ОБЪЕДИНЕННЫЙ
ИНСТИТУТ
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ИССЛЕДОВАНИЙ

Дубна

E2 - 5429

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ANGULAR MOMENTUM AND UNITARY SPINOR BASES OF THE LORENTZ GROUP

ЛАБОРАТОРИЯ ТЕОРЕТИЧЕСКОЙ ФИЗИКИ

1970

Хусар М.

E2-5429

Базисы углового момента и унитарных спиноров группы Лоренца

Матрица оператора "буста" имеет сложный вид в базисе углового момента, в то же время она сводится к диагональной форме $e^{i\alpha\nu}$ в базисе унитарных спиноров. Коэффициентами переразложения между двумя базисами являются комплексные коэффициенты Вигнера группы вращений. С помощью этих коэффициентов задается выражение для функции "буста" $d^{j_0\sigma}_{\ell\ell'\mu}(\alpha)$ в виде степенного ряда.

Препринт Объединенного института ядерных исследований.
Дубна, 1970

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E2-5429

Angular Momentum and Unitary Spinor Bases of the Lorentz Group

The matrix of the boost operator has a rather complicated form in angular momentum basis while it reduces to a diagonal form $e^{i\alpha\nu}$ in the unitary spinor basis. The overlap coefficients between the two bases turn out to be complex Wigner coefficients of the rotation group. With the aid of these coefficients an expression for the boost function $d^{j_0\sigma}_{\ell\ell'\mu}(\alpha)$ is derived in terms of power series in $e^{-2\alpha}$.

Preprint. Joint Institute for Nuclear Research.
Dubna, 1970

1356

E2 - 5429

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ANGULAR MOMENTUM AND UNITARY SPINOR BASES OF THE LORENTZ GROUP

Submitted to Acta Physica Hungarica

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It has been shown in [I] and in the revised versions [2],[3], that generalizing the notion of spinor to the unitary case, the unitary representations of the Lorentz group can be expressed in a simple form as compared to those in $O(3)$, $O(2,I)$, $E(2)$ bases. In the spinor basis the Lorentz group figures as an $SO(3, C)$ group (the group of three-dimensional complex rotations) isomorphic to the proper Lorentz group. (The transformation properties of three-dimensional complex vectors have been treated by H.Joos [4] and F.I.Fedorov [5]). Such an interpretation is possible since there exists a combination of the generators satisfying the commutators of two independent angular momenta. Similar decomposition holds for the $O(4)$ group. Though in the above spinor basis Lie Algebras of the $O(3,I)$ and the $O(4)$ groups coincide, for unitary representations all the generators of the latter group are Hermitean in contrast to the Lorentz group where the generators of the two complex angular momenta are pairwise adjoint to each other. In the spinor basis the unitary representations of the $O(4)$ group is simply a product of two D -functions and the transition to the angular momentum basis can be accomplished by means of the Wigner-coefficients of the real rotation group. It seems then natural that the overlap coefficients between $O(3)$ and $O(2, C)$ (spinor) bases of the Lorentz group can be interpreted as an analytic continuation of the familiar Wigner coefficients. This idea has been proposed by J.Smorodinsky and the author in [2] . The overlap coefficients have been derived earlier in a special case in [6] .

The expression for the boost function $d^{j_0\sigma}_{\ell\ell'\mu}(\alpha)$ in the angular momentum basis is rather complicated. On the other hand in the spinor basis the matrix of the boost operator along z -axis is diagonal and has the form $e^{i\alpha\nu}$. The purpose of the paper is to show that complicated formulas containing multiple sums for $d^{j_0\sigma}_{\ell\ell'\mu}(\alpha)$ -function can be reduced to the product of two overlap coefficients which accomplish the transition between $O(3)$ and $O(2, C)$. These coefficients are essentially complex Wigner coefficients and can be treated easily since they possess a number of properties well-known from the rotation group.

In Section I the eigenfunctions of the $O(3)$ and $O(2, C)$ bases are constructed. In Section 2 the overlap coefficients between the two bases are calculated and in Section 3 it is shown that they are essentially complexified Wigner coefficients of the real rotation group. In Section 4 an explicit form for the boost function is presented in terms of power series of the form $\sum_k a_k e^{-2k\alpha}$.

I) The Basis Functions

Denoting the infinitesimal generators of spatial rotations and boosts along k -axis (k = I,2,3) by M_k , N_k respectively, the combinations $\frac{1}{2}(M_k + iN_k) = J_k$ and $\frac{1}{2}(M_k - iN_k) = K_k$ satisfy

$$[J_k, J_l] = i\varepsilon_{klm} J_m, \; [K_k, K_l] = i\varepsilon_{klm} K_m, \; [J_k, K_l] = 0.$$

For unitary representations $J_k = K_k^+$ holds. Casimir operators $\vec{J}^2$, $\vec{K}^2$, of the two complex angular momenta $\vec{J}$, $\vec{K}$ are the Casimir operators of the Lorentz group:

$$\vec{J}^2 | \;\rangle = j(j+1) | \;\rangle \;, \quad \vec{K}^2 | \;\rangle = j^*(j^*+1) | \;\rangle \;, \tag{1.1}$$

where

$$j = \tfrac{1}{2}(j_0 - 1 + i\sigma) \tag{1.2}$$

($j_0 = 0, \pm\frac{1}{2}, \pm 1, \ldots \quad 0 \le \sigma < \infty$ for the principal series). Irreducible unitary representations can be characterized by (j_0, σ) or by (j, j^*) .

In unitary spinor basis the generators M_3 and N_3 are diagonal:

$$\begin{aligned} M_3 | j j^*; \mu\nu \rangle &= \mu | j j^*; \mu\nu \rangle \\ N_3 | j j^*; \mu\nu \rangle &= \nu | j j^*; \mu\nu \rangle \;, \end{aligned} \tag{1.3}$$

where μ takes integer (half-integer) values for single-(double-) valued representations, while ν is continuous real with the range $-\infty<\nu<\infty$. In terms of $\mathcal{J}_3$ and K_3 eq. (1.3) reads:

$$\begin{aligned} \mathcal{J}_3 | \jmath \jmath^*; m m^* \rangle &= m | \jmath \jmath^*; m m^* \rangle \\ K_3 | \jmath \jmath^*; m m^* \rangle &= m^* | \jmath \jmath^*; m m^* \rangle \end{aligned} \tag{1.4}$$

with

$$m = \tfrac{1}{2}(\mu + i\nu) \quad , \quad m^* = \tfrac{1}{2}(\mu - i\nu). \tag{1.5}$$

Since m plays a part in the complex rotation group analogous to that of the third component of the usual angular momentum, we reserve the notation m for the eigenvalue of the complex angular momentum $\mathcal{J}_3$, and the eigenvalue of the real angular momentum M_3 will be denoted by μ .

Eigenfunctions in the $O(3)$ basis in addition to (1.1.) satisfy:

$$\begin{aligned} \vec{M}^2 | \jmath \jmath^*; \ell \mu \rangle &= \ell(\ell+1) | \jmath \jmath^*; \ell \mu \rangle \\ M_3 | \jmath \jmath^*; \ell \mu \rangle &= \mu \; | \jmath \jmath^*; \ell \mu \rangle. \end{aligned} \tag{1.6}$$

In order to evaluate the overlap coefficients between the two bases it is convenient to choose such a representation of the basis vectors which makes the calculation simple. One of the possible choices is the hyperboloid or the cone $p^2 = 0$ .Whichever of them is considered for the generators

$M_{\mu\nu} = i(p_\mu \partial_\nu - p_\nu \partial_\mu)$ $\mathrm{Im}\, j(j+1) - \frac{1}{2} j_0 \sigma$ is equal to zero automatically, so we are restricted to the special case $j_0 = 0$, or $\sigma = 0$. However, it was pointed out by J.Lomont and H.E.Moses [7] , that on the cone there exists a canonical transformation which yields generators realizing arbitrary, characteristic for the principal series, values of j_0 and σ . This is due to the fact that on the cone the helicity $\lambda = \vec{M}\vec{p}/|\vec{p}|$ is a Poincaré and Lorentz-invariant quantity. The generators given in [7] are:

$$\vec{M} = \vec{M}^{(0)} + \vec{M}^{(1)} , \qquad \vec{N} = \vec{N}^{(0)} + \vec{N}^{(1)} , \tag{1.7}$$

where

$$\vec{M}^{(0)} = \frac{1}{i}(\vec{p} \times \nabla_{\vec{p}}) , \quad \vec{N}^{(0)} = \frac{1}{i} p \nabla_{\vec{p}} \quad (p \equiv |\vec{p}| = p^0)$$

$$\vec{M}^{(1)} = \left(\lambda \frac{p^1}{p+p^3} , \lambda \frac{p^2}{p+p^3} , \lambda\right)$$

$$\vec{N}^{(1)} = \left(-\lambda \frac{p^2}{p+p^3} , \lambda \frac{p_1}{p+p^3} , 0\right) . \tag{1.8}$$

Choose two coordinate systems on the cone:

$$S \text{ system: } p^0 = e^u , p^1 = e^u \sin\vartheta \cos\varphi , p^2 = e^u \sin\vartheta \sin\varphi , p^3 = e^u \cos\vartheta$$

$$\tag{1.9}$$

$$C \text{ system: } p^0 = e^{u'} \mathrm{ch}\beta , p^1 = e^{u'} \cos\varphi , p^2 = e^{u'} \sin\varphi , p^3 = -e^{u'} \mathrm{sh}\beta .$$

In terms of these parameters the Casimir operators are

$$\vec{J}^2 = \frac{1}{4}\left(\frac{\partial^2}{\partial a^2} + 2(1-\lambda)\frac{\partial}{\partial a} + \lambda^2 - 2\lambda\right), \quad \vec{K}^2 = \frac{1}{4}\left(\frac{\partial^2}{\partial a^2} + 2(1+\lambda)\frac{\partial}{\partial a} + \lambda^2 + 2\lambda\right) \tag{1.10}$$

(The same expressions are valid for the C system with the substitution $a \to a'$).

The eigenfunction is: $e^{-(1+i\sigma)a}$, provided $\lambda = j_0$. (The other solution is connected with the equivalent representation $(-j_0, -\sigma)$). The remaining operators we need, are:

$$\vec{M}^2 = \vec{M}^{(0)2} + \frac{2\lambda\mu}{1+\cos\vartheta} \tag{1.11}$$

$$M_3 = \frac{1}{i}\frac{\partial}{\partial\varphi} + \lambda \tag{1.12}$$

$$N_3 = -\frac{1}{i}\frac{\partial}{\partial\beta}. \tag{1.13}$$

Using (1.6), (1.11), (1.12) we find the normalized eigenfunctions in $O(3)$ basis : [*)]

$$\begin{aligned}\langle a\,\vartheta\varphi | l\mu\rangle &= \sqrt{2\pi(2l+1)}\, e^{-(1+i\sigma)a} e^{i\varphi(\mu-\lambda)} d^{l}_{\lambda\mu}(\vartheta) = \\ &= \sqrt{2\pi(2l+1)}\, e^{-(1+i\sigma)a}\, D^{l}_{\lambda\mu}(\varphi,\vartheta,-\varphi) \quad (\lambda \equiv j_0)\end{aligned} \tag{1.14}$$

*) The scalar product is defined by

$$\langle f|g\rangle = \frac{1}{(2\pi)^3}\int \frac{d^3p}{2p^0}\langle p|f\rangle^* \langle p|g\rangle.$$

In a similar way we get the spinor basis in the form: [*)]

$$\langle \alpha' \beta \varphi | \mu \nu \rangle = \eta \sqrt{2}\, e^{-(1+i\sigma)\alpha'} e^{i(\mu-\lambda)\varphi} e^{-i\nu\beta} \quad (\lambda \equiv j_0) \tag{1.15}$$

Here η is a phase factor:

$$\eta = \begin{cases} 2^{i\sigma} \eta_0 & \text{if } j_0 = \lambda \geq \mu;\ \lambda+\mu \geq 0 \text{ or } \lambda+\mu < 0 \\ \\ 2^{i\sigma} \eta_0 \dfrac{\sin\pi(j-m)}{\sin\pi(j^*-m^*)} & \text{if } \mu \geq \lambda = j_0,\ \lambda+\mu \geq 0, \text{ or } \lambda+\mu < 0 \end{cases} \tag{1.16}$$

with

$$\eta_0 = \Gamma \begin{bmatrix} -j+j^*+\ell+1, & j-m+1, & j+m+1 \\ j-j^*+\ell+1, & j^*-m^*+1, & j^*+m^*+1 \end{bmatrix}^{1/2},$$

(j and m are defined by (1.2) and (1.5)).
In what follows we shall consider 4 cases according to the signs of $\lambda+\mu$ and $\lambda-\mu$. Inequalities $\lambda+\mu \leq 0$, $\lambda+\mu < 0$ in eq. (1.16) merely indicate that the phase factors in these cases coincide.

2). The Overlap Coefficients

After performing an integration we obtain the overlap coefficients in the form : [**)]

---

*) $$\Gamma \begin{bmatrix} a, b, \ldots x \\ u, v, \ldots z \end{bmatrix} = \frac{\Gamma(a)\Gamma(b)\ldots\Gamma(x)}{\Gamma(u)\Gamma(v)\ldots\Gamma(z)}$$

**) In what follows we shall omit the trivial $\delta(\sigma'-\sigma)$ factor.

$$\langle \mu\nu | \ell\mu \rangle = \eta^* \sqrt{\frac{2\ell+1}{4\pi}}\, \Gamma \begin{bmatrix} \ell+\lambda'+1, & \ell-\mu'+1 \\ \ell-\lambda'+1, & \ell+\mu'+1 \end{bmatrix}^{1/2} \cdot$$

$$\Gamma \begin{bmatrix} \frac{1}{2}(\lambda'-1+i\sigma)+1-\frac{\mu'+i\nu}{2}, & \frac{1}{2}(\lambda'-1+i\sigma)+1+\frac{\mu'+i\nu}{2} \\ \lambda'-\mu'+1, & \lambda'+i\sigma+1 \end{bmatrix} {}_3F_2 \begin{bmatrix} -\ell+\lambda', \ \ell+\lambda'+1, \ \frac{1}{2}(\lambda'-1+i\sigma)+1-\frac{\mu'+i\nu}{2} \\ \lambda'-\mu'+1, \ \lambda'+i\sigma+1 \end{bmatrix}, \qquad (2.1)$$

where

$$\lambda' = \tfrac{1}{2}\left(|\lambda+\mu| + |\lambda-\mu|\right), \quad \mu' = \tfrac{1}{2}\left(|\lambda+\mu| - |\lambda-\mu|\right). \qquad (2.2)$$

and ${}_3F_2$ is the generalized hypergeometric function of unit argument. No problem of the convergence arises since the ${}_3F_2$ function entering eq. (2.1) really is a terminating series.

According to eq. (2.2) we have 4 cases, namely:

1) $j_0 = \lambda \geq \mu, \ \lambda+\mu \geq 0$. Then eq. (2.1) takes the form:

$$\langle \mu\nu | \ell\mu \rangle_1 = \eta^* \sqrt{\frac{2\ell+1}{4\pi}}\, \Gamma \begin{bmatrix} j+j^*+\ell+2, & \ell-\mu+1 \\ \ell-j-j^*, & \ell+\mu+1 \end{bmatrix}^{1/2} \cdot$$

$$\Gamma \begin{bmatrix} j-m+1, & j+m+1 \\ j+j^*-\mu+2, & 2j+2 \end{bmatrix} {}_3F_2 \begin{bmatrix} j+j^*-\ell+1, \ j+j^*+\ell+2, \ j-m+1 \\ j+j^*-\mu+2, \ 2j+2 \end{bmatrix}. \qquad (2.3)$$

2) $\mu \geq \lambda = j_0, \quad \lambda+\mu \geq 0$

$$\langle \mu\nu | \ell\mu \rangle_2 = \eta^* \sqrt{\frac{2\ell+1}{4\pi}}\ \Gamma\begin{bmatrix} -j-j^*+\ell, & \ell+\mu+1 \\ j+j^*+\ell+2, & \ell-\mu+1 \end{bmatrix}^{1/2} \cdot$$

$$\Gamma\begin{bmatrix} -j^*+m^*, & j+m+1 \\ -j-j^*+\mu, & j-j^*+\mu+1 \end{bmatrix} {}_3F_2\begin{bmatrix} -\ell+\mu, \ell+\mu+1, -j^*+m^* \\ -j-j^*+\mu, j-j^*+\mu+1 \end{bmatrix} . \tag{2.4}$$

3) $\lambda = j_0 > \mu, \quad \lambda+\mu < 0$

$$\langle \mu\nu | \ell\mu \rangle_3 = \eta^* \sqrt{\frac{2\ell+1}{4\pi}}\ \Gamma\begin{bmatrix} j+j^*+\ell+2, & \ell-\mu+1 \\ -j-j^*+\ell, & \ell+\mu+1 \end{bmatrix}^{1/2} \cdot$$

$$\Gamma\begin{bmatrix} j-m+1, & -j^*-m^* \\ j+j^*-\mu+2, & j-j^*-\mu+1 \end{bmatrix} {}_3F_2\begin{bmatrix} -\ell-\mu, \ell-\mu+1, j-m+1 \\ j+j^*-\mu+2, j-j^*-\mu+1 \end{bmatrix} \tag{2.5}$$

4) $j_0 = \lambda < \mu, \quad \lambda+\mu < 0$

$$\langle \mu\nu | \ell\mu \rangle_4 = \eta^* \sqrt{\frac{2\ell+1}{4\pi}}\ \Gamma\begin{bmatrix} -j-j^*+\ell, & \ell+\mu+1 \\ j+j^*+\ell+2, & \ell-\mu+1 \end{bmatrix}^{1/2} \cdot$$

$$\Gamma\begin{bmatrix} -j^*+m^*, & -j^*-m^* \\ -j-j^*+\mu, & -2j^* \end{bmatrix} {}_3F_2\begin{bmatrix} -j-j^*-\ell-1, -j-j^*+\ell, -j^*+m^* \\ -j-j^*+\mu, -2j^* \end{bmatrix} \tag{2.6}$$

There exist two fundamental relations between the $_3F_2$ hypergeometric functions of unit argument [8] . From the three-term relation: (see Appendix)

$$\frac{\sin \pi\beta_{45} F_p(0;45)}{\Gamma(\alpha_{012}, \alpha_{013}, \alpha_{023})} + \frac{\sin\pi\beta_{50} F_p(4;05)}{\Gamma(\alpha_{124}, \alpha_{134}, \alpha_{234})} + \frac{\sin\pi\beta_{04} F_p(5)}{\Gamma(\alpha_{125}, \alpha_{135}, \alpha_{235})} \qquad (2.7)$$

we obtain:

$$\langle \mu\nu | \ell\mu \rangle_1 = \langle \mu\nu | \ell\mu \rangle_2 \ .$$

In a similar way the two-term relation

$$F_p(0;45) = F_p(0, 14)$$

yields

$$\langle \mu\nu | \ell\mu \rangle_1 = \langle \mu\nu | \ell\mu \rangle_3 \ , \ \langle \mu\nu | \ell\mu \rangle_2 = \langle \mu\nu | \ell\mu \rangle_4$$

and thus the overlap coefficients (2.3),(2.4),(2.5),(2.6) are equal. It is worth mentioning that for an arbitrary choice of the phase the overlap coefficients 1,2,3,4 differ by a phase factor. We have chosen η (see eq. (1.16)) in such a way that the equality of the $\langle \mu\nu | \ell\mu \rangle_{1,2,3,4}$ should hold.

3) Relation to the Wigner Coefficients

One of the possible definitions of the Wigner coefficients of the three-dimensional real rotation group is [9]:

$$C(j_1 j_2 j_3; m_1 m_2) = \sqrt{2j_3+1}\ \Gamma\begin{bmatrix} j_1+m_1+1,\ j_2-m_2+1,\ j_3-m_3+1,\ j_3+m_3+1 \\ j_1-m_1+1,\ j_2+m_2+1,\ j_1+j_2-j_3+1,\ j_1+j_2+j_3+2 \end{bmatrix}^{1/2} .$$

$$\Gamma(j_1-j_2+j_3+1,\ -j_1+j_2+j_3+1)^{1/2} \frac{1}{\Gamma(j_3-j_2+m_1+1,\ j_3-j_1-m_2+1)} . \qquad (3.1)$$

$$\cdot\ {}_3F_2\begin{bmatrix} -j_1-j_2+j_3,\ -j_1+m_1,\ -j_2-m_2 \\ -j_1+j_3-m_2+1,\ -j_2+j_3+m_1+1 \end{bmatrix} .$$

Using the relation [8] :

$$\frac{\sin\pi\beta_{23}}{\pi\,\Gamma(\alpha_{023})} F_p(0;45) = \frac{F_n(2)}{\Gamma(\alpha_{134},\alpha_{135},\alpha_{345})} - \frac{F_n(3;24)}{\Gamma(\alpha_{124},\alpha_{125},\alpha_{245})}$$

(3.1) can be written in the form

$$C(j_1 j_2 j_3; m_1 m_2) = e^{\mp i\pi(j_2+m_2)}\,\Gamma\begin{bmatrix} j_1+m_1+1,\ j_1-m_1+1,\ j_3-m_3+1 \\ j_2+m_2+1,\ j_2-m_2+1,\ j_3+m_3+1 \end{bmatrix}^{1/2} .$$

$$\Gamma\begin{bmatrix} j_1+j_2-j_3+1,\ j_1-j_2+j_3+1,\ j_1+j_2+j_3+2 \\ -j_1+j_2+j_3+1 \end{bmatrix}^{1/2} \frac{1}{\Gamma(j_1-j_2-m_3+1,\ 2j_1+2)} . \qquad (3.2)$$

$${}_3F_2\begin{bmatrix} j_1-j_2-j_3,\ j_1-j_2+j_3+1,\ j_1-m_1+1 \\ j_1-j_2-m_3+1,\ 2j_1+2 \end{bmatrix} .$$

This form of the Wigner coefficients will be complexified with the prescription: $\mp$ for $\operatorname{Im}(j_2+m_2) \gtrless 0$ .Namely, let us introduce the notations:

$$
\begin{array}{ll}
j_1 \equiv j = \frac{1}{2}(j_0 - 1 + i\sigma) & m_1 \equiv m = \frac{1}{2}(\mu + i\nu) \\
j_2 \equiv -j^* - 1 = \frac{1}{2}(-j_0 - 1 + i\sigma) & m_2 \equiv m^* = \frac{1}{2}(\mu - i\nu) \\
j_3 \equiv \ell & m_3 \equiv \mu
\end{array}
\tag{3.3}
$$

$$
(\mu, j_0 = 0, \pm\tfrac{1}{2}, \pm 1, \ldots, \; 0 \le \sigma < \infty, \; -\infty < \nu < \infty, \; \ell = 0, \tfrac{1}{2}, 1, \ldots).
$$

Then comparing (3.2) with the overlap coefficient (2.3) we find:

$$
\langle \mu\nu | \ell\mu \rangle = e^{\pm i\pi(j_2+m_2)} \frac{1}{2} \left[\frac{\sin\pi(-j_1 - j_2 + j_3)}{\sin\pi(j_2+m_2)\sin\pi(j_2-m_2)} \right]^{1/2} C(j_1 j_2 j_3, m_1 m_2) .
$$

Thus the overlap coefficients are proportional to the complexified Wigner coefficients defined in (3.2). This result is quite clear, since the spinor basis is the eigenstate of $\vec{J}^2$, J_3 ; $\vec{K}^2$, K_3 and we want to obtain the angular momentum basis which is the eigenstate of $(\vec{J}+\vec{K})^2 = \vec{M}^2$ and $J_3 + K_3 = M_3$. It is to be emphasized that eq. (3.2) we do not consider as the definition of the Wigner coefficients for arbitrary complex values of the parameters, since it would require further considerations. Eq. (3.2) has the sense discussed above and it is defined for the values of parameters given by eq. (3.3).

In view of the relations between the ${}_3F_2$ functions the overlap coefficients possess a number of symmetry properties which are the complex counterparts of those of the Wigner coefficients. Here we mention that as a consequence of $|j_0| \leq \ell$ we have the triangle inequality for complex angular momenta: $|j_1 - j_2| \leq j_3$. Permutation of 1,2,3, of course, is not allowed now.

4). Matrix Elements of the Boost Operator in Angular Momentum Basis

The overlap coefficients make it possible to obtain an expression for the boost function $d^{j_0\sigma}_{\ell'\ell\mu}(\alpha)$ which is defined by

$$d^{j_0\sigma}_{\ell\ell'\mu}(\alpha) = \langle \ell'\mu | e^{i\alpha N_3} | \ell\mu \rangle .$$

Since the spinor basis is an eigenstate of N_3 we get the integral representation

$$d^{j_0\sigma}_{\ell\ell'\mu}(\alpha) = \int_{-\infty}^{\infty} d\nu\, e^{i\alpha\nu} \langle \mu\nu | \ell\mu \rangle^* \langle \mu\nu | \ell'\mu \rangle . \qquad (4.1)$$

The relation of eq. (4.1) to the O(4) group will be discussed elsewhere by J.Smorodinsky and G.I. Shepelov. As it is seen from (2.3) the integrand of (4.1) has four types of poles, on the complex ν plane, namely

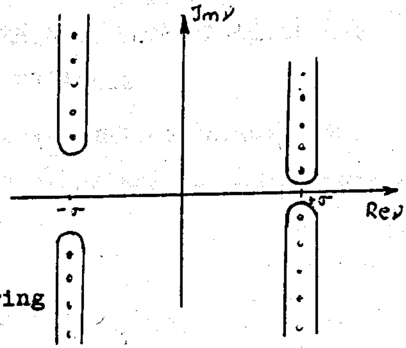
a) $\nu = i(j_0+1+\mu+2k)-\sigma$

b) $\nu = i(-j_0-1+\mu-2k)+\sigma$

c) $\nu = i(-j_0-1-\mu-2k)-\sigma$

d) $\nu = i(j_0+1-\mu+2k)+\sigma \qquad (k=0,1,2,\ldots).$

In the case $j_0 \geq \mu \geq 0$ the situation of the poles is shown in the figure.



Before closing the contour of integration the asymptotic behaviour of the integrand for large ν is required. We find for the overlap coefficients the following asymptotic formula:

$$\langle \mu\nu | l\mu \rangle = \sqrt{(2l+1)\pi}\,\frac{1}{\Gamma(2j+2,\ j+j^*-\mu+2)}\,\Gamma\begin{bmatrix} j+j^*+l+2,\ l-\mu+1 \\ -j-j^*+l,\ l+\mu+1 \end{bmatrix}\cdot$$

$$\cdot\, {}_2F_2\begin{bmatrix} j+j^*-l+1,\ j+j^*+l+2 \\ 2j+2,\ j+j^*-\mu+2 \end{bmatrix};1\Big]\left(\frac{i\nu}{2}\right)^{2j+1+l-j_0} e^{\mp\frac{\pi\nu}{2}}\,(1+o(1))$$

$$(\mathrm{Re}\,\nu \gtrless 0).$$

(We have omitted an irrelevant phase factor).

Then the leading term of the integrand is $e^{-\pi|\mathrm{Re}\,\nu|-\alpha\,\mathrm{Im}\,\nu}$ so for $\alpha>0$ the contour can be closed on the upper half plane. In the case $j_0 \geq \mu \geq 0$ the contour picks up the poles of the a) and d) types only, while in the remaining cases we loose some of the first poles of the a) and d) types and we have to include a finite number of poles of the c) and b) types which are situated on the upper half-plane. Finally, we obtain:

$$\frac{1}{2\pi i} d^{j_0\sigma}_{\ell\ell'\mu}(\alpha) = \sum_{k=0}^{\infty} (-1)^k \frac{(j_0+\mu+k)!}{k!} \left. \frac{\langle \mu\nu | \ell'\mu \rangle \langle \mu\nu | \ell\mu \rangle^*}{\Gamma(j+m+1)\Gamma(j^*+m^*+1)} \right|_{\nu=\nu_k} e^{i\alpha\nu_k} +$$

$$+ \sum_{k=0}^{\infty} (-1)^k \frac{(j_0-\mu+k)!}{k!} \left. \frac{\langle \mu\nu | \ell'\mu \rangle \langle \mu\nu | \ell\mu \rangle^*}{\Gamma(j-m+1)\Gamma(j^*-m^*+1)} \right|_{\nu=\nu_k'} e^{i\alpha\nu_k'}$$

where

$$\nu_k = i|j_0+1+\mu+2k| - \sigma$$

$$\nu_k' = i|j_0+1-\mu+2k| + \sigma .$$

For $\alpha < 0$ we can obtain an analogous formula by closing the contour on the lower half-plane, but it is easier to use the unitarity $d^{j_0\sigma}_{\ell\ell'\mu}(-\alpha) = d^{j_0\sigma}_{\ell'\ell\mu}(\alpha)^*$, which yields that we have to substitute simply $\nu_k \to \nu_k^*, \nu_k' \to \nu_k'^*$.

Acknowledgements

The author is deeply indebted to Professor J.Smorodinsky for helpful discussions and comments.

Appendix

The Thomae - Wipple Function

For the readers convenience we cite the definition of the F_ρ and F_n functions [8]. Consider 6 quantities r_i, for which

$$\sum_{i=0}^{5} r_i = 0$$

holds and define

$$\alpha_{\ell mn} = \tfrac{1}{2} + r_\ell + r_m + r_n, \quad \beta_{mn} = 1 + r_m - r_n .$$

Then the Thomae - Wipple functions are defined by

$$F\rho(\ell; mn) = \frac{1}{\Gamma(\alpha_{ghj}, \beta_{m\ell}, \beta_{n\ell})}\, {}_3F_2\left[\begin{matrix}\alpha_{gmn}, \alpha_{hmn}, \alpha_{jmn} \\ \beta_{m\ell}, \beta_{n\ell}\end{matrix}; 1\right]$$

$$Fn(\ell; mn) = \frac{1}{\Gamma(\alpha_{\ell mn}, \beta_{\ell m}, \beta_{\ell n})}\, {}_3F_2\left[\begin{matrix}\alpha_{\ell hj}, \alpha_{\ell gj}, \alpha_{\ell gh} \\ \beta_{\ell m}, \beta_{\ell n}\end{matrix}; 1\right],$$

where g, h, j represent those 3 numbers out of the six integers, 0, 1,...5, not already represented by ℓ, m and n.

For these functions two fundamental relations hold. According to the first one $F\rho(\ell; mn)$ functions with fixed ℓ and with different m, n are all equal.

The second fundamental theorem is a three-term relation. One of its possible forms is given by eq. (2.7). In our case the last term in eq. (2.7) becomes zero in view of the Γ -function in the denominator.

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Received by Publishing Department
on October 26, 1970.