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ON THE STRUCTURE OF THE TEST FUNCTION ALGEBRA

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ON THE STRUCTURE OF THE TEST FUNCTION ALGEBRA

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1. Introduction

In a previous paper [1] the concept of an AO^* -algebra, which is a generalization of the concept of B^* -algebra was developed. Here we prove that the $*$ -algebra $R = C \oplus s^4 \oplus s^8 \oplus \ldots$ of test functions (Borchers algebra) equipped with a certain topology τ_∞ becomes an AO^* -algebra. The topology τ_∞ is weaker than the direct sum topology $\tau_\otimes$ (tensor product topology) in R and the multiplication $f, g \to f.g$ in R is jointly continuous with respect to τ_∞, what does not hold with respect to the topology $\tau_\otimes$. From the Theorem proved in this paper it could follow certain continuity properties of the representations of R. This problem is discussed in short in section 3. One result of this kind is given in [2]. In section 2 a special system of norms defining the topology of the Schwartz' space s is introduced and some propositions about different systems of norms for s are formulated without proofs. In section 3 the definition of the $*$ -algebra R is recalled, the two different topologies $\tau_\otimes$ and τ_∞ are defined and compared and the Theorem is stated. The proof of the

Theorem is given in the last three sections. For that in section 4 some special representations of R analog to the free field representations are introduced. The main part of the proof is concentrated in section 5, but the essential Lemma 1 is only proved in section 6.

2. Special Norms for the Schwartz' Space Topology

For our purpose we need some relations between equivalent systems of seminorms defining the topology in the Schwartz' space s^n of all quickly decreasing functions $f(\xi)$ in n variables $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$. For this we define

$$P^J = (1+\xi_1^2)^{J_1} \ldots (1+\xi_n^2)^{J_n}$$

$$D^L = \partial_{\xi_1}^{L_1} \ldots \partial_{\xi_n}^{L_n} \qquad (1)$$

$$N_i = 1 + \xi_i^2 - \partial_{\xi_i}^2$$

$$N = N_1 \ldots N_n,$$

where J, L are n-tubels of nonnegative integres. Further we put yet $|J| = \max(J_1, \ldots, J_n)$ and $\|f\|^2 = \int |f(\xi)|^2 d\xi$, the L_2-norm. Then it holds the following proposition, which we give without proof.

Proposition 2.1

The following three systems of norms in s^n

$$p_{J,L}(f) = \sup_{\xi} |P^J D^L f| \qquad \forall J, L$$

$$q_{J,L}(f) = \| P^J D^L f \| \qquad \forall J, L$$

$$\|f\|_{\nu} = \| N^{\nu} f \| \qquad \nu = 0,1,2,...$$

are equivalent and defining the well-known topology of the Schwartz' space s^n. Specially it holds the estimates

$$p_{J,L}(f) \leq K q_{J+1,L+1}(f)$$

$$q_{J,L}(f) \leq C \| N^{2|J|+|L|} f \| \qquad (3)$$

$$\|N^{\nu} f\| \leq \| N^{\nu+1} f \|,$$

where the constants C, K still depend on L, J and where $J+1 = (J_1+1,\dots,J_n+1)$.

A well-known fact is also the following one:

Proposition 2.2

If $n_{\nu}(f)$, $\nu = 0,1,2,...$ is an arbitrary system of norms defining the topology of s^n and μ_{ν} an arbitrary sequence of positive numbers, then one can find in s^n an element g such that

$$n_{\nu}(g) > \mu_{\nu}, \qquad \forall_{\nu}.$$

In special one can find an g such that

$\|N^{\nu} g\| > \mu_{\nu}$, $\forall \nu$ ..

Next we define the $*$-algebra R of test functions (Borchers algebra) /3,4/. For this we put $R_0 = C$, the field of complex numbers, and $R_n = s^{4n}$ and define

$$R = \bigoplus_n R_n \quad \text{(algebraic direct sum).}$$

The elements $a \subseteq R$ have from $a = \sum_{n \geq 0} a_n$ (formal sum), where $a_n = a_n(x_1, \ldots, x_n) \subseteq R_n$ are the components of a. For every a only a finite number of components differes from zero. It is $(x_1, \ldots, x_n) = (\xi_1, \ldots, \xi_{4n})$. The linear space R becomes a $*$-algebra if we define the multiplication by $a_n \cdot b_m = a_n \cdot b_m(x_1, \ldots, x_{n+m}) = a_n(x_1, \ldots, x_n) b_m(x_{n+1}, \ldots, x_{n+m})$, where $a_n \subseteq R_n$, $b_m \subseteq R_m$ and the $*$-operation by $(a_n^*)(x_1, \ldots, x_n) = \overline{a_n(x_n, \ldots, x_1)}$ for every $a_n \subseteq R_n$. By linearity the so defined operation can be uniquely extended to the whole R. In every R_n it is defined the linear operator $N = N_1 \ldots N_{4n}$ (1) and consequently we obtain a linear operator in R which we also denote by N. From the definition immediately follows

Proposition 2.3

N is a $*$-endomorphism of R, i.e. it holds

1. N is a linear operator in R
2. $N(a \cdot b) = Na \cdot Nb$
3. $N(a^*) = (Na)^*$.

For two homogeneous components $f \in R_m$, $g \in R_n$ it holds. $\|f \cdot g\|_{\nu} = \|N^{\nu} f \cdot N^{\nu} g\| = \|N^{\nu} f\| \, \|N^{\nu} g\| = \|f\|_{\nu} \|g\|_{\nu}$.

3. Topologies in R

In this section we regard different topologies τ in R such that $R[\tau]$ becomes a topological algebra /5/. Because every R_n is a linear topological space, there exists immediately a natural topology in R, namely the direct sum topology, which we denote by $\tau_\otimes$ (the topology of the tensor product). This topology for R is taken in /4/. It is defined by the system of all norms

$$\tau_\otimes : \|f\|_{(\gamma_n)(\nu_n)} = \sum_{n\geq 0} \gamma_n \|f_n\|_{\nu_n} \quad ; \; f = \sum_{n\geq 0} f_n \in R ,$$

where (γ_n) is an arbitrary sequence of positive numbers and $0 = \nu_0 \leq \nu_1 \leq \nu_2 \leq \ldots$ an arbitrary sequence of integers, $\|f_0\|_0 = |f_0|$. With respect to this topology R becomes a topological locally convex $*$-algebra, but the multiplication $f, g \to f \cdot g$ is not jointly continuous with respect to this topology. To demonstrate, we choose the norm

$$\|f\|_{(1)(n)} = \sum_{n\geq 0} \|f_n\|_n .$$

If the multiplication is jointly continuous in R, there would exist another norm $\| \; \|_{(\gamma_n)(\nu_n)}$ such that

$$\|f \cdot g\|_{(1)(n)} \leq \|f\|_{(\gamma_n)(\nu_n)} \|g\|_{(\gamma_n)(\nu_n)} \tag{2}$$

holds. Now we take an arbitrary sequence $g^{(r)} \neq 0$ of elements of R such that $g^{(r)} \in R_r$ for $r = 1,2,\ldots$ and put

$$a_r = \| g^{(r)} \|_{(\gamma_n)(\nu_n)} = \gamma_r \| g^{(r)} \|_{\nu_r} \quad ;$$

By Prop. 2.2 we can choose a $f \in R_1 = s^4$ such that $\| f \|_\nu = \| N^\nu f \| \geq \nu\, a_\nu \, \| g^{(\nu)} \|_\nu^{-1}$. If the inequality (2) is right, it would follow

$$a_r \| f \|_{(\gamma_n),(\nu_n)} \geq \| f \cdot g^{(r)} \|_{(1),(n)} = \| f \cdot g^{(r)} \|_{r+1} =$$

$$= \| N^{r+1} f \cdot N^{r+1} g^{(r)} \| = \| N^{r+1} f \| \, \| N^{r+1} g^{(r)} \| \geq$$

$$\geq \| N^r f \| \, \| N^r g^{(r)} \| \geq r \cdot a_r$$

for all r (see Prop. 2.3), but this is a contradiction for large r.

If one regards the convergent problem only for enumerable sequences, it is easy to prove that following statement holds:

Proposition 3.1

A sequense $f^{(r)}$, $r = 1,2,\ldots$, of elements of R converges to zero with respect to $r_\times$ if and only if

1. the components $f_n^{(r)}$ tend to zero in R_n,
2. above a certain degree n_0, which is independent of r, all components $f_n^{(r)}$ are equal to zero, i.e. $f_n^{(r)} = 0$ for $n \geq n_0$ and all r.

If $f^{(r)}$, $g^{(r)}$, $r = 1,2,\ldots$, are two sequences tending to zero, then also $f^{(r)} \cdot g^{(r)} = h^{(r)}$ converges to zero.

The last statement follows immediately from 1. and 2., which are well-known facts.

Now, as we shall see, the topology $\tau_{\otimes}$ is not determined by the properties expressed in Prop. 3.1. If one demands only the above called properties 1. and 2. of convergence of usual sequences, e.a. so it is done in /3/, then one can work with a weaker topology than $\tau_{\otimes}$. Such a topology is the topology τ_{∞} defined by the following system of norms

$$\tau_{\infty} : \| f \|_{(\gamma_n),\nu} = \sum_{n \geq 0} \gamma_n \| f_n \|_{\nu} \tag{3}$$

(γ_n) is an arbitrary sequence of positive numbers and $\nu = 1,2,\ldots$. In difference to the topology $\tau_{\otimes}$ (1) the degrees of the norms $\| f_n \|_{\nu}$ of the components f_n are fixed in (3).

<u>Proposition 3.2</u>

R equipped with the topology τ_{∞} becomes a complete locally convex topological $*$-algebra and the multiplication $f, g \to f \cdot g$ is jointly continuous in $R[\tau_{\infty}]$.

<u>Proof:</u> The completeness of $R[\tau_{\infty}]$ can be proved by standard considerations. That the multiplication is jountly continuous we see from

$$\| f \cdot g \|_{(\gamma_n),\nu} = \sum_{n \geq 0} \gamma_n \| \sum_{k+\ell=n} f_k \, g_\ell \|_{\nu}$$

$$\leq \sum_{n \geq 0} \gamma_n \sum_{k+\ell=n} \| f_k \|_{\nu} \, \| g_\ell \|_{\nu}$$

(see Prop. 2.3). Namely, let β_ν be a sequence of positive numbers satisfying the inequalities $\gamma_n \leq \beta_k \beta_\ell$; $k + \ell = n$ than it follows

$$\| f \cdot g \|_{(\gamma_n),\nu} \leq \| f \|_{(\beta_n),\nu} \; \| g \|_{(\beta_n),\nu} \; .$$

Now we formulate the main result of this paper:

<u>Theorem</u>

The locally convex topological $*$-algebra $R[\tau_\infty]$ is an AO^*-algebra, i.e. it is algebraically and topologically $*$-isomorphic to a $*$-algebra $(\mathfrak{A}, D)$ of operators equipped with the uniform topology τ_D.

The proof of the Theorem is given in the next sections. First we repeat the definitions of the concepts used in the formulation of the Theorem, which are introduced in /1/. A $*$-algebra of operators $(\mathfrak{A}, D)$, Op^*-algebra, is given by a unitary space D with the scallar product $< , >$ and an algebra $\mathfrak{A}$ of (unbounded) linear operators from D into D, such that for every $A \in \mathfrak{A}$ there exists an $A^+ \in \mathfrak{A}$ with $< \phi, A\psi > = < A^+\phi, \psi >$. We always assume $\mathfrak{A}$ to contain the unity operator $\mathbb{1}$. The uniform topology τ_D of $\mathfrak{A}$ is defined by all seminorms

$$\tau_D : \quad \| A \|_{\mathfrak{M}} = \sup_{\phi,\psi \in \mathfrak{M}} |< \phi, A\psi >| ,$$

where $\mathfrak{M} \subset D$ is an arbitrary $\mathfrak{A}$-bounded set, i.e. a set for which $\sup\{ \| B\phi \| ; \phi \in \mathfrak{M} \} < \infty$ for every $B \in \mathfrak{A}$. $\mathfrak{A}[\tau_D]$, i.e. $\mathfrak{A}$ equipped with the topology τ_D, becomes a locally convex $*$-algebra and is called O^*-algebra when it is complete.

A topological $*$-algebra, which is algebraically and topologically $*$-isomorphic to an 0^*-algebra we call $A0^*$-algebra. By reason of the considerations in /1/ one may expect that the concept of a 0^*-algebra ($A0^*$- algebra) is a suitable generalization of the concept of a C^*-algebra (B^*-algebra). That suggests the following

<u>Conjecture</u>: Let $f \to A(f)$ be a week continuous $*$-representation of the $A0^*$-algebra $R[\tau_\infty]$ with the domain $D \subset \mathcal{H}$ and let $\mathfrak{A}$ be the algebra of all operators $A(f)$, then the representation $f \to A(f)$ is also uniformly continuous, i.e. it is continuous as mapping from $R[\tau_\infty]$ onto $\mathfrak{A}[\tau_D]$. As is very well-known, such a statement is right for every B^*-algebra, even for every Banach $*$-algebra.

4. The "Free Field" Representations of R

We denote by $\mathcal{H}_n$ the Hilbert space $L_2(R^{4n})$, $n = 1,2,\ldots$, $\mathcal{H}_0 = C$ the field of the complex numbers, and construct

$$\mathcal{H} = \sum_n \dot{+} \mathcal{H}_n \qquad \text{(Hilbert direct sum)}$$

(1)

$$D = \sum_n \mathcal{H}_n \qquad \text{(algebraic direct sum)}$$

For every $f \in R_1$ we define the operators $C^-(f), C^+(f)$ on D by

$$C^{-}(f)\phi_n = \int f(x)\phi_n(x, x_1, \dots, x_{n-1})dx, \quad n \geq 1$$

$$= 0 \qquad \text{for } n = 0 \tag{2}$$

$$C^{+}(f)\phi_n = f(x_1)\phi(x_2, \dots, x_{n+1}).$$

For any $f_m(x_1, \dots, x_m) = f^1(x_1) f^2(x_2) \dots f^m(x_m)$ and a m-tuble $\epsilon = (\epsilon_1, \dots, \epsilon_m)$, $\epsilon_i = \pm 1$ we put

$$C^{\epsilon}(f_m) = \prod_{i=1}^{m} C^{+}(f^{(i)})^{\frac{1+\epsilon_i}{2}} C^{-}(f^{(i)})^{\frac{1-\epsilon_i}{2}}. \tag{3}$$

By linear continuation one defines $C^{\epsilon}(f_m)$ for every

$$f_m(x_1, \dots, x_m) = \sum_i f^{1,i}(x_1) \dots f^{m,i}(x_m). \tag{4}$$

Statement 1

It holds

$$\| C^{\epsilon}(f_m)\phi_n \| \leq \rho_m \| f_m \|_5 \| \phi_n \| \tag{5}$$

where $\phi_n \in \mathcal{H}_n$ and ρ_m is a constant not depending on n. Consequently, $C^{\epsilon}(f_m)$ does not depend on the concrete representation (4) of f_m and so it can be continued to the whole $R_m = s^{4m}$.

Proof: The action of $C^{\epsilon}(f_m)$ to ϕ_n is a combination of the following three fundamental actions, which we discuss for the special case : $f_2(x_1, x_2)$:

I. $\epsilon = (1,1)$: $C^{\epsilon}(f_2)\phi_n = f_2(x_1 x_2)\phi_n(x_3, \dots, x_{n+2})$.

It follows immediately

$$\| C^{\epsilon}(f_2)\phi_n \| = \| f_2 \| \| \phi_n \| .$$

II. $\epsilon = (1,-1)$: $C^{\epsilon}(f_2)\phi_n = \int f_2(x_1, x)\phi_n(x, x_2, \dots, x_n)\,dx$ (=0 for n = 0).

It holds

$$| C^{\epsilon}(f_2)\phi_n |^2 \leq \int | f_2(x_1, x) |^2 dx \int | \phi_n(x, x_2, \dots, x_n) |^2 dx$$

and consequently $\| C^{\epsilon}(f_2)\phi_n \| \leq \| f_2 \| \| \phi_n \|$.

III. $\epsilon = (-1,1)$: $C^{\epsilon}(f_2)\phi_n = \int f_2(x, x)\,dx\, \phi(x_1, \dots, x_n)$

$$| \int f_2(x, x)\,dx | \leq \sup_x | Q\, f_2(x, x) | \int Q^{-1} dx \leq \alpha\, p_{(1)(0)}(f_2),$$

$$Q = (1 + \xi_1^2) \dots (1 + \xi_4^2), \qquad \text{where } x = (\xi_1, \dots, \xi_4)$$

$(1) = (1, \dots, 1)$, $(0) = (0, \dots, 0)$ (see Prop. 2.1, (2)).

By Prop. 2.1 (3) it follows finally

$$\| C^{\epsilon}(f_2)\phi_n \| \leq \alpha\, p_{(1),(0)}(f_2) \| \phi_n \| \leq k\alpha\, q_{(2),(1)}(f_2) \| \phi_n \|$$

$$< k c \alpha \| N^5 f_2 \| \| \phi_n \| = \rho \| f_2 \|_5 \| \phi_n \| .$$

So we have proved that for every $\epsilon = (\epsilon_1, \epsilon_2)$

$$\| C^{\epsilon}(f_2)\phi_n \| \le \rho_2 \| f_2 \|_5 \, \| \phi_n \|$$

holds with a certain constant ρ_2 .
It is easy to see that in the general case, with an arbitrary m , one can estimate quite analog and so the statement is proved.
For an arbitrary sequence $d = (d_1, d_2, \dots)$ of positive numbers we define a representation $f \to C_d(f)$ of R in $D = \sum_n \mathcal{H}_n \subset \mathcal{H}$

$$C_d(f_0)\phi = f_0\phi \ , \qquad f_0 \in R_0$$

$$C_d(f_1)\phi_n = [d_{n+1}C^+(f_1) + d_n C^-(f_1)]\phi_n \ , \ \phi_n \in \mathcal{H}_n \ , \ d_0 = 0 \tag{6}$$

and then by continuation for each $f = \sum_{n \ge 0} f_n \in R$. It is easy to see, analog to the free field representation, that $f \to C_d(f)$ is a $*$ -representation of R , i.e. it holds $< \phi, C_d(f)\psi > = < C_d(f^*)\phi, \psi >$ for all $\phi, \psi \in D$ and for a $\phi_n \in \mathcal{H}_n$, $f_m \in R_m = s^{4m}$ we obtain

$$C_d(f_m)\phi_n = \sum_{\epsilon} [{}^{d}_{\epsilon}]_n C^{\epsilon}(f_m)\phi_n \tag{7}$$

$$\epsilon = (\epsilon_1, \dots, \epsilon_m) \quad \text{arbitrary,}$$

where the $[{}^{d}_{\epsilon}]_n$ are certain coefficients. From the definition of these coefficients we see immediately the following relations :

$$[{}^{(a)}_{\epsilon}]_n = a^m \ , \quad \epsilon = (\epsilon_1 \dots, \epsilon_m), \tag{8}$$

where $(\alpha) = (\alpha_1 \alpha_1 \alpha_1 \dots)$ is a constant sequence,

$$[{}^{d \cdot K}_{\epsilon}]_n = [{}^{d}_{\epsilon}]_n [{}^{K}_{\epsilon}]_n , \tag{8}$$

where $d \cdot K = (d_0 K_0, d_1 \cdot K_1, \dots)$ and

$$[{}^{d \cdot K}_{\epsilon}]_n \leq \{\sup (K_{n-m}, \dots, K_{n+m}\}^m [{}^{d}_{\epsilon}]_n . \tag{10}$$

Now we define in $R_m = s^{4m}$ the following norm

$$\|f_m\|' = \sup \| \sum_{|\epsilon|=r} [{}^{\lambda}_{\epsilon}]_n C^{\epsilon} (f_m) \phi_n \| , \tag{11}$$

where $n \geq m$, the sum runs over all ϵ with $|\epsilon| = \epsilon_1 + \dots + \epsilon_m = r$, and the supremum is taken over all $r = -m, -m+1, \dots, m-1, m$, all $\phi_n \in \mathcal{H}_n, \|\phi_n\| \leq 1$ and all sequences $\lambda = (\lambda_1, \lambda_2, \dots)$ of positive numbers with $|\lambda_1| \leq 1$. It is immedietely to see that $\|f_m\|'$ is independent of n.

<u>Statement 2</u>

It holds

$$\|f_m\| \leq \|f_m\|' < \alpha_m \|f_m\|_5 , \tag{12}$$

where $\|f_m\| = \|f_m\|_{L_2}$ and $\|f_m\|_5 = \|N^5 f_m\|$ (see Prop. 2.1). α_m is a constant only depending on m.

<u>Proof:</u>

Because of (8) and (10) it holds $[{}^{\lambda}_{\epsilon}]_n \leq 1$ and consequently

$\| f_m \|' \le \sup_{r, \phi_n} \sum_{|\epsilon|=r} \| C^{\epsilon}(f_m)\phi_n \|'$. Then from Stat. 1 it follows $\| f_m \|' \le \| f_m \|_5 \cdot a_m$ with a certain constant a_m. On the other hand one of the $C^{\epsilon}(f_m)$ is the operator $C^{\epsilon}(f_m)\phi_n = = f_m(x_1, \dots, x_m)\phi(x_{m+1}, \dots, x_{m+n})$, where $\epsilon = (1) = (1,1,1 \dots, 1)$ and it holds $\| C^{\epsilon}(f_m)\phi_n \| = \| f_m \| \, \| \phi_n \|$. Since $\epsilon = (1)$ is the unique ϵ with $|\epsilon| = m$ this implies $\| f_m \|' \ge \| f_m \|$. From the last remark in the foregoing proof we still obtain the following

Statement 3

Every representation $f \to C_d(f)$ is an isomorphism, i.e. $C_d(f) = 0$ holds only for $f = 0$.

Finally, from the foregoing considerations, especially from Stat. 2, it follows the last statement of this section:

Statement 4

If we put $\| f_m \|'_{\nu} = \| N^{\nu} f_m \|'$ for $f_m \in R_m$, then it holds

$$\| f_m \|_{\nu} \le \| f_m \|'_{\nu} \le \| f_m \|_{\nu+5} \tag{13}$$

and consequently the topology τ_{∞} in $R = \underset{n \ge 0}{+} R_n$ is also defined by all the norms

$$\| f \|'_{(\gamma_n), \nu} = \sum_{n \ge 0} \gamma_n \| f_n \|'_{\nu} \tag{14}$$

$f = \sum_{n \ge 0} f_n \in R$, γ_n positive numbers, $\nu = 0, 1, 2, \dots$

5. The Proof of the Theorem

The proof will be given in some steps. Let $\lambda = (\lambda_1, \lambda_2, \dots)$ be an arbitrary sequence of positive numbers with $|\lambda_n| \leq 1$ and $\lambda \cdot \mu = (1 \cdot \lambda_1, 2 \cdot \lambda_2, \dots,), \mu = 1,2,3\dots)$. For any λ we take a Hilbert space

$$\mathcal{H}^\lambda = \sum_{n \geq 0} + \mathcal{H}_n^\lambda$$

and the dense domain

$$D^\lambda = \sum_{n \geq 0} \mathcal{H}_n^\lambda$$

as in (4.1) and construct

$$\mathcal{H} = \sum_\lambda + \mathcal{H}^\lambda \qquad (1)$$

$$D = \sum_\lambda D^\lambda .$$

In D^λ we take the representation $f \to C_{\lambda\mu}(f)$ of R (see 4.6) and construct the direct sum of all these representations,

$$T(f) = \sum_\lambda C_{\lambda\mu}(f) \qquad (2)$$

with the domain D dense in $\mathcal{H}$.

Let $\mathcal{T}$ be the $*$-algebra of all the operators $T(f)$. $\mathcal{T}$ with the domain D is an Op^*-algebra. It holds the following Lemma, which we will prove in the next section:

Lemma 1

For an arbitrary sequence $(\gamma_n)_{n=0,1,2\dots}$ of positive numbers there exists an $\mathcal{T}$-bounded set $\mathfrak{M}$ in D (see Sect. 3), such that for every $f = \sum_{m \geq 0} f_m \in R$ it holds

$$\sum_{m \geq 0} \gamma_m \| f_m \|' \leq \| T(f) \|_{\mathfrak{M}} = \tag{3}$$
$$= \sup_{\phi, \psi \in \mathfrak{M}} | < \psi , T(f) \phi > | .$$

Further, for every $\nu = 0, 1, 2, \ldots$ we define the representation

$$f \to A_\nu (f) = T(N^\nu f) \tag{4}$$

of R . N is the endomorphism of Prop. 2.3. This is a representation of R in a certain space $\mathcal{H}^{(\nu)}$ (one exemplar of (1)). We put

$$f \to A(f) = \sum_{\nu \geq 0} A_\nu (f) \tag{5}$$

the direct sum of all these representations $A_\nu(f)$, which is defined in

$$D = \sum_\nu D^{(\nu)} = \sum_{\nu, \lambda, n} \mathcal{H}_n^{\lambda, (\lambda)} \tag{6}$$

dense in

$$\mathcal{H} = \sum_\nu + \mathcal{H}^{(\nu)} = \sum_{\nu, \lambda, n} \oplus \mathcal{H}_n^{\lambda, (\nu)} . \tag{7}$$

By $\mathfrak{A} = (\mathfrak{A} , D)$ we denote the Op^* -algebra of all the operators $A(f)$. Then from Lemma 1 we obtain

Lemma 2

For every norm $\| f \|'_{(\gamma_n),\nu} = \sum_{n \geq 0} \gamma_n \| f_n \|'_\nu$ (see Stat. 4.3) we find a $\mathfrak{A}$-bounded set $\mathfrak{M}$ in $\mathbf{D}$, such that

$$\| f \|'_{(\gamma_n),\nu} \leq \| A(f) \|_{\mathfrak{M}} = \sup_{\psi,\phi \in \mathfrak{M}} | < \psi , A(f) \phi > | .$$

We need only to take the corresponding set $\mathfrak{M}$ of Lemma 1 in the subspace $\mathbf{D}^{(\nu)} \subset \mathcal{H}^{(\nu)}$.

Further, from Lemma 2 we see that $f \to A(f)$ is an algebraic $*$-isomorphism, i.e. if $A(f)=0$, then $f=0$ (a fact, which in consequence of Stat. 3 also follows direct from the construction of $A(f)$).

In other words, taking into account Stat. 4, by Lemma 2 it is proved

Lemma 3

The inverse $*$-homomorphism $A(f) \to f$ is a continuous mapping from $\mathfrak{A}[\tau_D]$ onto $R[\tau_\infty]$.

This holds, because the system of all the norms $\| A(f) \|_{\mathfrak{M}}$ defines the topology τ_D (see the remarks after Theorem).

Finally we prove

Lemma 4

The algebraic $*$-isomorphism $f \to A(f)$ is a continuous mapping of $R[\tau_\infty]$ onto $\mathfrak{A}[\tau_D]$.

Proof:

Let $\mathfrak{M}$ be an arbitrary $\mathfrak{A}$-bounded set of $\mathbf{D}$. Then there exists a ν_0 such that $\mathfrak{M} \subset \sum_{\nu=0}^{\nu_0} \mathbf{D}^{(\nu)}$. If that is not so, we could find a sequence $\phi_r \subseteq \mathfrak{M}$, $r = 1,2,\ldots$; such that ϕ_r has a component $\phi_r^{(\nu_r)}$ in $D^{(\nu_r)}$ with $\nu_r \to \infty$. Then, in consequence of Prop. 2.2 one could find a $f \subseteq R_1 = s^4$ such that

$$\| A(f)\, \phi_r \| \approx \| N^{\nu_r} f \| \; \| \phi_r^{(\nu_r)} \| \to \infty$$

for $r \to \infty$, i.e. $\mathfrak{M}$ would not be $\mathfrak{A}$-bounded. Now, an arbitrary element $\phi \in \mathfrak{M}$ has the decomposition $\phi = \sum_{\nu,\lambda,n} \phi_n^{\lambda,(\nu)}$ (see (6)), $\nu \leq \nu_0$. Because of $\sup_{\phi \subseteq \mathfrak{M}} \| A(f_m)\phi \| < \infty$ from the construction of the $A(f_m)$ it follows

$$\sum_{\lambda,n} (n + m)^m \, \| \phi_n^{\lambda,(\nu)} \|^2 \leq k_m^{\nu} \tag{8}$$

for all $\phi \in \mathfrak{M}$, where k_m^{ν} are constants depending on $\mathfrak{M}$. Further we put yet $\sigma = \sup_{\phi \subseteq \mathfrak{M}} \| \phi \| < \infty$.
If $\phi, \psi \subseteq \mathfrak{M}$, then it holds

$$|\langle \psi, A(f_m)\phi \rangle|^2 \leq \sigma^2 \| A(f_m)\, \phi \|^2 ,$$

$$\| A(f_m)\phi \|^2 = \sum_{\nu=0}^{\nu_0} \sum_{\lambda} \| \sum_{n} \sum_{\epsilon} [{}^{\lambda\mu}_{\epsilon}]_n \, C^{\epsilon} (N^{\nu} f_m)\, \phi_n^{\lambda,(\nu)} \|^2$$

$$\leq \sum_{\nu=0}^{\nu_0} \sum_{\lambda} \Big(\sum_{n,\epsilon} \frac{1}{n^2} \Big) \sum_{n,\epsilon} n^2 (n+m)^m \, \| C^{\epsilon} (N^{\nu} f_m)\, \phi_n^{\lambda,(\nu)} \|^2$$

$$\leq K_m \sum_{\nu=0}^{\nu_0} \| f_m \|^2_{\nu+5} \, k^{\nu}_{m+2} \leq \delta_m^2 \| f_m \|^2_{\nu_0+5} ,$$

where K_m, δ_m are certain constants. In the last estimation we have applied (4.10), Stat. 1 and (8).

Consequently, it holds for $\psi, \phi \in \mathfrak{M}$

$$|<\psi, A(f)\phi>| \leq \sum_{m\geq 0} |<\psi, A(f_m)\phi>| \leq \sum_m \sigma\delta_m \|f_m\|_{\nu_0+5}$$

and from that it follows

$$\|A(f)\|_{\mathfrak{M}} \leq \sum_{m\geq 0} \sigma\delta_m \|f_m\|_{\nu_0+5}$$

i.e. $f \to A(f)$ is continuous as mapping from $R[r_\infty]$ onto $\mathfrak{A}[r_D]$.

By Lemmas 3 and 4 it is given the proof of the Theorem.

All that remains for us to prove is Lemma 1, what will be done in the next section.

6. The Proof of Lemma 1

By $\mathfrak{M}_1^{\lambda}(\rho_1)$ we denote the ball $\{\phi_1 \in \mathcal{H}_1^{\lambda} : \|\phi_1\| \leq \rho_1\}$ and construct

$$\mathfrak{M}^{\{\lambda^n\}} = \sum_{n\geq 0} \mathfrak{M}_{s_n}^{\lambda^n}(\rho_n) \quad ,$$

$$\mathfrak{N}^{\{\lambda^n\}} = \sum_{n\geq 0} \sum_{r=-n}^{+n} \mathfrak{M}_{s_n+r}^{\lambda^n}(\rho_n) \tag{1}$$

(algebraic direct sums), where $1 \leq s_0 < s_1 < s_2 < \dots$ is sequence of integers satisfying

$$s_n \geq s_{n-1} + 2n \tag{2}$$

and

$$\rho_n = 2^{-n} s_n^{-\alpha_n}, \quad \alpha_n = \frac{2n-1}{4},$$

$\{\lambda^n\} = \{\lambda^0, \lambda^1, \dots\}$ is a system of sequences λ.

Property 1

The sets $\mathfrak{M}^{\{\lambda^n\}}, \mathfrak{N}^{\{\lambda^n\}}$ are uniformly $\mathfrak{T}$-bounded, more precisely it holds

$$\sup_{\phi \in \mathfrak{M}^{\{\lambda^n\}}} \| T(f_m) \phi \| \leq k_m \| f_m \|_5$$

$$\sup_{\phi \in \mathfrak{N}^{\{\lambda^n\}}} \| T(f_m) \phi \| \leq k_m \| f_m \|_5 ,$$

where k_m depends only on m, but is independent of $\{\lambda^n\}$.

For if $\phi = \Sigma \phi_{s_n} \in \mathfrak{M}^{\{\lambda^n\}}$ it holds

$$\begin{aligned} \| T(f_m) \phi \| &\leq \sum_n \| C_{\lambda^n \mu}(f_m) \phi_{s_n} \| \\ &\leq \sum_n \| \sum_\epsilon [{}^{\lambda^n \mu}_{\epsilon}]_{s_n} C^\epsilon (f_m) \phi_{s_n} \| \\ &\leq \alpha \| f_m \|_5 \sum_n (s_n + m)^m \rho_n \leq G < \infty . \end{aligned} \tag{3}$$

This follows from Stat. 1 and the estimation $[{}^{\lambda^n \mu}_{\epsilon}]_{s_n} \leq (s_n + m)^n$ (see 4.8-10). α, G are constants independent of $\{\lambda^n\}$.

The series on the right-hand side of (3) converges, because

$(s_n + m)^m \leq m^m s_s^{m-\alpha_n} 2^{-n}$ and $m - \alpha_n \to -\infty$ for $n \to \infty$. So we have proved that the $\mathfrak{M}^{\{\lambda^n\}}$ are uniformly $\mathfrak{J}$-bounded. Quite analog one proves the uniform $\mathfrak{J}$-boundedness of the $\mathfrak{N}^{\{\lambda^n\}}$. Because of Property 1 also the set

$$\mathfrak{M} = \bigcup_{\{\lambda^m\}} \left(\mathfrak{M}^{\{\lambda^m\}} \cup \mathfrak{N}^{\{\lambda^m\}} \right) \tag{4}$$

is $\mathfrak{J}$-bounded.

Next we prove

Property 2

The sequence s_n can be chosen in such a way that

$$\sum_{m \geq 0} \gamma_m \| f_m \|' \leq \| T(f_m) \|_{\mathfrak{M}}$$

holds, where $\mathfrak{M}$ is the set (4).

For if

$$\phi = \sum_n \phi_{s_n} \in \mathfrak{M}^{\{\lambda^n\}}$$
$$\psi = \sum_n \sum_{r=-n}^{n} \psi_{s_n + r} \in \mathfrak{N}^{\{\lambda^n\}} \tag{5}$$

(finite sums) we regard

$$|<\psi, A(f)\phi>| = |\sum_{m,n,k} < \sum_{r=-n}^{n} \psi_{s_n+r}, A(f_m)\phi_{s_k} >| \geq$$

$$\geq |\sum_{m} < \sum_{r=-m}^{m} \psi_{s_m+r}, A(f_m)\phi_{s_m} >| - \qquad (S)$$

$$- \sum_{m} \sum_{n,k<m-1} \sum_{r=-n}^{n} |< \psi_{s_n+r}, A(f_m)\phi_{s_k} >| \qquad (V) \qquad (6)$$

$$- \sum_{m} \sum_{n\geq m+1} |< \sum_{r=-n}^{n} \psi_{s_n+r}, A(f_m)\phi_{s_n} >| \qquad (N)$$

S, V, N –are the three different expressions on the right-hand side, taken with the positive sign.

For the expression N we obtain the following estimation:

First it holds (because of $n \geq m+1$)

$$|< \sum_{r=-n}^{rn} \psi_{s_n+r}, A(f_m)\phi_{s_n} >| = |< \sum_{r=-m}^{+m} \psi_{s_n+r}, A-(f_m)\phi_{s_n} >| \leq$$

$$\leq | < \sum_{r=-m}^{m} \psi_{s_n+r}, \sum_{r=-m}^{+m} \sum_{|\epsilon|=r} [{\lambda^n \mu \atop \epsilon}]_{s_n} C^{\epsilon}(f_m)\phi_{s_n} >|$$

$$\leq \sum_{r=-m}^{m} (s_m + m)^m |< \psi_{s_n+r}, \sum_{|\epsilon|=r} [{\tilde{\tilde{\lambda}} \atop \epsilon}]_{S_n} C^{\epsilon}(f_m)\phi_{s_n} >|$$

$$\leq 2m (s_n + m)^m \rho_n^2 \|f_m\|'$$

$|\epsilon| = \epsilon_1 + \dots + \epsilon_m$. In the last estimation we have applied the inequality $|\psi_{s_n+r}\| \cdot \|\phi_{s_n}\| \leq \rho_n$ and the relation

$$[\lambda^n_\epsilon \mu]_{s_n} = [\tilde{\tilde{\lambda}}_\epsilon]_{s_n} (s_n + m)^m ,$$

where $\tilde{\tilde{\lambda}} = (\ldots, \tilde{\tilde{\lambda}}_{s_n - m}, \tilde{\tilde{\lambda}}_{s_n - m+1}, \ldots, \tilde{\tilde{\lambda}}_{s_n + m}, \ldots)$ is a certain sequence with $0 < \tilde{\tilde{\lambda}}_i \leq 1$ (see 4.8–10).

$$\| \sum_{|\epsilon| = r} [\tilde{\tilde{\lambda}}_\epsilon]_{s_n} C^\epsilon (f_m) \phi_{s_n} \| \leq \rho_n \| f_m \|'$$ holds in

consequence of the definition (4.11) of $\| f_m \|'$.
Because of $m - 2\alpha_n < 0$ for $n \geq m+1$ we obtain

$$2m (s_n + m)^m \rho_n^2 \leq 2m\, 2^m s^{m-2\alpha_n}\, 2^{-2n} \leq m 2^{m+1-2n}$$

and consequently it is proved

$$N \leq \sum_{m \geq 0} \sum_{n \geq m+1} m 2^{m+1-2n} \| f_m \|' \leq \sum_{m \geq 0} \| f_m \|' . \qquad (7)$$

By analog considerations one proves

$$V \leq \sum_{m \geq 0} \beta_m \| f_m \|' , \qquad (8)$$

where for each m β_m is constant depending on $s_0, s_1, \ldots, s_{m-1}$, but independent of s_m, $s_{m+1} \ldots$.
The estimations for N and V were given for an arbitrary sequence $s_0, s_1, s_2, \ldots$ satisfying (2). Now, we determine the sequence $\{s_n\}_{n=0,1,\ldots}$ by induction in the following way: Suppose that the $s_0, s_1, \ldots, s_{m-1}$ are already chosen, then we take s_m

so large that

$$(s_m - m)^m \rho_m^2 \geq 2(\gamma_m + \beta_m + 1). \tag{9}$$

That is possible because of

$$(s_m - m)^m \rho_m^2 \geq \left(\frac{s_m}{2}\right)^m 2^{-2m} s_m^{-2\alpha_m} = 2^{-3m} s_m^{1/2}$$

and because β_m is independent of s_m. With this determination of the sequence $\{s_n\}$ the $\mathfrak{G}$-bounded set $\mathfrak{M}$ (4) is completely determined also.

Let $f = \sum_{m\geq 0} f_m$ be a certain element of R. Then it is $f_m = 0$ for $m > m_0$. In dependence on this f we chose $\phi, \psi \in \mathfrak{M}$ (5) in the following way: For $m \leq m_0$ we determine $-m \leq r_m \leq m$, λ^m, ϕ_{s_m}, $\psi_{s_m + r_m}$ in such a way that

$$\langle \psi_{s_m + r_m}, \sum_{|\epsilon| = r_m} [{}^{\lambda^m \mu}_{\epsilon}]_{s_m} C^\epsilon (f_m) \phi_{s_m} \rangle \geq \tag{10}$$

$$\geq \frac{1}{2} \rho_m^2 (s_m - m)^m \| f_m \|'.$$

This is possible in consequence of the definition (4.11) of $\| f_m \|'$ and the fact that

$$[{}^{\lambda^m \mu}_{\epsilon}]_{s_m} = (s_m - m)^m [{}^{\hat{\lambda}^m}_{\epsilon}]_{s_m}, \quad \epsilon = (\epsilon_1, \ldots, \epsilon_m)$$

holds with a certain $\hat{\lambda} = (\hat{\lambda}_1), \; 0 < \hat{\lambda}_1 \leq 1$ (see 4.8–10).

Now we need only to put

$$\phi = \sum_{n=0}^{m_0} \phi_{s_n} , \qquad \psi = \sum_{n=0}^{m_0} \psi_{s_n + r_n} \subset \mathfrak{M} . \tag{11}$$

With these ϕ, ψ then it follows from (9) and (10)

$$S = \Big| \sum_m < \psi_{s_m + r_m} , A(f_m) \phi_{s_m} > \Big| =$$

$$= \Big| \sum_m < \psi_{s_m + r_m} , \sum_{|\epsilon| = r_m} [\lambda^m_\epsilon \mu]_{s_m} C^\epsilon (f_m) \phi_{s_m} > \Big| \tag{12}$$

$$\geq \sum_m \frac{1}{2} \rho^2_m (s_m - m)^m \, || f_m ||'$$

$$\geq \sum_m (\gamma_m + \beta_m + 1) \, || f_m ||' .$$

Because of (7), (8) and (12) from the last inequality it follows (see (6))

$$< \psi , A(f) \phi > \geq \Sigma \gamma_m \, || f_m ||' , \tag{13}$$

where $\psi, \phi \subset \mathfrak{M}$ are the elements (11). Thus, it is shown

$$|| A(f) ||_{\mathfrak{M}} \geq \sum_{m \geq 0} \gamma_m \, || f_m ||'$$

and the Lemma 1 is completely proved.

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