal{H}_{\kappa,\alpha}$ then $\mathcal{H}_{\kappa,\alpha} = \mathcal{H}^{R(0)}_{\Psi_{\kappa,\alpha}}$ for any $\Psi_{\kappa,\alpha} \in \mathcal{M}_{\kappa,\alpha}$. Hence it follows due to (3.8):

$$\mathcal{H}_{\kappa,\beta} \doteq V_{\beta\alpha}(0)\mathcal{H}_{\kappa,\alpha} = V_{\beta\alpha}(0)\,\mathcal{H}^{R(0)}_{\Psi_{\kappa,\alpha}} = \mathcal{H}^{R(0)}_{V_{\beta\alpha}(0)\Psi_{\kappa,\alpha}}$$

From here we see that projection on $\mathcal{H}_{\kappa,\beta}$, $P_{\kappa,\beta} \in R(0)'$. Further, the operator $V_{\beta\alpha}(0)$ being isometric on $\mathcal{H}_\alpha$, the pair-wise orthogonality of all $P_{\kappa,\alpha}$ implies pair-wise orthogonality of all $P_{\kappa,\beta}$. This means that $V_{\beta\alpha}(0)$ transforms the system of projections $\{P_{\kappa,\alpha}\}_{\kappa \in \mathcal{K}_\alpha} \subset R(0)'_\alpha$ into the system of mutually orthogonal, non-zero projections

$\{P_{\kappa,\beta}\}_{\kappa\in\mathcal{K}_\alpha} \subset R(0)'_\beta$ with the sum equal to I. By the other side, there exists in $R(0)'_\beta$ the system of projections $\{P_{\ell,\beta}\}_{\ell\in\mathcal{K}_\beta}$ with the properties described in the criterion (D), i.e. such that, in particular, all the algebras $R(0)'_{P_{\ell,\beta}}$ are countably decomposable. In such a situation there exists necessarily a definite relation between the cardinals of the sets $\mathcal{K}_\alpha$ and $\mathcal{K}_\beta$, according to the following lemma by Dixmier (DW-235):

Let R be von Neumann algebra, $\{E_i\}_{i\in J}$ infinite family of projections from R, with upper bound I and such that all R_{E_i} are countably decomposable; $\{F_k\}_{k\in\mathcal{K}}$ family of non-zero pair-wise orthogonal projections from R. Then

$$\text{card } K \leq \text{card } I.$$

Applying this lemma to the systems $\{P_{\kappa,\beta}\}_{\kappa\in\mathcal{K}_\alpha}$ and $\{P_{\ell,\beta}\}_{\ell\in\mathcal{K}_\beta}$ in the algebra $R(0)'_\beta$ we obtain

$$\text{card}\,\mathcal{K}_\alpha = \aleph'_\alpha \leq \text{card}\,\mathcal{K}_\beta = \aleph'_\beta \qquad (3.9)$$

It is not difficult to deduce an inverse inequality too. Conjugate operator $V_{\beta\alpha}(0)^*$ realizes an isometrical mapping of $\mathcal{H}_\beta$ on $\mathcal{H}_\alpha$ which possesses all the same properties as the mapping $V_{\beta\alpha}(0)$. Therefore the system $\{P_{\ell,\beta}\}_{\ell\in\mathcal{K}_\beta} \subset R(0)'_\beta$ is transformed by $V_{\beta\alpha}(0)^*$ into a system of non-zero pair-wise orthogonal projections $\{P_{\ell,\alpha}\}_{\ell\in\mathcal{K}_\beta} \subset R(0)'_\alpha$ where $P_{\ell,\alpha} \doteq \doteq V_{\beta\alpha}(0)^* P_{\ell,\beta} V_{\beta\alpha}(0)^{*-1}$ Application of Dixmier's lemma to the systems $\{P_{\kappa,\alpha}\}_{\kappa\in\mathcal{K}_\alpha}$ and $\{P_{\ell,\alpha}\}_{\ell\in\mathcal{K}_\beta}$ of projections in the algebra $R(0)'_\alpha$ gives us the inverse inequality to (3.9).

The theorem 3.3 is completely proved.

This theorem means, in particular, that in the general case of a theory with arbitrary set of coherent sectors, local unitary equivalence is not guaranteed by the Haag-Araki axioms, in contrast to equivalence properties 1, 2, 3, 5. Any two locally unitarily equivalent sectors obey, besides the axioms, some definite constraints, which express themselves in terms of sector invariants and so are the conditions on R and not on R(0). In other words, the oonstraints put by local unitary equivalence are of the global but not local nature. Generally speaking, these constraints require the definite relationship between dimensions of H-images of pure vector states belonging to locally unitarily equivalent sectors. It is interesting to note that abelian coherent sector ($\mathcal{K}' = I$) appears to be locally unitarily equivalent not only to all sectors with finite-dimensional H-images of pure states ($\mathcal{K}_{\alpha}' < \aleph_0$) but also to all sectors with separable H-images ($\mathcal{K}_{\alpha}' = \aleph_0$). Let us remind also, that in particular case of superselection theory connected with a compact gauge group /5/ (in suoh a theory only the values $\mathcal{K}_{\alpha}' = N_{\alpha} < \infty$ are possible) the parameters $\mathcal{K}_{\alpha}'$ possess a physical interpretation as multiplicity of a partiole multiplet covariant under corresponding irreducible representation of the group. Since in usual field theories the action of the field does not ohange this multiplioity, that particular example shows already that the existence of a field requires further restrictions.

Further, according to the theorem, all the set of coherent superselection sectors can be divided into two classes. The first of these classes includes all sectors with $\mathcal{K}_{\alpha}' \leq \aleph_0$, the second one the sectors with $\mathcal{K}_{\alpha}' > \aleph_0$, and the sectors belonging to different classes can never be connected by local unitary equivalence. The first class sectors possess more usual properties and are characterized in detail in proposition I.10. Since the property of local unitary equivalence can with assurance be expected from any realistic theory, all coherent sectors in such theories belong to the same class. If in addition the theory contains a vacuum sector $\mathcal{H}_{\Omega}$, then the class of the theory is defined by the parameter $\varkappa'_{\Omega}$, the dimension of the H-image of the vacuum state, or, if we want to put it into "more physical" terms, by the "degree of vacuum degeneration". In this connection some interesting problems arise, such as are the theories possible to exist, where the vacuum state (and so all other coherent sectors) belong to the "exotic" class $\varkappa' > \aleph_0$? And what specific properties do such theories possess?

4. FIELD-LIKE PROPERTIES OF QUANTUM THEORIES IN $\mathcal{H} = \mathcal{H}_{\rho}$: UNBOUNDED REGIONS

In this section we shall study (the most important for the F-problem) equivalence properties of the representations π_{α} restricted to C^*-algebras of unbounded regions. We relate to such regions the following two kinds of C^*-algebras:

$$\mathfrak{A}^c(O) \doteq R(O)' \cap \mathfrak{A} \; ; \quad \mathfrak{A}(O') \doteq \bigvee_{\hat{O} \subset O'} R(\hat{O}) \, , \quad \hat{O} \in B(M)$$

The corresponding restrictions of the representations π_{α} are

$$\pi_{\alpha}^c \doteq \pi_{\alpha} \Big|_{\mathfrak{A}^c(O)} \quad ; \quad \pi_{\alpha}^{(O')} \doteq \pi_{\alpha} \Big|_{\mathfrak{A}(O')}$$

At first we consider the algebras $\mathfrak{A}^c(O)$ and the representations π_α^c. Their structure is closely connected with the weak duality condition /4, 5/.

Definition 4.I

We shall say that in theory of the $\mathcal{H}=\mathcal{H}_p$ class the condition of weak duality (resp., sectorial weak duality) is fulfilled, if

$$\overline{\mathfrak{A}^c(O)}^w = R^c(O) \doteq R(O)' \cap R \qquad (4.I)$$

or respectively

$$\overline{\mathfrak{A}^c(O)_\alpha}^w = R^c(O)_\alpha \doteq R(O)'_\alpha \cap R_\alpha \qquad (4.I=\alpha)$$

Theorem 4.I

The theory of the $\mathcal{H}=\mathcal{H}_p$ class satisfies the weak duality condition if and only if the sectorial weak duality is fulfilled in every coherent sector and all the representations π_γ^c are pair-wise disjoint. I.e.

$$\overline{\mathfrak{A}^c(O)}^w = R^c(O) \Longleftrightarrow \overline{\mathfrak{A}^c(O)_\alpha}^w = R^c(O)_\alpha \text{ and } \forall_{\alpha\neq\beta}\ \pi_\alpha^c \,\delta\, \pi_\beta^c$$

Proof. Necessity. If (4.I) holds, then it is clear that

$$\overline{\mathfrak{A}^c(O)_\alpha}^w = R^c(O)_{P_\alpha} = (R(O)' \cap R)_{P_\alpha}.$$

For every $A_\alpha \in R^c(O)_{P_\alpha}$ there exists $B \in R^c(O)$ such that $A_\alpha = P_\alpha B|_{\mathcal{H}_\alpha}$. Hence $B \in R^c(O)$ implies $A_\alpha = P_\alpha B|_{\mathcal{H}_\alpha} \in R(O)'_\alpha$ and $B \in R$ implies $A_\alpha = P_\alpha B|_{\mathcal{H}_\alpha} \in R_\alpha$ what gives $A_\alpha \in R(O)'_\alpha \cap R_\alpha$, i.e.

$$R^c(O)_{P_\alpha} \subset R(O)'_\alpha \cap R_\alpha . \qquad (4.2)$$

Now let us take $A_\alpha \in R(O)'_\alpha \cap R_\alpha$. It follows from $P_\alpha \in R(O)' \cap R$ that $B \doteq P_\alpha A_\alpha P_\alpha \in R(O)' \cap R$ and $A_\alpha = P_\alpha B|_{\mathcal{H}_\alpha} \in (R(O)' \cap R)_{P_\alpha}$, i.e.

$$R^c(O)_{P_\alpha} \supset R(O)'_\alpha \cap R_\alpha$$

Together with (4.2) this gives the sectorial weak duality (4.I- α).

To prove the disjointness of π_α^c let us note that the central support $P_\alpha^c \doteq \mathfrak{Z}^c(0)$-supp π_α^c $\left(\mathfrak{Z}^c(0) \doteq R'(0) \cap R^c(0)'\right)$ of the representation π_α^c can be represented as $P_{\mathcal{H}_\alpha}^{\mathfrak{A}^c(0)'}$ and the projection P_α as $P_\alpha = P_{\mathcal{H}_\alpha}^{R^c(0)'}$ The condition (4.I) clearly implies

$$P_\alpha^c = P_{\mathcal{H}_\alpha}^{\mathfrak{A}^c(0)'} = P_{\mathcal{H}_\alpha}^{R^c(0)'} = P_\alpha$$

and orthogonality of P_α and P_β with $\alpha \neq \beta$ gives the desired disjointness of π_α^c .

<u>Sufficiency</u>. If all π_α^c are pair-wise disjoint, their central supports enjoy the property $P_\alpha^c P_\beta^c = \delta_{\alpha\beta} P_\beta^c$. Next, for every $\alpha \in \sigma$

$$P_\alpha^c = P_{\mathcal{H}_\alpha}^{\mathfrak{A}^c(0)'} \geqslant P_{\mathcal{H}_\alpha}^{R^c(0)'} = P_\alpha$$

whence $P_\beta^c P_\alpha = \delta_{\alpha\beta} P_\alpha$ Hence it follows

$$P_\beta^c = P_\beta^c \sum_{\alpha\in\sigma} P_\alpha = \sum_{\alpha\in\sigma} P_\beta^c P_\alpha = \sum_{\alpha\in\sigma} \delta_{\alpha\beta} P_\alpha = P_\beta$$

If in addition (4.I- α) holds, then for every $A \in R^c(0)$, $A_\alpha \doteq P_\alpha A|_{\mathcal{H}_\alpha} \in \overline{\mathfrak{A}^c(0)_{P_\alpha}}^w = R^c(0)_{P_\alpha}$. Hence with the account for $P_\alpha^c \in \overline{\mathfrak{A}^c(0)}^w$ we obtain $A = \sum_{\alpha\in\sigma} P_\alpha A P_\alpha =$

$$= \sum_{\alpha\in\sigma} P_\alpha^c A_\alpha P_\alpha^c \in \overline{\mathfrak{A}^c(0)}^w \text{ i.e. } R^c(0) \subset \overline{\mathfrak{A}^c(0)}^w.$$

Taken together with the trivial inclusion $\mathfrak{A}^c(0) \subset R^c(0)$ this gives us (4.I).

The problem of finding the necessary and sufficient criterion of weak duality is of great interest. This problem is not solved until now, but we can point out a very general sufficient condition of global nature.

Theorem 4.2

Any coherent sector $\mathcal{H}_\alpha$ such that $\varkappa'_\alpha \leq \aleph_0$ satisfies the sectorial weak duality. If $\varkappa'_\alpha \leq \aleph_0$ for all coherent sectors, and in addition the set $\mathcal{T}$ of all sectors is countable (i.e., there are only discrete superselection rules) then the theory satisfies weak duality. I.e.

$$\varkappa'_\alpha \leq \aleph_0 \implies \overline{\mathcal{O}\mathcal{L}^c(O)_\alpha}^w = R^c(O)_\alpha \qquad (4.3\text{-}\alpha)$$

$$\forall_{\alpha \in \mathcal{T}} \varkappa'_\alpha \leq \aleph_0 \,, \quad \operatorname{card} \mathcal{T} \leq \aleph_0 \implies \overline{\mathcal{O}\mathcal{L}^c(O)}^w = R^c(O) \qquad (4.3)$$

Proof. We shall prove only $(4.3\text{-}\alpha)$, the proof of (4.3) being completely analogous.

Let the region $O \in B(M)$ be given and $\hat{O} \in B(M)$ belong to O'. According to corollary 2.18, in the case $\varkappa'_\alpha \leq \aleph_0$ there is a non-void set $\mathcal{M}_{\alpha,\infty} \subset \mathcal{H}_\alpha$ consisting of analytical for the energy vectors cyclic for $R(\hat{O})_\alpha$. Since $R(\hat{O})_\alpha \subset \subset \mathcal{O}\mathcal{L}^c(O)_{P_\alpha} \subset R^c(O)_{P_\alpha}$ such vectors are also cyclic for $\mathcal{O}\mathcal{L}^c(O)_{P_\alpha}$ and $R^c(O)_{P_\alpha}$. Hence it follows that for every $\Psi_\alpha \in \mathcal{M}_{\alpha,\infty}$, $P_\alpha = P^{\mathcal{O}\mathcal{L}^c(O)}_{\Psi_\alpha} = P^{R^c(O)}_{\Psi_\alpha}$. By the other side, proposition 2.17 implies that Ψ_α is separating for $R^c(O)_{P_\alpha}$. In this situation the desired result directly follows from

Lemma 4.3

Let W^*-algebra R be acting in $\mathfrak{H}$ and containing a $*$-subalgebra $\mathcal{A}$ with the identity operator. Let further $\mathcal{A}$ satisfy the following condition: there exists a vector $\Psi \in \mathfrak{H}$ separating for R and such that $P^R_\Psi = P^{\mathcal{A}}_\Psi$. Then $R = \overline{\mathcal{A}}^w$.

Proof. For every $A \in R$ there exists a sequence $\{A_n\}_{n=1}^\infty$ of elements $A_n \in \mathcal{A}$ such that $A\Psi = \text{s-}\lim_{n\to\infty} A_n \Psi$. Hence it follows for every $T \in R'$ that $AT\Psi = \text{s-}\lim_{n\to\infty} A_n T\Psi$.

Ψ being cyclic for R′, this equality implies that our sequence $\{A_n\}_{n=1}^{\infty}$ converges strongly to A on the set $R'\Psi$ dense in $\mathfrak{H}$. Further, for every $\Phi \in \mathfrak{H}$ and $\varepsilon > 0$ there can be found $\tilde{\Psi} \in R'\Psi$ such that $\| A_n(\Phi - \tilde{\Psi})\| < \varepsilon$ for all n. Hence it follows that

$$\forall_{\Phi \in \mathfrak{H}} \quad \sup_n \| A_n \Phi\| \le \sup_n \| A_n(\Phi - \tilde{\Psi})\| + \sup_n \| A_n \tilde{\Psi}\| < \infty .$$

This means that the sequence $\{A_n\}_{n=1}^{\infty}$ satisfies the condition of the s-convergence criterion (/24/, ch. II, § I) and so $A_n \xrightarrow{s} A$. Thus every element of R belongs to the strong closure $\bar{\mathcal{A}}^s$ of the algebra $\mathcal{A}$, i.e. $R \subset \bar{\mathcal{A}}^s = \bar{\mathcal{A}}^w$.

This proves lemma 4.3 and at the same time the theorem 4.2.

<u>Corollary 4.4.</u>

Let quantum theory in $\mathcal{H} = \mathcal{H}_p$ be given satisfying the axioms I-VI as well as the conditions

$$\forall_{\alpha \in \sigma} \; \varkappa'_\alpha \le \aleph_0 \quad , \qquad \text{card}\; \sigma \le \aleph_1 .$$

Then all the representations π_α^c are pair-wise disjoint.

<u>Proof</u> represents the obvious combination of the theorems 4.I and 4.2.

As a result we find that under fairly general conditions the representations π_α^c do not generate any intertwining operators of coherent sectors. This means that study of them is hardly interesting from the viewpoint of obtaining field-like properties of the theory.

Finally, let us proceed to the analysis of the representations $\pi_\alpha^{(o')}$. In this point we shall consider as a rule the regions-diamonds, O = D = D′′. According to corollary 2.5, the algebras $R(D')_\alpha = \Pi_\alpha^{(D')}(\mathfrak{U}(D'))$ are factors and by the property of the primary representations, only two situations are possible:

$$\forall_{\alpha,\beta\in\sigma} \quad \Pi_\alpha^{(D')} \approx \Pi_\beta^{(D')} \quad \text{or} \quad \Pi_\alpha^{(D')} \,\delta\, \Pi_\beta^{(D')}. \quad (4.4\text{-}a,b)$$

The case when the asymptotical quasiequivalence takes place for one part $\sigma_1 \subset \sigma$ of coherent sectors, and the asymptotical disjointness for the another part σ_2, reduces also to (4.4), because the theory splits into two independent theories with the sets of sectors σ_1 and σ_2. We shall develop successively the description of the situations (4.4-a) and (4.4-b) and shall compare their properties. At first we consider the case (b).

Proposition 4.5

Let quantum theory in $\mathcal{H} = \mathcal{H}_p$ be given and the axioms I-VI be satisfied. Then the following conditions are equivalen:

1) All coherent sectors $\mathcal{H}_\alpha$ are asymptotically disjoint

$$\forall_{\alpha,\beta\in\sigma} \quad \Pi_\alpha^{(D')} \,\delta\, \Pi_\beta^{(D')} \qquad \text{for every diamond } D$$

2) All superselection operators are "asymptotical observables"

$$\forall_{\alpha\in\sigma} \; \forall_{D\subset M} \quad P_\alpha \in R(D')$$

3) $$\forall_{D\subset M} \quad R(D') = \bigoplus_{\alpha\in\sigma} R(D')_\alpha$$

4) $$R_{as} \doteq \bigcap_{O\in B(M)} R(O') = \bigoplus_{\alpha\in\sigma} \mathcal{C}(\mathcal{H}_\alpha)$$

Proof. 2 $\Rightarrow$ I is obvious, since for all $O\subset M$, $P_\alpha \in R(O)'$ and then 2) gives that $\mathfrak{Z}(D')$-supp $\Pi_\alpha^{(D')} = P_\alpha$. 3 $\Rightarrow$ 2 is obvious since due to 3), $\mathfrak{Z}(D') = \oplus_{\alpha\in\sigma} \mathcal{L}(\mathcal{H}_\alpha) \ni P_\alpha$. 2 $\Longrightarrow$ 3 follows from the property of inductions of any W^*-algebra R : if projection P belongs to the centre of R then $R = R_P \oplus R_{I-P}$. 4 $\Longrightarrow$ 2 is obvious right away. As a result, we have to establish only I $\Longrightarrow$ 4. We start with the formula $\mathfrak{Z}(D')$-supp $\pi_\alpha^{(D')} = P^{\mathfrak{Z}(D')}_{\mathcal{H}_\alpha} = P^{R(D')'}_{\mathcal{H}_\alpha}$. If $\alpha\in\sigma$ is fixed, then due to I) the projection $P^{R(D')'}_{\mathcal{H}_\alpha} \geqslant P_\alpha$ is orthogonal to all $P^{R(D')'}_{\mathcal{H}_\beta}$, $\beta\neq\alpha$ (and the latter are in their turn orthogonal to each other) and so to their sum $\sum_{\beta\neq\alpha} P^{R(D')'}_{\mathcal{H}_\beta} \geqslant I - P_\alpha$. As a consequence, I) implies that $P^{R(D')'}_{\mathcal{H}_\alpha} = P_\alpha$ or, equivalently $P_\alpha \in R(D')$ (the property 2)). Thus we have deduced $P_\alpha \in \bigcap_{D\subset M} R(D') \doteq R^{(D)}_{as}$ and now we shall demonstrate that $R_{as} = R^{(D)}_{as}$. $R_{as} \subset R^{(D)}_{as}$ by definition. Inverse inclusion will be deduced from the fact that every bounded region $O\in B(M)$ belongs to some diamond $D(O)$ and conversely, in every diamond D there exists some bounded region (for instance, $O = D$). Hence the desired result follows readily

$$R(O) \subset R(D(O)); \quad R(O)' \supset R(D(O))'$$

$$R_{as} \supset \bigcap_{D(O)\subset M} R(D(O))' = \bigcap_{D\subset M} R(D') = R^{(D)}_{as}$$

and the proof is finished.

It is easy to verify that the "algebra of asymptotical observables" R_{as}, introduced in the proposition, belongs always to the centre $\mathfrak{Z}$ of global algebra. Indeed, by definition $R_{as} \subset R = R(M)$; by the other side, due to locality $R_{as} = \bigcap_{O\in B(M)} R(O') \subset \bigcap_{O\in B(M)} R(O)' = R'$ so that $R_{as} \subset \mathfrak{Z}$. This means that R_{as} belongs to the type of "asymptotical central subalgebras" studied by Haag, Kadison and Kastler /25/. According to general results of these authors, properties of representations of C^*-algebra with a net of local subalgebras, generalized with respect to R_{as} in the sense of § I in /25/, characterize the asymptotical properties of the theory. Our concrete choice of R_{as} somewhat differs from that of Haag-Kadison-Kastler but it coincides with the choice proposed hypothetically by them (cf. /25/ , /26/, p. 29) for relativistic quantum theory not satisfying duality. The results of propositions 4.5 and 4.6 confirm that such a choice is a reasonable one. However, these results do not give yet detailed picture of Π_α properties with respect to R_{as} and so the problem still remains (although rather simple one) to give a complete description of these properties in the spirit of /25/.

Now we proceed to the physically more interesting situation (4.4-β). Sufficiently complete characterization of its properties gives the following proposition.

<u>Proposition 4.6</u>

The following conditions are equivalent:

1) all coherent sectors are asymptotically quasiequivalent

$$\forall_{\alpha,\beta\in\sigma} \quad \forall_{D\subset M} \quad \Pi_\alpha^{(D')} \simeq \Pi_\beta^{(D')}$$

2) W^*-algebras of the regions D' are factors

$$\mathfrak{Z}(D') = \mathcal{C}(\mathcal{H}).$$

3) The algebra of asymptotical observables is trivial

$$R_{as} = \mathcal{C}(\mathcal{H}).$$

Besides this, conditions (I = 3) are equivalent to conditions obtained from those (2 - 6) of proposition 3.2 by changing in them region O by D'.

4) induction $R(D') \longrightarrow R(D')_\alpha$ is an isomorphism,

5) all $\Psi \in \mathcal{H}_\alpha$ are separating for $R(D')$,

6) there exists $\Psi \in \mathcal{H}_\alpha$ separating for $R(D')$,

7) all $\Psi \in \mathcal{H}_\alpha$ are separating for $\mathfrak{Z}(D')$

8) there exists $\Psi \in \mathcal{H}_\alpha$ separating for $\mathfrak{Z}(D')$.

Proof. Firstly we shall demonstrate the equivalence of conditions I = 3 by proving the implications I $\Rightarrow$ 2 3 $\Rightarrow$ I and 2 $\Rightarrow$ 3. To obtain I $\Rightarrow$2 we note that under the condition I), $P_\alpha \notin \mathfrak{Z}(D')$ for all $\alpha \in \sigma$, since otherwise $\Pi_\alpha^{(D')} \& \widetilde{\Pi}_\beta^{(D')}$ (proposition 4.5). By the other side, $\mathfrak{Z}(D') \subset \mathfrak{Z}$ (corollary 2.5) and both facts taken together mean the absence of non-trivial projections in $\mathfrak{Z}(D')$ i.e., $\mathfrak{Z}(D') = \mathcal{C}(\mathcal{H})_3$ $\Rightarrow$ I follows from proposition 4.5 too, since this proposition implies that $R_{a_3} = \mathcal{C}(\mathcal{H})$ excludes $\Pi_\alpha^{(D')} \& \widetilde{\Pi}_\beta^{(D')}$ and consequently ensures the fulfillment of I). 2 $\Rightarrow$ 3 follows from the inclusions $R_{a_3} \subset R' \subset R(D')'$ and $R_{a_3} \subset R(D')$ for all $D \subset M$, which give in total: $R_{a_3} \subset \mathfrak{Z}(D')$.

Further, the proof of the equivalence of the conditions I) to (4 - 8) goes exactly as in proposition 3.2. The only difference is that we now use not the usual version of the Reeh-Schlieder theorem, but its extension to unbounded regions, obtained in the proposition 2.15.

In addition to the proposition just proved we now shall obtain the conditions under which asymptotical quasiequivalence implies asymptotical unitary equivalence. It is easy to prove that $\Pi_\alpha^{(O')} \simeq \widetilde{\Pi}_\beta^{(O')}$ follows from $\Pi_\alpha^{(O')} \succsim \widetilde{\Pi}_\beta^{(O')}$ exactly under the same (necessary and sufficient) conditions, under which $\Pi_\alpha^{(O)} \simeq \widetilde{\Pi}_\beta^{(O)}$ follows from $\Pi_\alpha^{(O)} \succsim \widetilde{\Pi}_\beta^{(O)}$.

and which were formulated in detail in the theorem 3.3. To make sure of this it is sufficient to note that due to $R' \subset R(O')'$ for all $O \subset M$, the algebras $R(O')'_{\alpha}$ contain, exactly as it was the case for $R(O)'_{\alpha}$, the families of projections $\{P_{k,\alpha}\}_{k \in K_{\alpha}}$ and these families anjoy with respect to the algebras $R(O')'_{\alpha}$ all the same properties that with respect to $R(O)'_{\alpha}$ In other words, we obtain the property

$$\left.\begin{array}{c} \Pi_{\alpha}^{(O')} \simeq \Pi_{\beta}^{(O')} \\ \Pi_{\alpha}^{(O)} \simeq \Pi_{\beta}^{(O)} \end{array}\right\} \Longrightarrow \Pi_{\alpha}^{(O')} \simeq \Pi_{\beta}^{(O')} \qquad (4.5)$$

Comparing the propositions 4.5 and 4.6 with each other we can easily elucidate physical distinctions of both situations. Namely, we see that in the case $\Pi_{\alpha}^{(D')} \simeq \Pi_{\beta}^{(D')}$ all the principal properties of the algebras $R(D')$ repeat the corresponding properties of $R(D)$. Conversely, if $\Pi_{\alpha}^{(D')} \not\simeq \Pi_{\beta}^{(D')}$ then the properties of $R(D')$ are rather close to those of the global algebra $R=R(M)$. Further, the analysis of the proposition 4.5 and 4.6 as well as their consequences strongly assures us that both situations (4.4-α) and (4.4-β) are compatible with all axioms I-VI, although we have no rigorous proof of this for the time being. If this supposition is really true then both cases give us the correct axiomatic theory of local observables, but only in one case, when $\Pi_{\alpha}^{(D')} \simeq \Pi_{\beta}^{(D')}$ there exist intertwining operators between coherent sectors, which enjoy local properties, and so the construction of a field can be possible. In other words, the main difference between field theory and general theory of local observables is that the latter enjoys in general case the greater arbitrariness in its asymptotical behaviour.

Then the subclass of field theories should be singled out from the set of all local observable theories by means of some "asymptotical condition", which forbids asymptotical disjointness of coherent sectors and ensures their asymptotical unitary equivalence. In contrast to the local unitary equivalence conditions found in § 3, such asymptotical conditions appear to include not only global constraints. We prove below one form of asymptotical condition, which shows clearly that, indeed, these conditions represent restrictions on asymptotical behaviour of states.

Our condition will be formulated in terms of the so called strictly localized states. We shall use the following convenient definition of them /5/:

Definition 4.2

We shall say that the state ω on observable algebra $\mathfrak{A}$ is strictly localized in region $O \subset M$, if the values of ω on all observables belonging to O' coincide with corresponding values of the vacuum functional ω_Ω.

For the theory in $\mathcal{H} = \mathcal{H}_P$, possessing the unique vacuum sector, the set of vectors representing vector states strictly localized in O, will be denoted as $S\mathcal{L}(O)$

$$S\mathcal{L}(O) \doteq \left\{ \Phi \in \mathcal{H} \mid \omega_\Phi \big|_{R(O')} = \omega_\Omega \big|_{R(O')} \right\} \quad (4.6)$$

The vectors $\Phi \in S\mathcal{L}(O)$ will be called strictly localized vectors.

The asymptotical character of the strict localizability is clearly displayed by the following property. If we introduce the notations $S\mathcal{L} \doteq \bigcup_{O \subset M} S\mathcal{L}(O)$ and $A_a \doteq U(a) A U(-a)$ for all $A \in R$, $a \in M$, then

$$\forall_{\Phi \in S\mathcal{L}} \ \forall_{A \in \mathfrak{A}} \ \lim_{\substack{a \to \infty \\ a^2 < 0}} \left| \omega_\Phi(A_a) - \omega_\Omega(A) \right| = 0$$

(This relation holds automatically for all $A \in \mathfrak{A}$ and extends to $\mathfrak{A}$ by means of obvious estimates).

The very useful tool of studying strictly localized states we find in our notion of the H-image (see § 1). In fact, the formula (4.6) means exactly that every strictly localized vector must belong to the H-image $\mathfrak{M}^{\frac{1}{2}}_{\Omega}(O')$ of the vacuum state ω_Ω with respect to algebra $R(O')$:

$$S\mathcal{L}(O) = \mathfrak{M}^{\frac{1}{2}}_{\Omega}(O'). \qquad (4.7)$$

Such re-writing of (4.6) immediately tells us, what is the subspace spanned by all vectors strictly localized in $O \subset M$. Namely, according to the formula (I.2)

$$\mathcal{H}_{S\mathcal{L}(O)} \doteq \overline{L\{S\mathcal{L}(O)\}} = \mathcal{H}^{R(O')'}_{\Omega} \qquad (4.8)$$

The set of all strictly localized in O vectors from coherent sector $\mathcal{H}_\alpha$ will be denoted as $S\mathcal{L}(O)_\alpha$ and the subspace spanned by this set as $\mathcal{H}_{S\mathcal{L}(O)_\alpha}$

$$S\mathcal{L}(O)_\alpha \doteq S\mathcal{L}(O) \cap \mathcal{H}_\alpha = \mathfrak{M}^{\frac{1}{2}}_{\Omega}(O') \cap \mathcal{H}_\alpha$$

$$\mathcal{H}_{S\mathcal{L}(O)_\alpha} \doteq \overline{L\{\mathfrak{M}^{\frac{1}{2}}_{\Omega}(O') \cap \mathcal{H}_\alpha\}}. \qquad (4.9)$$

It is important that in general case only the inclusion

$$\mathcal{H}_{S\mathcal{L}(O)_\alpha} \subset \mathcal{H}_{S\mathcal{L}(O)} \cap \mathcal{H}_\alpha$$

takes place, but not the equality. The formulas (4.7 - 9) will be our main tool in proving the following basic result.

<u>Theorem 4.7</u>

Quantum theory of the $\mathcal{H} = \mathcal{H}_p$ class possesses the asymptotical unitary equivalence if and only if in every coherent sector $\mathcal{H}_\alpha$ and for every $O \in B(M)$ there exists a total set of strictly localized vectors and in addition the (global) conditions of the local unitary equivalence are fulfilled:

$$\left.\begin{array}{l}(\alpha)\ \forall_{\alpha\in\sigma}\ \forall_{O\in B(M)}\ \mathcal{H}_{S\mathcal{L}(O)_\alpha}=\mathcal{H}_\alpha \\ (\beta)\ \forall_{\alpha,\beta\in\sigma}\ \forall_{O\in B(M)}\ \Pi_\alpha^{(O)}\simeq\Pi_\beta^{(O)}\end{array}\right\}\Longleftrightarrow \forall_{\alpha,\beta\in\sigma}\ \forall_{O\in B(M)}\ \Pi_\alpha^{(O')}\simeq\Pi_\beta^{(O')}$$

Proof. Necessity. First of all we note that for all $O\in B(M)$

$$\forall_{\alpha\in\sigma}\ \mathcal{H}_{S\mathcal{L}(O)_\alpha}=\mathcal{H}_\alpha \Longrightarrow \mathcal{H}_{S\mathcal{L}(O)}=\mathcal{H} \qquad (4.10)$$

(the converse being not true in general case). Indeed, by definition $\mathcal{H}_{S\mathcal{L}(O)_\alpha}\subset\mathcal{H}_{S\mathcal{L}(O)}$ and hence $\bigoplus_{\alpha\in\sigma}\mathcal{H}_{S\mathcal{L}(O)_\alpha}\subset$ $\subset\mathcal{H}_{S\mathcal{L}(O)}$ whence (4.10). Next, it follows from the definition 1.2 that every H-image $\mathcal{M}^1_\varphi$ of a vector state on any W^*-algebra enjoys the property

$$\psi\in\mathcal{M}^1_\varphi \Longrightarrow \mathcal{H}^{R'}_\psi=\mathcal{H}^{R'}_\varphi .$$

Using this for the H-image $\mathcal{M}^1_\Omega(C')$ and taking into account formulas (4.8), (4.9) and (4.10) we obtain

$$\forall_{\varphi\in S\mathcal{L}(O)_\alpha}\ \mathcal{H}^{R(O')'}_\psi=\mathcal{H}^{R(O')'}_\Omega=\mathcal{H} .$$

This means that all vectors from the (non-void by supposition) set $S\mathcal{L}(O)_\alpha\subset\mathcal{H}_\alpha$ are separating for $R(O')$. Owing to the point 6) in proposition 4.6 this implies $\Pi_\alpha^{(O')}\approx\Pi_\beta^{(O')}$. This result together with the formula (4.5) and the condition (β) gives $\Pi_\alpha^{(O')}\simeq\Pi_\beta^{(O')}$.

Sufficiency. We shall verify firstly that total set of strictly localized vectors exists always in the vacuum coherent sector. This follows from another simple property of H-images on any W^x-algebra $\mathcal{B}$: if $\mathcal{B}^{'U}$ is the set of all unitary operators from $\mathcal{B}'$, then

$$\forall_{\Phi \in \mathcal{H}} \quad \mathfrak{M}^{1}_{\Phi} \supset \mathcal{B}'^{U} \Phi \qquad (4.11)$$

On account of locality this means in our case that

$$S\mathcal{L}(O)_{\Omega} = \mathfrak{M}^{1}_{\Omega}(O') \cap \mathcal{H}_{\Omega} \supset [R(O')^{U}\Omega]' \cap \mathcal{H}_{\Omega} \supset R(O)^{U}\Omega$$

and correspondingly

$$\mathcal{H}_{S\mathcal{L}(O)_{\Omega}} \supset \overline{L\{R(O)^{U}\Omega\}} = \overline{R(O)\Omega} .$$

Taking into account that Ω is analytic for the energy and is in our context an arbitrary vector from the H-image $\mathfrak{M}^{1}_{\Omega}$ of the vacuum state $\omega_{\Omega} \in R^{*+}_{1}$ we obtain

$$\forall_{\Omega_K \in \mathfrak{M}^{1}_{\Omega}} \quad \mathcal{H}_{S\mathcal{L}(O)_{\Omega}} \supset \mathcal{H}^{R}_{\Omega_K} \qquad (4.12)$$

Further, the vacuum state being pure (proposition 2.20) we can apply to it the formula (30) from /I/:

$$\forall_{\Phi \in \mathcal{P}} \quad \mathcal{H}^{3'}_{\Phi} = \overline{L\{\bigcup_{\Psi \in \mathfrak{M}^{1}_{\Phi}} \mathcal{H}^{R}_{\Psi}\}} .$$

Taking this together with (4.12), we have the desired result:

$$\mathcal{H}_{S\mathcal{L}(O)_{\Omega}} = \mathcal{H}^{3'}_{\Omega} = \mathcal{H}_{\Omega} .$$

From here and using $\pi^{(O')}_{\alpha} \simeq \pi^{(O')}_{\beta}$ it is not difficult to deduce the condition (α) of the theorem. Let us use Kadison's criterion /27/ of the unitary equivalence:

$$\pi^{(O')}_{\alpha} \simeq \pi^{(O')}_{\beta} \iff V(\pi^{(O')}_{\alpha}) = V(\pi^{(O')}_{\beta}) .$$

Let us take here $\pi_{\beta} = \pi_{\Omega}$ and an arbitrary $\alpha \in \sigma$, and let $V_{\alpha\Omega}(O')$ be partially isometric operator realising the unitary equivalence of $\pi_\alpha^{(0')}$ and $\pi_\Omega^{(0')}$. Using the isometry of $V_{\alpha\Omega}(0')$ on $\mathcal{H}_\Omega$ we obtain:

$$\mathcal{H}_\alpha = V_{\alpha\Omega}(0')\mathcal{H}_\Omega = V_{\alpha\Omega}(0')\overline{L\{S\mathcal{L}(0)_\Omega\}} = \overline{L\{V_{\alpha\Omega}(0')S\mathcal{L}(0)_\Omega\}} \qquad (4.13)$$

By the other side, the property of $V_{\alpha\Omega}(0')$ as an intertwining operator: $V_{\alpha\Omega}(0')P_\Omega \in R(0')''$ and the formula (4.II) give us

$$V_{\alpha\Omega}(0')S\mathcal{L}(0)_\alpha \subset S\mathcal{L}(0)\cap\mathcal{H}_\alpha = S\mathcal{L}(0)_\alpha$$

The putting of this into (4.13) leads to (α):

$$\forall_{\alpha\in\sigma} \quad \mathcal{H}_\alpha = \mathcal{H}_{S\mathcal{L}(0)_\alpha}$$

To end the proof it is sufficient to note that due to the formula (4.5), (for all $0\in B(M)$) $\pi_\alpha^{(0')} \simeq \pi_\beta^{(0')}$ implies $\pi_\alpha^{(0)} \simeq \pi_\beta^{(0)}$.

Corollary 4.8

Let $\mathcal{H}_\alpha$ be coherent superselection sector such that $\pi_\alpha^{(0)} \simeq \pi_\Omega^{(0)}$. Then $\mathcal{H}_\alpha$ contains a total set of strictly localized vectors if and only if $\mathcal{H}_\alpha$ contains at least one such vector:

$$\mathcal{H}_{S\mathcal{L}(0)_\alpha} = \mathcal{H}_\alpha \iff S\mathcal{L}(0)_\alpha \neq \emptyset$$

Proof. The necessity is obvious, while the sufficiency follows from the proof of the theorem. In fact, we have seen that due to proposition 4.6 the existence in $\mathcal{H}_\alpha$ of only one strictly localized vector is already sufficient for $\pi_\alpha^{(0')} \simeq \pi_\beta^{(0')}$. This means that $\pi_\alpha^{(0)} \simeq \pi_\beta^{(0)}$ and $S\mathcal{L}(0)_\alpha \neq \emptyset$ imply $\pi_\alpha^{(0')} \simeq \pi_\Omega^{(0')}$, and this, according to the theorem, implies the existence in $\mathcal{H}_\alpha$ of a total set of strictly localized vectors.

<u>Note</u>. It can easily be shown that the vacuum Ω as well as any vector analytical for the energy cannot represent pure state on any local subalgebra R(O) or R(O′). On account of this and the property (I.3) of the H-image

$$S\mathcal{L}(O) = \mathcal{M}^{1}_{\Omega}(O') \subsetneq \mathcal{H}^{R(O)'}_{\Omega} \cap S^{1}$$

This opens the possibility of a situation, in which $\mathcal{H}_{S\mathcal{L}(O)} = \mathcal{H}^{R(O)'}_{\Omega} = \mathcal{H}$ but for some $\alpha \in \sigma$, the coherent sectors $\mathcal{H}_\alpha$ do not contain any strictly localized vector, i.e. $S\mathcal{L}(O)_\alpha = \emptyset$. By this reason the following implications take place:

$$\mathcal{H}_{S\mathcal{L}(O)} = \mathcal{H} \Longleftarrow \widetilde{\Pi}^{(O')}_{\alpha} \simeq \widetilde{\Pi}^{(O')}_{\beta} \Longleftarrow \forall_{\alpha \in \sigma} \mathcal{H}_{S\mathcal{L}(O)_\alpha} = \mathcal{H}_\alpha$$

(the left one follows from the analyticity of Ω for the energy, while the right one was established in the proof of the theorem), but no one of them can be replaced by equivalence relation. This means, in particular, that our formulation of the theorem 4.7 does not allow any essential simplification.

Now let us discuss in more detail the obtained necessary and sufficient condition of the asymptotical unitary equivalence. This condition is rather similar in form to our main initial condition $\mathcal{H} = \mathcal{H}_p$. However, the difference is that in the latter case the "full" condition is completely equivalent to the sum of sectorial ones:

$$\mathcal{H} = \mathcal{H}_p \Longleftrightarrow \mathcal{H}_\alpha = (\mathcal{H}_\alpha)_p$$

whereas it is not so for the "asymptotical condition" (α):

$$\mathcal{H} = \mathcal{H}_{S\mathcal{L}(0)} \quad \overset{\not\Longrightarrow}{\Longleftarrow} \quad \forall_{\alpha \in G} \; \mathcal{H}_{\alpha} = \mathcal{H}_{S\mathcal{L}(0)_{\alpha}} .$$

The physical content of the condition is also clear in general features. Like other asymptotical conditions introduced in quantum field theory (quasilocality condition by Haag /28/ space-like asymptotical condition by Ruelle /29/, etc.) this condition establishes for the expectation values of the theory a definite law of asymptotical behaviour in space-like directions. But the detailed investigation of this condition is not performed yet. In the sphere of this investigation fall several problems of significant interest, such as decompositions of arbitrary states into strictly localized states and/or relation of the class of strictly localized states to that of "states without long-range correlations" of Haag and Kastler /23/; relations between strict localizability, asymptotical abelianness end cluster properties, etc. The last point is of interest, in particular, because of possible dependence of cluster properties of the theory on the type of its statistics . And also always in the close analogy with the history of other asymptotical conditions, the problem is still open whether our asymtotical condition is the consequence of the Haag-Araki axioms or the independent additional restriction.

CONCLUSION

Here we shall compare results obtained with the results of Doplicher, Haag and Roberts /5/ (for the sake of brevity referred to below as DHR).

The general scheme of DHR represents the algebraic theory in Hilbert space $\mathcal{H}$, on which C^*-algebra $\mathfrak{A}$ of observables and C^*-algebra $\mathfrak{F}$ of fields are acting, both provided with nets of local subalgebras, $\{R(O)\}_{O\in B(M)}$ and $\{\mathfrak{F}(O)\}_{O\in B(M)}$ respectively. $\mathfrak{F}$ is irreducible ($\mathfrak{F}^w = \mathcal{B}(\mathcal{H})$) while $R = \overline{\mathfrak{A}}^w$ belongs to a certain subclass of theories in $\mathcal{H} = \mathcal{H}_p$. Namely, R coincides with the commutant $U(G)'$ of representation in $\mathcal{H}$ of some compact gauge group G, and as a consequense is of the form $R = \oplus_{\alpha\in\sigma} R_\alpha$, all the R being type I factors. However, in this case the algebra $R' = U(G)''$ possesses only finite-dimensional irreducible representations. This means that global algebra of the DHR scheme is algebra of the $\mathcal{H}-\mathcal{H}_p$ class, with the additional restriction: for all $\alpha\in\sigma$, R'_α is factor of the type I_{N_α}, $N_\alpha < \infty$; or equivalently, $\mathcal{H}'_\alpha = N_\alpha < \infty$. Apart from this, another distinction of our global structure is that for its deduction we do not assume the existence of either the group G or the algebra $\mathfrak{F}$, but use only a minimal necessary concretization: the object of our study is an arbitrary quantum system with superselection rules. Further, the DHR scheme includes also a great number of local additional conditions. In most part, they are of the form of relations connecting with each other the nets $\{R(O)\}$ and $\{\mathfrak{F}(O)\}$ and so they cannot even be formulated in the "usual" axiomatic theory where the input includes only one of these objects. Besides of this, such questions as the range of necessity of introduced conditions, their independence on each other and on the axioms etc. were not answered most often. For all these reasons we did not consider as superfluous after the works by DHR to return once more to the analysis of the F-problem.

Now we proceed to the consequtive comparison. The § I of our work contains the physical basing of structure used and so has no parallels in the DHR work. The content of § 2 is equally not connected with DHR. With respect to the F-problem (the only problem of DHR) the results of that section are either quite irrespective of or very remote. Attention paid by us to these results is explained by their independent interest and also by our hopes on other applications of our scheme, besides the F-problem. Next, the § 3 results (equivalence properties of coherent sectors for global algebras and local algebras of bounded regions) play only the preliminary part in the problem of relationship between fields and observables. For this reason and also, probably, due to the simplicity of their obtaining, they were practically not considered by DHR. In fact, the only complicated result in § 3 is the theorem 3.3, but its complexity is caused entirely by the taking into account arbitrary values of the parameters $\mathcal{X}_\alpha, \mathcal{X}'_\alpha$. In the DHR scheme where only the values $\mathcal{X}'_\alpha < \infty$ are allowed this theorem reduces to a semi-trivial assertion.

Further, analysis of the representations π_α^c and weak duality condition was undertaken by DHR in § 5 of /5/. In this point our results supplement and clear up the results of DHR and some of their additional conditions. Thus the theorem 4.I shows that the weak duality condition obtained by DHR in their theorem 5.2 is not only necessary, but also sufficient. Next, we prove the fulfillment of the weak duality in all coherent sectors $\mathcal{H}_\alpha$ with $\mathcal{X}'_\alpha \leq \aleph_0$ and also in the whole $\mathcal{H}$ for the theories with discrete superselection rules only; for the same theories the axiomatic proof of the disjointness of all π_α^c is obtained. These results have no overlapping with DHR. Let us mention also that the result of the theorem 5.6 by DHR (necessary condition of duality) was re-proved in the work by one of us /30/ in the slightly more general form and without the assumption of weak duality made by DHR. Finally, analyzing the representations $\tilde{\pi}_{\alpha}^{(\sigma')}$ DHR proved their quasi-equivalence with the aid of strong restrictions on the connection between the nets $\{R(O)\}$ and $\{\mathfrak{F}(O)\}$. The net $\{\mathfrak{F}(O)\}_{O \in B(M)}$ being not given in our case, we cannot exclude the alternative situation, $\pi_{\alpha}^{(\sigma')} \not\sim \pi_{\beta}^{(\sigma')}$ and so we perform the detailed analysis of both situations. Then we formulate in terms of local observables and prove the "asymptotical condition", which is necessary and sufficient for the unitary equivalence of all $\tilde{\pi}_{\alpha}^{(\sigma')}$. The necessity of this condition was stated by DHR without proof and under restrictions on $\{\mathfrak{F}(O)\}$.

As a result, from all the complex of properties, which is, according to DHR /6/, sufficient for the construction of field group and field operators, the following properties are not yet obtained in our scheme:

1) sufficient conditions of the fulfillment of duality in coherent sectors;

2) conditions of the existence of localized automorphisms.

In future we are intended to return to these properties.

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