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ЛАБОРАТОРИЯ ТЕОРЕТИЧЕСКОЙ ФИЗИКИ

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QUASI-POTENTIAL EQUATION FOR THE RELATIVISTIC HARMONIC OSCILLATOR

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§1. Introduction

With the development of the quasi-potential approach[1-3], particular interest is attracted to problems that can be considered as relativistic generalizations of the well-known exactly soluble problems of quantum mechanics. In the framework of the quasi-potential formalism, the relativistic Coulomb problem was investigated in refs.[4-6], and the problem of a relativistic particle in a potential well was considered in refs.[5,7]. Naturally the question arises: how to formulate the harmonic oscillator problem in the quasi-potential theory?

Provided the answer to this question is known, a relativistic quasi-potential version of the shell-model, for instance, can be developed, which in turn can be applied to the quark model.

The Hamiltonian of the three-dimensional isotropic oscillator has, in non-relativistic quantum mechanics, the well-known form:

$$H = \frac{\vec{p}^{\,2}}{2m} + \frac{m\omega^2 \vec{r}^{\,2}}{2} . \tag{1.1}$$

Because of the complete symmetry between momentum and coordinate operators in (1.1) it does not matter in which representation ($\vec{r}$ – or $\vec{p}$) one seeks the energy levels and wave functions of the oscillator.

If we choose the $\vec{p}$-representation, then the first term on the r.h.s. of eq. (1.1), the kinetic energy of the free non-relativistic particle, would be a c-number, whereas the second one, responsible for the interaction, would be proportional to the Laplace operator in $\vec{p}$-space:

$$\vec{r}^{\,2} = -\hbar^2 \frac{\partial^2}{\partial \vec{p}^{\,2}} . \qquad (1.2)$$

We would like to stress the fact that eq. (1.2) is the Casimir operator of group of motion of the three-dimensional Euclidean $\vec{p}$-space. This circumstance plays the main role in the relativistic generalization of the Hamiltonian (eq. (1.1)), which we will undertake below.

We recall first that in the quasi-potential picture we can treat the $\vec{p}$-space as a Lobachevsky - space /3/. The corresponding Laplace operator Δ_L is related to the square of the <u>relativistic</u> coordinate r^2 by the following equality (see ref. /3/):

$$\Delta_L = -\left(1 + \frac{m^2 c^2}{\hbar^2} \vec{r}^{\,2}\right). \qquad (1.3)$$

Taking into account the fact that the energy of the free relativistic particle is given by the relation

$$E = \sqrt{m^2 c^4 + \vec{p}^{\,2} c^2} , \qquad (1.4)$$

we could conjecture that the relativistic analogue of eq. (1.1) is:

$$H = \sqrt{m^2 c^4 + \vec{p}^2 c^2} - \frac{\omega^2 \hbar^2}{2 m c^2} \Delta_L . \qquad (1.5)$$

In spherical coordinates

$$\frac{p_0}{c} = m c \operatorname{ch} \chi$$

$$\vec{p} = mc \operatorname{sh} \chi \cdot \vec{n} \qquad (1.6)$$

$$(\vec{n} = (\sin\theta \cos\phi, \ \sin\theta \sin\phi, \ \cos\theta)) .$$

The Laplace-operator (1.3) has the form

$$\Delta_L = \frac{1}{\operatorname{sh}^2 \chi} \frac{\partial}{\partial \chi} (\operatorname{sh}^2 \chi \frac{\partial}{\partial \chi}) + \frac{\Delta_{\theta,\phi}}{\operatorname{sh}^2 \chi} , \qquad (1.7)$$

where

$$\Delta_{\theta,\phi} = \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} (\sin\theta \frac{\partial}{\partial\theta}) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} .$$

Thus the stationary quasi-potential Schroedinger-equation with the Hamiltonian (1.5) is written in the following way

$$\{ 2mc^2 \operatorname{ch}\chi - \frac{\omega^2 \hbar^2}{mc^2} [\frac{1}{\operatorname{sh}^2\chi} \frac{\partial}{\partial\chi} (\operatorname{sh}^2\chi \frac{\partial}{\partial\chi}) + \frac{\Delta_{\theta,\varphi}}{\operatorname{sh}^2\chi}] \} \Psi(\chi,\theta,\phi) = 2E\Psi(\chi,\theta,\phi) . \tag{1.8}$$

Note that in configuration space the equation (1.8) goes over to a differential-difference equation, because the free Hamiltonian $H_0 = \sqrt{m^2c^4 + \vec{p}^{\,2}c^2}$ in $\vec{r}$ space has the form/3/:

$$H_0 = [2 \operatorname{ch} i \frac{d}{dr} + \frac{2i}{r} \operatorname{sh} i \frac{d}{dr} - \frac{\Delta_{\theta,\phi}}{r^2} e^{i\frac{d}{dr}}] . \tag{1.9}$$

Clearly it is better to work with the differential equation than with the differential-difference one, because the theory of the latter is considerably less developed. In other words, it is reasonable to consider the problem of the relativistic oscillator described by the Hamiltonian (1.2) in the $\vec{p}$-representation. Separating in a standard way the variables in eq. (1.8), we arrive at the one-dimensional equation for the partial wave function:

$$[\frac{1}{\operatorname{sh}^2\chi} \frac{d}{d\chi} \operatorname{sh}^2\chi \frac{d}{d\chi} - \frac{\ell(\ell+1)}{\operatorname{sh}^2\chi} - \frac{4m^2c^4}{\omega^2\hbar^2} \operatorname{sh}^2\frac{\chi}{2} + \frac{2mc^2}{\omega^2 h^2} (E - mc^2)] \Psi_\ell(\chi) = 0 . \tag{1.10}$$

The investigation of equation (1.10) and its solutions is our main task in this paper.

§2 and §5. In §2 we formulate a modified procedure for the solution of the non-relativistic equation for the function $\Psi_\ell(p)$:

$$\left[\frac{1}{p^2}\frac{d}{dp}p^2\frac{d}{dp}-\frac{\ell(\ell+1)}{p^2}+\frac{2}{m\omega^2\hbar^2}\left(E_{nr}-\frac{p^2}{2m}\right)\right]\Psi_\ell(p)=0 \quad (1.11)$$

with boundary conditions

$$\Psi_\ell(0)<\infty \quad (1.12a)$$

$$\Psi_\ell(\infty)=0. \quad (1.12b)$$

This turns out to be useful in the subsequent analysis of the relativistic case. In §5 the quasi-potential equation with an oscillator type interaction, which has a form (1.8), is considered.

In the present work we do not consider any realistic physical applications (a separate work will be devoted to this problem), but we will concentrate on the methematical problems.

§2. Solution of the Non-Relativistic Problem

Let us transofrm in eq. (1.1) to the dimensionless quantities

$$\xi=\frac{p^2}{m\omega\hbar} \quad (2.1)$$
$$\lambda=\frac{E}{\omega\hbar}$$

and make the substitution:

$$\Psi_\ell (p) = \frac{U_\ell(\xi)}{\xi^{3/4}} . \tag{2.2}$$

For the function $U_\ell(\xi)$ we obtain the equation:

$$U''_\ell(\xi) - [-\frac{1}{4} + \frac{\lambda}{\xi} + \frac{\frac{1}{4} - s^2}{\xi^2}] U_\ell(\xi) = 0$$

$$(s \equiv \frac{2\ell + 1}{4}), \tag{2.3}$$

with boundary conditions

$$\xi^{-1/4} U_\ell(\xi) |_{\xi=0} = 0 \tag{2.4a}$$

$$U_\ell(\xi) < \infty \quad \text{if} \quad \xi \neq 0 . \tag{2.4b}$$

The equation (2.3) coincides with the equation for the Whittaker functions/9,10/ so that we can write its general solution in the form:

$$U_\ell(\xi) = C_1 M_{\lambda, s}(\xi) + C_2 M_{\lambda, -s}(\xi), \tag{2.5}$$

or

$$U_\ell(\xi) = B_1 W_{\lambda, s}(\xi) + B_2 W_{-\lambda, s}(-\xi), \tag{2.6}$$

where $M_{\lambda, s}(\xi)$ and $W_{\lambda, s}$ are the Whittaker functions.

Then, taking into account the fact that

$$M_{\lambda,-s}(\xi) = \xi^{\frac{-2\ell+1}{4}} e^{-\frac{\xi}{2}} \Phi(-\frac{2\ell-1}{4} - \lambda ; \frac{1}{2} - \ell ; \xi), \tag{2.7}$$

and the boundary condition (2.4a), we obtain that $C_2 = 0$. Thus[10],

$$U_\ell(\xi) = C_1 M_{\lambda, s}(\xi) =$$

$$= C_1 \left[\frac{\Gamma(\ell + 1/2) e^{-i\pi(\lambda - s - 1/2)}}{\Gamma(\lambda + s + 1/2)} W_{\lambda, s}(\xi) + \frac{\Gamma(\ell + 1/2) e^{-i\pi\lambda}}{\Gamma(1/2 + s - \lambda)} W_{-\lambda, s}(\xi) \right]. \tag{2.8}$$

Because of

$$W_{-\lambda, s}(-\xi) \approx \xi^{\lambda} e^{\frac{\xi}{2}} (1 + O(\xi^{-1})) \tag{2.9}$$

when $\xi \to \infty$ the boundary condition (2.4b) is equivalent to the requirement

$$\frac{1}{\Gamma(\frac{1}{2} + s - \lambda)} = 0 . \tag{2.10}$$

Eq. (2.10), because of the well-known properties of the Γ-function[9], gives the quantization rule for the energy levels of the non-relativistic oscillator:

$$E_n = \omega \hbar (2n + \ell + \frac{3}{2}), \tag{2.11}$$

$$n = 0, 1, 2, \ldots$$

The eigenfunctions corresponding to the spectrum (2.11) have the form:

$$U_\ell(\xi) = C_{n,\ell} W_{n+\frac{1}{2}(\ell+\frac{3}{2}), \frac{2\ell+1}{4}}(\xi) =$$

$$= D_{n,\ell} M_{n+\frac{1}{2}(\ell+\frac{3}{2}), \frac{2\ell+1}{4}}(\xi), \tag{2.12}$$

where $C_{n,\ell}$ and $D_{n,\ell}$ are normalization constants.

If we start with relation (2.6) and first take into account the boundary condition (2.4b), we would arrive at the equality $B_2 = 0$.

Then, using the representation

$$W_{\lambda,s}(\xi) = \frac{\Gamma(-2s)}{\Gamma(\frac{1}{2} - s - \lambda)} M_{\lambda,s}(\xi) + \frac{\Gamma(2s)}{\Gamma(\frac{1}{2} + s - \lambda)} M_{\lambda,-s}(-\xi), \tag{2.13}$$

we would obtain the energy spectrum (2.11)' as a consequence of the boundary condition (2.4a). That is, in the solutions (2.5) - (2.6) considered above in terms of Whittaker functions, the boundary conditions at the origin and at infinity are on an equal footing in the derivation of the energy levels. On the contrary, in the usual approach[8] the boundary conditions at the origin and at infinity have different roles. The boundary condition at infinity leads to two different energy quantization rules, one of which, namely (2.11), is then chosen when taking into account the condition at the origin[x].

The solutions of equation (1.11), which satisfy the normalization condition

$$\int_0^\infty \Psi_{n\ell}(p)\,\Psi_{n'\ell}(p)\,p^2\,dp = \delta_{nn'}, \tag{2.14}$$

in terms of Whittaker functions is written in the following manner

$$\Psi_{n\ell}(p) = \frac{1}{(m\omega\hbar\xi)^{3/4}}\sqrt{\frac{2(-1)^{n+\ell+1}\,\Gamma(-n-\ell-1/2)}{\pi\,n!}}\;W_{n+\frac{1}{2}(\ell+\frac{3}{2}),\,\frac{2\ell+1}{4}}(\xi)\,. \tag{2.15}$$

§3. The Relativistic Problem

Let us return now to equation (1.10). Putting:

$$\Psi_\ell(\chi) = \frac{\eta_\ell(\chi)}{(mc\,\mathrm{sh}\,\chi)^{3/2}}, \tag{3.1}$$

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[x] The boundary condition at the origin itself gives no formula for the energy levels.

we obtain the equation for the new function $\eta_\ell(\chi)$:

$$\{\eta_\ell''(\chi) - \mathrm{cth}\,\chi \cdot \eta_\ell'(\chi) + [2\frac{mc^2}{\omega^2\hbar^2}(E - mc^2) - \frac{3}{2} +$$

$$+ \frac{3}{4}\,\mathrm{ch}^2\chi - \frac{4m^2c^4}{\omega^2\hbar^2}\,\mathrm{sh}^2\frac{\chi}{2} - \frac{\ell(\ell+1)}{\mathrm{sh}^2\chi}]\;\eta_\ell(\chi)\}=0. \tag{3.2}$$

It is convenient to introduce a dimensionless variable

$$\xi = \frac{4mc^2}{\omega\hbar}\,\mathrm{sh}^2\frac{\chi}{2},$$

which is proportional to the relativistic binding energy

$$W = 2(E - mc^2) = \omega\hbar\xi\ . \tag{3.4}$$

In the non-relativistic limit, obviously,

$$\xi \to \frac{p^2}{m\omega\hbar}$$

(compare to (2.1)).

Then, introducing the notations x/

$$2\frac{mc^2}{\omega\hbar} \equiv k,$$

$$\frac{2\ell+1}{4} = s \, ,$$

$$\frac{E - mc^2}{2\omega\hbar} - \frac{3}{8k} \equiv \lambda \, , \tag{3.5}$$

we obtain finally, instead of (3.2):

$$\eta''_\ell(\xi) + \left[- \frac{\lambda}{4(1+\frac{\xi}{2k})} + \frac{\lambda}{\xi(1+\frac{\xi}{2k})} + \frac{\frac{1}{4} - s^2}{\xi^2(1+\frac{\xi}{2k})} \right] \eta_\ell(\xi) = 0 \tag{3.6}$$

the boundary conditions for the functions $\eta_\ell(\xi)$ being imposed now in the form:

$$\xi^{-1/4} \eta_\ell(\xi) \big|_{\xi=0} = 0 \tag{3.7a}$$

$$\eta_\ell(\xi) < \infty \quad \text{if} \quad \xi \neq 0 \, . \tag{3.7b}$$

Equation (3.6) has three singular points: regular points at $\xi = 0$ and at $\xi = -2k$, and an irregular point at $\xi = \infty$. From the theory of differential equations /11/, it is well-known that it is impossible to express the solutions of such equations in terms of elementary functions. If one seeks a solution in the form of an expansion in terms of known functions

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x/ In the non-relativistic region

$$\lambda \to \frac{2E_{nr}}{\omega\hbar} = \lambda_{nr}$$

$$\eta(\xi) = \sum a_n(\lambda)\phi_n(\xi), \tag{3.8}$$

then, for the coefficients $a_n(\lambda)$ one would obtain recurrence relations containing at least three terms. Such a situation exists, for instance, in the case of Mathieu equation, the equation for spheroidal functions.

The complexity of the resulting recurrence relations depends greatly on the appropriate choice of the functions $\phi_n(\xi)$. The solutions $a_n(\lambda)$ of these relations have the property of ensuring the convergence of the expansion (3.8) only for definite (eigen - !) values of the parameter λ.

In the non-relativistic limit $(k \to \infty)$ the boundary-value problem (3.6-3.7) goes over into the problem (2.3)-(2.4) considered above. The points $\xi = 0$ and $\xi = \infty$ are still also singular for the "degenerate" equation. Keeping in mind this fact in order to solve the relativistic problem, we choose the Whittaker functions as the basis $\phi_n(\xi)$ i.e. the solutions of the non-relativistic problem. In other words, the expansion formula (3.8) would have the form:

$$\eta_\ell(\xi) = \sum_{r=0}^{\infty} a_r W_{r+\frac{2\ell+3}{4},\frac{2\ell+1}{4}}(\xi). \tag{3.9}$$

Substituting (3.9) into (3.6) and taking into account (2.4), we obtain:

$$\sum_{r=0}^{\infty} a_r \left[(\lambda - \kappa) W_{\kappa,s}(\xi) + \frac{\lambda}{2k} \xi W_{\kappa,s}(\xi) - \right.$$

$$\left. - \frac{1}{8k} \xi^2 W_{\kappa,s}(\xi) + \frac{1}{k} \xi^2 W''_{\kappa,s}(\xi) + \frac{\xi^2}{4k^2} W''_{\kappa,s}(\xi) \right] = 0, \qquad (3.10)$$

where

$$\kappa = r + \frac{2\ell + 3}{4} = r + s + \frac{1}{2}. \qquad (3.11)$$

As is well-known, the Whittaker functions satisfy the following recurrence relations:

$$\xi W_{\kappa,s}(\xi) = 2\kappa W_{\kappa,s}(\xi) + W_{\kappa+1,s}(\xi) + [(\kappa - 1/2)^2 - s^2] W_{\kappa-1,s}(\xi), \qquad (3.12a)$$

$$\xi^2 W''_{\kappa,s}(\xi) - \frac{\xi^2 W_{\kappa,s}(\xi)}{8} = \frac{1}{8} [(\kappa - \frac{1}{2})^2 - s^2][(\kappa - \frac{3}{2})^2 - s^2] W_{\kappa-2,s}(\xi) -$$

$$- \frac{\kappa}{2} [(\kappa - \frac{1}{2})^2 - s^2] W_{\kappa-1,s}(\xi) + \frac{1}{4}(3s^2 - 5\kappa^2 - \frac{3}{4}) W_{\kappa,s}(\xi) - \qquad (3.12b)$$

$$- \frac{1}{2} W_{\kappa+1,s}(\xi) + \frac{1}{8} W_{\kappa+2,s}(\xi),$$

$$\xi^3 W''_{\kappa,s}(\xi) = \frac{1}{4}[(\kappa-1/2)^2-s^2][(\kappa-3/2)^2-s^2][(\kappa-5/2)^2-s^2]W_{\kappa-3,s}(\xi)+$$

$$+\frac{1}{4}[(\kappa-1/2)^2-s^2][(\kappa-3/2)^2-s^2](2\kappa-5)W_{\kappa-2,s}(\xi)+$$

$$+\frac{1}{4}[(\kappa-1/2)^2-s^2][s^2-\kappa^2-5\kappa+\frac{15}{4}]W_{\kappa-1,s}(\xi)- \qquad (3.12c)$$

$$-\frac{1}{4}[4\kappa(\kappa^2-s^2)-\kappa]W_{\kappa,s}(\xi)+\frac{1}{4}[s^2-\kappa^2+5\kappa+\frac{15}{4}]W_{\kappa+1,s}(\xi)+$$

$$+\frac{1}{4}(2\kappa+5)W_{\kappa+2,s}(\xi)+\frac{1}{4}W_{\kappa+3,s}(\xi).$$

Taking into account (3.12), we obtain the required recurrence relations for the coefficients α_r :

$$a_3^{(s)}\alpha_3+b_2^{(s)}\alpha_2+c_1^{(s)}\alpha_1+d_0^{(s)}\alpha_0=0;$$

$$a_4^{(s)}\alpha_4+b_3^{(s)}\alpha_3+c_2^{(s)}\alpha_2+d_1^{(s)}\alpha_1+e_0^{(s)}\alpha_0=0;$$

$$a_5^{(s)}\alpha_5+b_4^{(s)}\alpha_4+c_3^{(s)}\alpha_3+d_2^{(s)}\alpha_2+e_1^{(s)}\alpha_1+f_0^{(s)}\alpha_0=0;$$

$$a^{(s)}_{r+3}\,\alpha_{r+3}+b^{(s)}_{r+2}\,\alpha_{r+2}+c^{(s)}_{r+1}\,\alpha_{r+1}+d^{(s)}_{r}\,\alpha_{r}+ \tag{3.13}$$

$$+e^{(s)}_{r-1}\,\alpha_{r-1}+f^{(s)}_{r-2}\,\alpha_{r-2}+\frac{1}{16k^2}\,\alpha_{r-3}=0 \qquad (r \geq 3),$$

where

$$a^{(s)}_r=\frac{1}{16k}(r-2)(r-1)\,r\,(r-2+2s)(r-1+2s)(r+2s)\;;$$

$$b^{(s)}_r=\frac{(r-1)\,r(r-1+2s)(r+2s)}{8k}\left(1-\frac{r-2+s}{k}\right);$$

$$c^{(s)}_r=\frac{r(r+2s)}{2k}\left(\lambda-r-s-\frac{1}{2}-\frac{r^2+2rs+6r+6s-1}{8k}\right); \tag{3.14}$$

$$d^{(s)}_r=\lambda-\left(r+s+\frac{1}{2}\right)+\frac{\lambda}{k}\left(r+s+\frac{1}{2}\right)-\frac{1}{4k}(5r^2+2s^2+10rs+5r+5s+2)-$$

$$-\frac{r+s+1/2}{4k^2}(r^2+2sr+r+s);$$

$$e^{(s)}_r=\frac{1}{2k}\left(\lambda-1+\frac{6-r^2-2rs+4r+4s}{8k}\right);$$

$$f^{(s)}_r=\frac{1}{8k}\left(1+\frac{r+s+3}{k}\right); \qquad \left(s=\frac{2\ell+1}{4}\right).$$

The relations (3.13) together with the normalization condition for the wave function allow us, in principle, to determine the eigenvalues λ and the coefficients $a_r(\lambda)$ in the expansion (3.9). Let us calculate the values of λ with accuracy up to $1/k$. Putting

$$\lambda = \lambda_0 + \frac{\lambda'}{k}, \tag{3.15}$$

and substituting (3.15) into (3.13) we shall have, when comparing the coefficients of the terms of the same degree in k :

$$\lambda^0_{n,\ell} = n + s + \frac{1}{2} = n + \frac{1}{2}\left(\ell + \frac{3}{2}\right); \tag{3.16}$$

$$\lambda'_{n,\ell} = \frac{1}{4}\left[(n + s + 1/2)^2 - 3s^2 + \frac{3}{4}\right] =$$

$$= \frac{1}{32}\left[8n^2 - 4\ell^2 + 8n\ell + 12n + 9\right]. \tag{3.17}$$

On the other hand, when $k \to \infty$ it follows from (3.5) that

$$\lambda_{n,\ell} = \frac{E_{n,\ell} - mc^2}{2\omega\hbar} - \frac{3}{2k} \to \frac{E^{nr}_{n,\ell} + \Delta E^{(1)}}{2\omega\hbar} - \frac{3}{8k} =$$

$$= \lambda^0_{n,\ell} + \frac{\lambda'_{n\ell}}{k}. \tag{3.18}$$

We obtain from this relation the expression for the energy levels of the relativistic oscillator with accuracy up to $\sim \frac{1}{c^2}$

$$E_{n,\ell} = mc^2 + E^{(nr)}_{n,\ell} + \Delta E^{(1)}_{n\ell} , \qquad (3.19)$$

where

$$E^{nr}_{n\ell} = \omega\hbar\left(2n+\ell+\frac{3}{2}\right),$$

$$(3.20)$$

$$\Delta E^{(1)}_{n\ell} = \frac{\omega^2\hbar^2}{16\,mc}\left[\left(2n+\ell+\frac{3}{2}\right)^2 - 3\ell(\ell+1)+\frac{33}{4}\right].$$

As follows from (3.20), the relativistic correction removes the accidental degeneracy in angular momentum of the energy levels.

The formulae (3.19)-(3.20) can also be obtained in a different way. Namely, if we put (3.12) into (3.10) and retain in the expansion only the terms $W_{r+1/2+s,s}(\xi)$, the other terms can be neglected when $k\to\infty$. Then, keeping in mind (3.17), we get:

$$\sum_{r=0}^{\infty} a_r \left\{\lambda^0 - \kappa + \frac{1}{k}\left[\lambda^0\kappa + \lambda^{(1)} - \frac{1}{4}\left(5\kappa^2 - 3s^2 + \frac{3}{4}\right)\right]\right\} W_{\kappa,s}(\xi) = 0. \qquad (3.21)$$

Because of the independence of the functions $W_{\kappa,s}(\xi)$, each of the coefficients in expansion (3.21) is equal to zero, which leads us again to the formulae (3.19)-(3.20). This method of evaluating the eigen-functions is equivalent to perturbation theory.

§4. The s-Wave Equation

It is interesting to consider the equation (1.10) for $\ell = 0$ (the s-wave) separately, because its solutions are functions which are well-known in the mathematical literature as modified Matheu functions[12]. Note that the solution of the equation (1.10) for arbitrary ℓ could be treated as some adjont modified Mathieu functions[x].

Let us make in equation (1.10) for $\ell = 0$ the substitution of the unknown function

$$y(\chi) = \Psi_0(\chi)\, \mathrm{sh}\, \chi . \tag{4.1}$$

We shall obtain

$$\frac{d^2 y(\chi)}{d\chi^2} + \left[\frac{2mc^2}{\omega^2 \hbar^2} E - 1 - \frac{2m^2 c^2}{\omega^2 \hbar^2} \mathrm{ch}\, \chi\right] y(\chi) = 0 . \tag{4.2}$$

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[x] In ref.[13] the adjoint Mathieu functions are introduced on the basis of the equation for the spheroidal functions.

The non-relativistic equation corresponding to (4.2) has the form:

$$\frac{d^2y(p)}{dp^2} + [\frac{2E}{m\omega^2\hbar^2} - \frac{p^2}{m^2\omega^2\hbar^2}]y(p) = 0. \qquad (4.3)$$

(One obtains it from equation (1.11) for $\ell = 0$ after the substitution $y(p) = p\psi_0(p)$). Introducing into (4.2) the rew variable $x = \frac{\chi}{2}$ and putting

$$a(k) = 4(\frac{2mc^2}{\omega^2\hbar^2} E - 1) = 4(\frac{E}{\omega\hbar} k - 1), \qquad (4.4)$$

$$k = 2\frac{mc^2}{\omega\hbar},$$

we arrive at the equation

$$\frac{d^2y(x)}{dx^2} + (a - 2k^2 \operatorname{ch} 2x)y(x) = 0. \qquad (4.5)$$

The boundary conditions imposed on the function $y(x)$ have the following form

$$y(0) = 0 \qquad (4.6)$$

$$y(x) < \infty \quad \text{if} \quad x \neq 0. \qquad (4.7)$$

The investigation of the boundary-value problem (4.5)-(4.7) showed that its solution exists only for the values $a = a_{2n+1}(k^2)$ which satisfy the transcendental equation

$$a - 1 - q - \cfrac{q^2}{a - 9 - \cfrac{q^2}{a - 25 - \cdots \cdots \cfrac{}{\cdots \cdots a - (2r+1)^2 - \cfrac{q^2}{\cdots \cdots}}}} = 0. \tag{4.8}$$

The left-hand side of (4.8) is a continuous fraction. If $a = a_{2n+1}(k^2)$ the function $y(x)$ is expressed in terms of one of the modified Mathieu functions /9,12/

$$y(x) = Gek_{2n+1}(x, -k^2) =$$

$$= \frac{ce_{2n+1}(0,k^2)}{\pi k A_1^{2n+1}(k^2)} \, \mathrm{th}\,\chi \sum_{r=0}^{\infty} (-1)^r (2r+1) \cdot \tag{4.9}$$

$$\cdot A_{2r+1}^{2n+1}(k^2) K_{2r+1}(2k \,\mathrm{ch}\, x),$$

where $Ce_{2n+1}(x,k^2)$ is the Mathieu function of the second kind, and $K_{2r+1}(z)$ is the so-called McDonald function. The coefficients of the expansion (4.9) are defined by means of the recurrence relations:

$$(a-k^2-1)A_1^{2n+1} - k^2 A_3 = 0\,,$$

$$[a-(2r+1)^2]A_{2r+1}^{2n+1} - k^2(A_{2r+1}^{2n+1} + A_{2r+3}^{2n+1}) = 0\,. \tag{4.10}$$

In order to evaluate the relativistic corrections to the energy levels one needs to obtain asymptotic expressions for the parameters (4.10) when $k \to \infty$. Note that the asymptotic expansions for the eigenvalues of the modified Mathieu equation, which are given in the literature /9,11,12/, are wrong. The cause of this mistake is the fact that in the derivation of these expansions, symmetry properties of the eigen-values were used which are not valid for large values of k. Applying to the equation (4.5) the method developed in /14/, we shall have

$$a_n(k) \underset{k\to\infty}{\to} 2k^2 + 2(2n+1)k + \frac{2n^2+2n+1}{4}\,. \tag{4.13}$$

Then, taking into account that, according to (4.4)

$$a(k) \underset{k\to\infty}{\to} \frac{8mc^2}{\omega^2\hbar^2}(mc^2 + E^{nr} + \Delta E)\,, \tag{4.14}$$

where ΔE is the correction of order $\frac{1}{k}$ and equating (4.13) and (4.14) we get:

$$E^{nr} = \omega \hbar (2n + \frac{3}{2}),$$

$$\Delta E = \frac{\omega^2 \hbar^2}{16 m c^2} [(2n + \frac{3}{2})^2 + \frac{33}{4}]. \qquad (4.15)$$

Relations (4.15) are, obviously, a partucular case of (3.19) for $\ell = 0$.

§5. The Exactly Soluble Relativistic Problem

As is well known, the Hamiltonian (1.1) of the non-relativistic problem is invariant under transformations of the U(3) group. It can be shown that the requirement of U(3) symmetry in the case of the relativistic Hamiltonian leads us to a potential (in the system of units $\hbar = 2m = c = 1$):

$$V = \omega^2 (\Delta_{\theta,\phi} + r^{(2)}) e^{i \frac{d}{dr}} , \qquad (5.1)$$

where $\Delta_{\theta,\phi}$ is the angular part of the Laplace operator, and $r^{(2)}$ is the generalized function[x], which is defined by the formula /3/

[x] We prefer to introduce here the new terminology "generalized power function" instead of "generalized degree" used earlier /3/.

$$r^{(\lambda)} = i^{\lambda} \frac{\Gamma(-ir+\lambda)}{\Gamma(-ir)} . \qquad (5.2)$$

It is easy to show that in the non-relativistic limit

$$V(r) \to \omega^2 r^2 . \qquad (5.3)$$

The radial part of the Schroedinger equation with the potential (5.1) is written in the following way.

$$\left[2\,\mathrm{ch}\, i \frac{d}{dr} + \frac{\ell(\ell+1)}{r^{(2)}} e^{i\frac{d}{dr}} + \omega^2 (\ell(\ell+1) + r^{(2)}) e^{i\frac{d}{dr}} - 2E_q \right] \Psi_{q\ell}(r) = 0 .$$

As in the case of the non-relativistic oscillator, we shall seek the solution of equation (5.4) in the form:

$$\Psi_{q\ell}(r) = C(-r)^{(\ell+1)} M(r)\, \Omega_{n\ell}(r^2) . \qquad (5.5)$$

The factors $(-r)^{(\ell+1)}$ and

$$M(r) = \omega^{ir} \Gamma\left(ir + \frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{4}{\omega^2}}\right) \qquad (5.6)$$

are connected with the behaviour of the solution $\Psi_{q\ell}(r)$ at the points $r = 0$ and $r = \infty$ respectively, and $\Omega_{n\ell}(r^2)$ is a polynomial of the n-th degree (n - is the principal quantum number).

In the non-relativistic limit we have, obviously:

$$(-r)^{(\ell+1)} \to (-r)^{\ell+1}$$

$$M(r) \to e^{-\frac{\omega r^2}{2}} . \qquad (5.7)$$

Inserting (5.5), into eq. (5.4), we obtain the equation satisfied by the polynomials $\Omega_{n\ell}(r^2)$:

$$\{ A(r) e^{-i\frac{d}{dr}} + B(r) e^{i\frac{d}{dr}} - C(r) 2E \} \Omega_{\ell n}(r^2) = 0 , \qquad (5.8)$$

where the coefficients A, B, C are defined by means of the following relations:

$$A(r) = \omega^2 r^4 - i r^3 \omega^2 [2\ell + 1 + \sqrt{1 + \frac{4}{\omega^2}}] -$$

$$- r^2 [\omega^2 \ell(\ell+1) + \omega^2 (2\ell+1) \sqrt{1 + \frac{4}{\omega^2}} + 1] +$$

$$+ i r [\omega^2 \ell(\ell+1) \sqrt{1 + \frac{4}{\omega^2}} + (2\ell+1) + \ell(\ell+1)] ;$$

$$B(r) = -\omega^2 r^4 - 2i\omega^2 r^3 + r^2 [\omega^2 + \omega^2 \ell(\ell+1) - 1] \} - \qquad (5.9)$$

$$- i r [1 + \omega^2 \ell(\ell+1)] - \ell(\ell+1) ;$$

$$C(r) = - i r^3 \omega - \omega r^2 [\ell + \frac{1}{2} \sqrt{1 + \frac{4}{\omega^2}} - \frac{1}{2}] +$$

$$+ \frac{i r \ell \omega}{2} [\sqrt{1 + \frac{4}{\omega^2}} - 1] .$$

Straightforward, but rather lengthy calculations give the following exact formula for the energy levels of the relativistic oscillator

$$E_{\ell,n} = \omega\left(2n + \ell + \frac{3}{2} + \frac{1}{2}\sqrt{1 + \frac{4}{\omega^2}}\right). \qquad (5.10)$$

We see from eq. (5.10) that these levels differ from the corresponding non-relativistic levels in the value of the ground level oscillations, which deviates from the non-relativistic one by the quantity $\frac{\omega}{2}\sqrt{1+\frac{4}{\omega^2}}$. The degeneracy of the energy levels in formula (5.10) is just the same as the degeneracy of the energy levels (2.11) of the non-relativistic oscillator. This fact demonstrates that the relativistic oscillator described by equation (5.4) possesses a hidden "dynamical" U(3) symmetry. We note in this connection that it would be interesting to pass on the basis of equation (5.4) to the infinite component fields formalism, as was done in ref. /6/ for the Coulomb interaction.

Let us now write explicitly the polynomials $\Omega_{n\ell}(r^2)$ for $\ell = 0$ and several low values of n :

$$\Omega_{00}(r^2) = 1;$$

$$\Omega_{10}(r^2) = 1 - \frac{4}{5 + 3\sqrt{1+\frac{4}{\omega^2}}}\, r^2;$$

$$\Omega_{20}(r^2) = 1 - \frac{40\left(3+\sqrt{1+\frac{4}{\omega^2}}\right) r^2}{2\left(7+5\sqrt{1+\frac{4}{\omega^2}}\right)+5\left(5+3\sqrt{1+\frac{4}{\omega^2}}\right)\left(3+\sqrt{1+\frac{4}{\omega^2}}\right)} + \tag{5.11}$$

$$+ \frac{16\, r^4}{2\left(7+5\sqrt{1+\frac{4}{\omega^2}}\right)+5\left(5+3\sqrt{1+\frac{4}{\omega^2}}\right)\left(3+\sqrt{1+\frac{4}{\omega^2}}\right)} ;$$

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References

1. A.A. Logunov, A.N. Tavkhelidze. Nuovo Cimento., 29, 380 (1963).
   A.N. Tavkhelidze. Lectures on Quasipotential Method in Field Theory, Tata Institute of Fundamental Research., Bombay (1964). (The detailed list of references could be found there).
2. V.G. Kadyshevsky. Nuclear Physics., B6, 125 (1968).
   V.G. Kadyshevsky, M.D. Mateev. Nuovo Cimento., 55A, 275 (1968).

3. V.G. Kadyshevsky, R.M. Mir-Kasimov, N.B. Skachkov. Nuovo Cim., 55A, 233 (1968). V.G. Kadyshevsky, R.M. Mir-Kasimov, N.B. Skachkov. Yad. Fisika, 9, 212 (1969).
4. V.G. Kadyshevsky, R.M. Mir-Kasimov, M. Freeman. Yad. Fisika, 9, 646 (1969).
5. M. Freeman, M.D. Mateev, R.M. Mir-Kasimov. Nuclear Physics, B12, 197 (1969).
6. C. Itzykson, V.G. Kadyshevsky, I.T. Todorov. Preprint Insitute for Advanced Study, Princeton (1969) (Submitted to Phys. Rev).
7. V.G. Kadyshevsky, R.M. Mir-Kasimov, N.B. Skachkov. Yad. Fisika, 9, 462 (1969).
8. L.D. Landau, E.M. Lifshitz. Quantum Mechanics. Fizmatgiz, Moscow, 1963; A.S. Davydov. Quantum Mechanics. Fizmatgiz, Moscow, 1963.
9. H. Bateman, A. Erdelyi. Higher Transcendental Functions. 1-3, Nauka, Moscow, 1965-1967.
10. E.T. Whittaker, G.N. Watson. A Course of Modern Analysis. Fizmatgiz, Moscow, 1963.
11. Ph.M. Morse, H. Feshbach. Methods of Theoretical Physics. Inostrannaya Literatira, Moscow, 1958.
12. N.W. McLachlan. Theory and Application of Mathieu Functions. Inostrannaya Literatura, Moscow, 1953.
13. J. Meixner, F.W. Schafke. Mathieusche Functionen und Spharoidfunktionen, Springer, 1954; C. Flammer. Spheroidal Wave Functions. Stanford, 1957; J.A. Stratton, P.M. Morse, L.J. Chu, J.D.C. Little, F.J. Corbato. Spheroidal Wave Functions. (Chapman and Hall, Ltd., London, 1956).
14. R. Sips. Trans.Am.Math.Soc., 66, 93 (1949).

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