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A METHOD OF COHERENT STATES AND DIAGRAM TECHNIQUE FOR DUAL AMPLITUDES

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A METHOD OF COHERENT STATES AND DIAGRAM TECHNIQUE FOR DUAL AMPLITUDES

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The study of finite energy sum rules /1-4/ and their solutions leads to construction of dual amplitudes for the elastic and quasi-elastic scattering processes of hadrons /5-9/.

In the papers /10,11/ a method of constructing dual amplitudes with arbitrary numbers of interacting particles was developed. Recently a number of papers have been published, concerned with the formulation of a diagram technique for such amplitudes /12/. The diagram technique is built up by using a possibility of factorization in the external momenta, through a set of an infinite number of operators which obey oscillator type commutators. The use of this infinite set of operators leads to an infinite degeneration of the resiques of the single poles in the scattering amplitude, thus making the physical interpretation rather complicated. So a problem arises to factorize the dual amplitudes in terms of a finite oscillator type operators.

In the present paper we show that the factorization with a finite set of oscillators could be done by applying the method of coherent states /13/ to describe the two-particle resonance systems.

1. Factorization of the Dual "Four Point" Function

Let us consider the elastic scattering of two scalar particles with masses m. We denote by p_1, p_2, q_1, q_2 the momenta of the particles (Fig. 1).

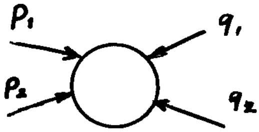

Fig. 1.

The Veneziano amplitude, as is known, is represented by the Euler beta function:

$$B(1-\alpha(s),\,1-\alpha(t)) = \int_0^1 dx\, x^{-\alpha(s)}(1-x)^{-\alpha(t)}, \tag{1.1}$$

where $\alpha(s)$ and $\alpha(t)$ are linear Regge trajectories in the s - and t -channels, respectively:

$$\alpha(s) = \alpha' s + \alpha(0) \qquad \alpha(t) = \alpha' t + \alpha(0). \tag{1.2}$$

Expressing now the t - variable in terms of the scalar product (p_1, q_1) we obtain:

$$\alpha(t) = 2\alpha'(p_1 q_1) - \beta + 1, \tag{1.3}$$

where $\beta = 1-\alpha(2m^2)$, $0<\beta<1$, $0<\alpha(2m^2)<1$. Let us rewrite the formula (1.1) in the following form

$$B(1-\alpha(s),\ 1-\alpha(t)) = \sum_{n=0}^{\infty} \frac{\Gamma(1-\alpha(t))(-1)^n}{\Gamma(1-\alpha(t)-n)n!}\ \frac{1}{1+n-\alpha(s)} \tag{1.4}$$

the residue of the function (1.1) in the pole $\alpha(s) = n+1$

$$\frac{\Gamma(1-\alpha(t))(-1)^n}{\Gamma(1-\alpha(t)-n)n!} \equiv \frac{(-1)^n\,\Gamma(-2\alpha'(p_1 q_1)+\beta)}{n!\,\Gamma(-2\alpha'(p_1 q_1)+\beta-n)} \tag{1.5}$$

$$\cos\theta = \frac{(\vec{p}_1\,\vec{q}_1)}{|\vec{p}_1||\vec{q}_1|}$$

is a polynomial function of degree n depending on $\cos\theta$. Therefore we can expand it in terms of the Legendre polynomials with $0 \leq \ell \leq n$. If θ_1, ϕ_1 and θ_2, ϕ_2 are the polar angles of the momenta p_1 and q_1, respectively, then the residue (1.5) can be represented as a scalar product of $\Psi_n(p_1)$ and $\Psi_n(q_1)$ with the components $\Psi^n_{h\ell m}(p_i \theta_i \phi_i) = R^n_h(p_i) Y_{\ell m}(\theta_i \phi_i)$. We shall show that these vectors can be determined in the space of the oscillator type representations of the group $U(3,2)$ and in this way we obtain an operator factorization for the B-function using a finite set of operators.

The oscillator type representations of the group $U(3,2)$ are obtained by using the five boson creation and annihilation operators a^+_μ, b^+, a_μ, b

$$[a^+_\mu a_\nu] = g_{\mu\nu} \qquad g_{\mu\nu} = 0 \quad \mu \neq \nu$$

$$[b^+ b] = 1 \qquad g_{00} = -g_{11} = -g_{22} = -g_{33} = 1 \tag{1.6}$$

(the other commutators are equal to zero).

The generators of the oscillator type representation $\hat{A}^a$ are determined as follows:

$$\hat{A}^a = \bar{\phi} A^a \phi, \tag{1.7}$$

where 5×5 matrices A^a are generators of the fundamental representation of the group $U(3,2)$, and ϕ and ϕ are five-dimensional vector operators

$$\phi = \begin{pmatrix} a_\mu \\ b \end{pmatrix} \qquad \bar{\phi} = (-g^{\mu\nu} a^+_\nu, -b^+). \tag{1.8}$$

Since the unit matrix commutes with all the A^a it is obvious that

$$H = \bar{\phi}\phi = -g^{\mu\nu} a^+_\mu a_\nu - b^+ b \tag{1.9}$$

is invariant under the action of the group.

We define now the operators (1.7) as operators in the Fock space. For this purpose we define the vacuum state

$$a_\mu | 0 > = b | 0 > = 0$$

and then the basic vectors of the irreducible representation can be written in the form:

$$| \chi_{n_\mu n_5} > = \frac{\prod_{i=0}^{3} (a_i^+)^{n_i} (b^+)^{n_5}}{\sqrt{n_0! \, n_1! \, n_2! \, n_3! \, n_5!}} | 0 > , \tag{1.10}$$

where $n_5 + \Sigma n_\mu \equiv n$ is an eigenvalue of the operator H.

The space of the functions (1.10) has an indefinite metric because it is a finite dimensional tensor representation of the noncompact group $U(3,2)$.

Consider the vectors

$$| \Psi_n (p) > = \Sigma \frac{1}{k! (n-k)!} (a^+ p)^k (b^+)^{n-k} | 0 > \equiv \frac{1}{n!} (a^+ p + b^+)^n | 0 > \tag{1.11}$$

which are eigenfunctions of the operator H with eigenvalue n, belonging to a fixed irreducible representation.

The scalar product of two such vectors

$$< \Psi_n (p) | \Psi_m (q) > = \frac{(-1)^n}{n!} [1 + (pq)]^n \, \delta_{mn} \tag{1.12}$$

does not determine a positive norm. But in the space R we can redefine the scalar product so that the norm of the vectors (1.11) becomes positive. For this purpose we define the coefficients $D_k^n(\alpha', \beta)$ in the following decomposition

$$\frac{\Gamma(-2\alpha' x + \beta)}{\Gamma(-2\alpha' x + \beta - n)} = \Sigma \, D_k^n (\alpha', \beta) x^k . \tag{1.13}$$

We shall list some properties of these coefficients:

i) the matrix

$$A_{kk'}(\beta) = \frac{k!}{k'!(k-k')!}\left(-\frac{\beta}{2a'}\right)^{k-k'}$$

gives a translation for the second variable in $D_k^n(a',\beta)$

$$\sum_k D_k^n(a',\beta)A_{kk'}(\gamma) = D_{k'}^n(a',\beta+\gamma). \tag{1.14}$$

Therefore the $D_k^n(a',0)$ completely determine $D_k^n(a',\beta)$ for arbitrary β.

ii) $D_k^n(a',0)$ satisfy identically the recurrence relation

$$D_k^{n+1}(a',0) = -2a' D_{k-1}^n(a',0) - (n+1)D_k^n(a',0). \tag{1.15}$$

So $D_k^n(a',0)$ are related to $s(n,k)$ – the Stirling numbers of the first kind /14/:

$$D_k^n(a',0) = (-2a')^k s(n+1,k+1). \tag{1.16}$$

iii) The following identity holds

$$\sum_n \frac{D_m^n(a',\beta)D_s^{k-n}(a',\gamma)}{n!(k-n)!} = \frac{(m+s)!}{m!k!s!} D_{m+s}^k(\beta+\gamma-1). \tag{1.17}$$

iv) There is a contour integral representation in the form

$$\frac{D_k^n(a',\beta)}{n!} = -\frac{1}{4\pi}\int_{C_1}\int_{C_2} dx\, dy\, (1-x)^{-2a'y+\beta-1}(-x)^{-n-1}y^{-k-1}. \tag{1.18}$$

Both the contours are circles with centers at the origin and radii $\rho_1,\rho_2 < 1$.

v) $D_k^n(a',\beta)$ are related also with the expansions

$$(1+x)^{-2a'y+\beta-1} = \sum_{n,k}\frac{x^n}{n!}D_k^n(a',\beta)y^k$$

$$\frac{1}{k!}(-2a')^k \ln^k(1+x) = \sum\frac{x^n}{n!}D_k^n(a',1). \tag{1.19}$$

The proof of these properties is given in appendix A , together with some additional formulae.

Let us introduce the operator $D(\alpha',\beta)$

$$D(\alpha',\beta)=\Gamma(\hat{K}+1)\,\Gamma(\hat{L}+1)\times$$

$$\cdot\frac{(-1)}{4\pi}\int_{C_1}\int_{C_2} dx\, dy\,(1-x)^{-2\alpha' y+\beta-1}\,(-x)^{-H-1}\,y^{\hat{K}-1}, \tag{1.20}$$

where $\hat{K}=-g^{\mu\nu}a^+_\mu a_\nu$, $\hat{L}=-b^+b$.

Obviously $D(\alpha',\beta)$ is nondegenerate operator which commutes with H . Hence it is a diagonal operator in the spece of the oscillator type representations of the group

$$(\Psi(p),\Psi(q))=\langle\Psi(p)\,|\,D(\alpha',\beta)\,|\,\Psi(q)\rangle. \tag{1.21}$$

using $D(\alpha',\beta)$ as a metric tensor. Consequently it is easy to verify that for the vectors (1.11) we have

$$(\Psi_n(p)\,\Psi_m(q))=\frac{(-1)^n}{n!}\,\frac{\Gamma(-2\alpha'(pq)+\beta)}{\Gamma(-2\alpha'(pq)+\beta-n)}\,\delta_{nm}\,. \tag{1.22}$$

If $p=q$ and $pq=p^2=m^2$ this scalar product determines a positive norm of the vectors $\Psi_n(p)$. Indeed in this case

$$\frac{(-1)^n}{n!}\,\frac{\Gamma(-2\alpha' m^2+\beta)}{\Gamma(-2\alpha' m^2+\beta-n)}=\frac{1}{n!}\,\alpha(1m^2)\,[\alpha(1m^2)+1]\,\ldots[\alpha(1m^2)+n+1)>0.$$

Thus the equality (1.22) gives the required factorization of the residue (1.5).

The vectors (1.11) could be interpreted as wave functions of the two-particle resonance states, satisfying the following eq.:

$$(1-\alpha(s)+H)\,\Psi_n(p_1)=0. \tag{1.23}$$

Hence we obtain an energy spectrum $\alpha(s)=n+1$. We denote the Green-function of the equation (1.23) by $G(s)$

$$G(s) \equiv (1-\alpha(s)+H)^{-1} = \int_0^1 dx\, x^{H-\alpha(s)} \quad . \tag{1.24}$$

Then taking into account (1.4) and (1.22) it is possible to write the amplitude (1.1) in the following form:

$$B(1-\alpha(s), 1-\alpha(t)) = (\Psi(p_1), G(s)\Psi(q_1)), \tag{1.25}$$

where

$$|\Psi(p_1)\rangle = \sum_{n=0}^{\infty} |\Psi_n(p_1)\rangle = e^{a^+p+b^+} |0\rangle \tag{1.26}$$

is a coherent state of the oscillator (1.23) and the scalar product is defined in (1.20)

$$(\Psi(p_1), G(s)\Psi(q_1)) = \langle 0 | e^{ap_1+b} \frac{D(\alpha',\beta)}{1-\alpha(s)+H} e^{a^+q_1+b^+} | 0\rangle . \tag{1.27}$$

Let us note some features of the amplitude (1.27): the factorization of the amplitude is obtained in the space of the finite - dimensional representations of the group U(3,2) using five creation and annihilation operators. According to (1.23) the vacuum corresponds to the lowest resonance state with energy a priori different from zero. The coherent states (1.26) can be considered as free two-particle states. The dependence on the other particle is effectively taken into account with a suitable choice of the resonance spectrum.

2. Factorization of the Five Point Dual Diagram

Suppose that in five-particle systems the interaction occurs in the following manner: (i) two particles transform into a resonance state, (ii) further this configuration interacts with the third incoming particle, forming a new resonance state, (iii) finally – a decay into two free particles. This can be represented graphically by the following diagram:

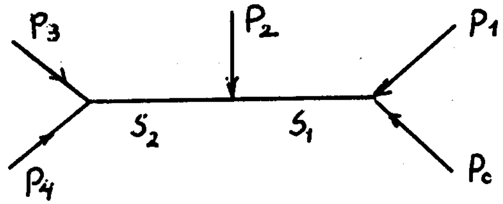

Fig. 2.

Let us define the energy variables s_1 and s_2 for the resonance state as follows:

$$s_1 = (p_0 + p_1)^2 = (p_2 + p_3 + p_4)^2$$
$$s_2 = (p_0 + p_1 + p_2)^2 = (p_3 + p_4)^2 . \tag{2.1}$$

In the case of identical particles the resonance state in the system of the two fixed particles (0-1 for example) is simultaneously a resonance state in the three-particle system (2-3-4). Thus the resonance wave function besides the two-particle equation (1.23) must satisfy some three-particle equation of the same type. Such a three-particle equation can be obtained from Eq. (1.23) when the operator H is now the Casimir operator for the group $U(3,2) \times U(3,2)$ in the oscillator representation

$$H = H_1 + H_2 , \tag{2.2}$$

where

$$H_i = -g^{\mu\nu} a^+_{(i)\mu} a_{(i)\nu} - b^+_{(i)} b_{(i)} , \quad i = 1, 2 . \tag{2.3}$$

The operators $(a^+_{1\mu}, b^+_1)$ and $(a^+_{2\mu}, b^+_2)$ commute with each other.

Hence we have the following equations for the resonances state function in the system of (0-1)-particles.

$$[H_1 - \alpha(s_1)]\,|\Psi_n(p_1)> = 0 \tag{2.4a}$$

$$[H_1 + H_2 - \alpha(s_1)]\,|\Psi_n(p_1)> = 0. \tag{2.4b}$$

Analogously, for the resonance state in the system of the particles (3-4) we require that

$$<\Psi_n(p_3)|\,[H_1 + H_2 - \alpha(s_2)] = 0 \tag{2.5a}$$

$$<\Psi_n(p_3)|\,[H_2 - \alpha(s_2)] = 0. \tag{2.5b}$$

From here we obtain the following solutions

$$|\Psi_n(p_1)> = \frac{1}{n!}[p_1 a_1^+ + b_1^+]^n\,|0>$$

$$<\Psi_n(p_3)| = \frac{1}{n!} <0|\,[p_3(a_1 + a_2) + b_1 + b_2]^n. \tag{2.6}$$

The corresponding coherent states have the form (1.26):

$$|\Psi(p_1)> \equiv \sum_n |\Psi_n(p_1)> = e^{p_1 a_1^+ + b_1^+}\,|0>$$

$$<\Psi(p_3) \equiv \sum_n <\Psi_n(p_3)| = <0|\,e^{p_3(a_1 + a_2) + b_1 + b_2}. \tag{2.7}$$

In the representation space of the group $U(3,2) \times U(3,2)$ the metric tensor D which gives a positive norm of the states (2.6) is the direct product of the metric tensors D_1 and D_2 corresponding to the first and the second factors.

$$D = D_1(\alpha_1', \beta)\,D_2(\alpha', \beta_2). \tag{2.8}$$

Now we shall seek an expression for the five-point dual diagram in the form:

$$B_5 \equiv (\Psi(p_3), G_2(s_2)\Gamma(p_2)G_1(s_1)\Psi(p_1)), \tag{2.9}$$

where

$$G_1(s_1) \equiv [H_1 - \alpha(s_1) + 1]^{-1} = \int_0^1 dx_1 x_1^{-\alpha(s_1)+H_1} \tag{2.10a}$$

$$G_2(s_2) \equiv [H_1 + H_2 - \alpha(s_2) + 1]^{-1} = \int_0^1 x_2^{\cdot\alpha(s_2)+H_1+H_2} dx \tag{2.10b}$$

and $\Gamma(p_2)$ is the vertex, which we shall determine, requiring an agreement of (2.9) with the well known expression for B_5

$$B_5 = \int dx_1 dx_2 x_1^{-\alpha(s_1)} x_2^{-\alpha(s_2)} (1-x_1)^{-\alpha_{12}} (1-x_2)^{-\alpha_{23}} (1-x_1 x_2)^{p_{13}}. \tag{2.11}$$

Here we comply the following notations for the trajectories:

$$\alpha_{i\,i+1} = \alpha'(p_i + p_{i+1})^2 + \alpha(0)$$

$$p_{13} = -2\alpha'(p_1 p_3) + \gamma_{13}, \tag{2.12}$$

where γ_{13} is a constant.

Let us introduce the operator

$$V(p\alpha; b; \beta) = \sum_{n,k} \frac{1}{n!} D^n_k(\alpha'\beta)(p\alpha)^k b^{n-k}, \tag{2.13}$$

where the coefficients $D^n_k(\alpha',\beta)$ have been defined in the first section (1.13). Accoring to the first of the equations (1.19) we have a formal representation

$$V(pa; b; \beta) = (1+b)^{-2\alpha'\frac{pa}{b}+\beta-1} \tag{2.14}$$

Two properties of this operator are of interest. The first one connects it with the metric tensor

$$<0<e^{pa+b} \qquad D(\alpha',\beta)=<0|\,V(p\,a;b;\beta). \tag{2.15}$$

This equation shows that in the sense of the metric $D(\alpha',\beta)$ the resonance state on the right-hand side is contravariant to the coherent state on the left-hand side. Therefore, the operators V and e^{pa+b} are related with each other. Indeed the formal equality (2.14) determines V as a finite difference analog of the exponent /15/. Another property which confirms the above analogy is contained in the following identity

$$V(pa;b;1+\beta)V(qa;b;1+\gamma)=V((p+q)a;b;1+\beta+\gamma). \tag{2.16}$$

(The proof is given in appendix B). Formally it can be obtained directly using (2.14). The equality (2.16) determines the operator V as a representation of the five-dimensional translation group and in the same sense it is an analog of e^{pa+b}.

Now using (2.15) we rewrite (2.9) in the form:

$$B_5=<0|\,V(p_3a_2;b_2;\beta_2)V(p_3a_1;b_1;\beta_1)G_2(s_2)\Gamma(p_2)G_1(s_1)e^{p_1a_1^++b_1^+}|0>. \tag{2.17}$$

Using (2.10a) we can evaluate that

$$G_1(s_1)e^{p_1\tilde{a}_1^++b_1^+}|0>=\int_0^1 x_1^{-\alpha(s_1)}e^{x_1p_1a_1^++x_1b_1^+}dx_1|0>. \tag{2.18}$$

Let us multiply this equality on the left by the operator $V(p_2a_1;b_1;\beta_1)$. Then we obtain

$$V(p_2a_1;b_1;\beta_1)G_1(s_1)e^{p_1\tilde{a}_1^++b_1^+}|0>=$$

$$=\int_0^1 dx_1x_1^{-\alpha(s_1)}(1-x_1)^{-\alpha_{12}}\exp(x_1p_1a_1^++x_1b_1^+)|0>. \tag{2.19}$$

Obviously if we choose $\Gamma(p_2)$ in the form

$$\Gamma(p_2)=e^{p_2a_2^++b_2^+}V(p_2a_1;b\;;\beta_1) \tag{2.20}$$

then the expression (2.17) coincides with (2.11). Finally we have:

$$B_5 = (\Psi(p_3), G_2(s_2) e^{p_2 a_2^+ + b_2^+} V(p_2 a_1; b_1; \beta_1) G_1(s_1) \Psi(p_1)). \quad (2.21)$$

3. Factorization of the N -Point Dual Diagram

The results obtained in the previous two sections can be easily generalized to the N -point functions.

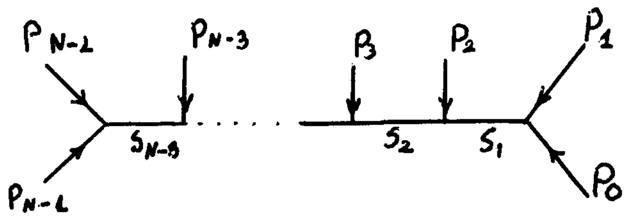

Fig. 3.

Here p_i are the momenta of the i -th particle. The definition of the variables $s_1, s_2 \dots s_{N-3}$ is:

$$s_i = \Big(\sum_{k=0}^{i} p_k\Big)^2. \quad (3.1)$$

Slightly modifying B_4 and B_5 we get:

$$B_4 = \langle 0 | e^{p_2 a_2^+ + b_2^+} V_1(p_2) G_1(s_1) e^{p_1 a_1^+ + b_1^+} | 0 \rangle$$

$$B_5 = \langle 0 | e^{p_3 a_3^+ + b_3^+} V_2(p_3) V_1(p_3) G_2(s_2) e^{p_2 a_2^+ + b_2^+} V_1(p_2) G_1(s_1) e^{p_1 a_1^+ + b_1^+} | 0 \rangle. \quad (3.2)$$

Here we use denotations

$$V_i(p_k) = V((p_k a_i); b_i; \beta_i). \quad (3.3)$$

The additional left-hand exponent in the equality (3.2) is rewritten down only for symmetry reasons. (The left-hand vacuum state transformed it into unity).

The operator

$$e^{p_i a_i^+ + b_i^+} \equiv e^{\xi_i} \tag{3.4}$$

must be put into correspondence with i -th external line of the diagram.

The operator

$$\Gamma_i = \prod_{k=1}^{i-1} V_k (p_i) \tag{3.5}$$

corresponds to the vertex with the incoming i -th particle.

To every internal line with variable s_i we put into correspondence the propagator

$$G_i (s_i) \equiv [\tilde{\tilde{H}}_i - \alpha(s_i) + 1]^{-1} = \int_0^1 dx_i \, x_i^{-\alpha(s_i) + \tilde{\tilde{H}}_i} \quad , \tag{3.6}$$

where

$$\tilde{\tilde{H}}_i = \sum_{k=1}^{i} H_k \tag{3.7}$$

and

$$H_k = - g^{\mu\nu} a^+_{(k)\mu} a_{(k)\nu} - b^+_{(k)} b_{(k)} . \tag{3.8}$$

Thus, multiplying the operators (3.4), (3.5) and (3.6) in the order corresponding to the elements of the diagram (Fig.3) and taking a vacuum expection value we obtain

$$B_N = \langle 0 | e^{\xi_{N-2}} \Gamma_{N-2} G_{N-3}(s_{N-3}) \ldots G_2(s_2) e^{\xi_2} \Gamma_2 G_1(s_1) e^{\xi_1} | 0 \rangle \tag{3.9}$$

which is a correct expression for the 4- and 5-point functions (1.1), (2.11).

We shall show that the expression (3.9) for all N coincides with the known dual amplitude /16/. To do this we substitute into (3.9) the operator $G_i(s_i)$ from the expression (3.6). Then we get $N-3$ -dimensional integral with the integrand:

$$\prod_{i=1}^{N-3} x_i^{-\alpha(s_i)} \langle 0 | e^{\xi_{N-2}} \Gamma_{N-2} x^{\tilde{\tilde{H}}_{N-3}} \dots x^{\tilde{\tilde{H}}_2} e^{\xi_2} \Gamma_2 x^{\tilde{\tilde{H}}_1} e^{\xi_1} | 0 \rangle . \qquad (3.10)$$

Note that the operator e^{ξ_i} commutes with all Γ_k and $\tilde{\tilde{H}}_\ell$ when $k \leq i$ and $\ell < i$. Therefore instead of (3.10) we can write

$$\prod_{i=1}^{N-3} x_i^{-\alpha(s_i)} \langle 0 | \Gamma_{N-2} x_{N-3}^{\tilde{\tilde{H}}_{N-3}} \dots x_2^{\tilde{\tilde{H}}_2} \Gamma_2 x_1^{\tilde{\tilde{H}}_1} e^{\sum_{k=1}^{N-3} \xi_k} | 0 \rangle . \qquad (3.11)$$

Separating from every vertex Γ_i the $(i-1)$-th factor, i.e.

$$\Gamma_i = \prod_{k=1}^{i-2} V_k(p_i) V(p_i a_{i-1}; b_{i-1}; \beta_{i-1}) \qquad (3.12)$$

and taking its commutators with the all $x^{\tilde{\tilde{H}}_k}$ $k < i$ we obtain

$$\prod_{i=1}^{N-3} x_i^{-\alpha(s_i)} \langle 0 | \prod_{k_1=1}^{N-4} V_{k_1}(p_{N-2}) x_{N-3}^{\tilde{\tilde{H}}_{N-3}} \prod_{k_2=1}^{N-5} V_{k_2}(p_{N-3}) x_{N-4}^{\tilde{\tilde{H}}_{N-4}} \dots$$

$$\dots x_2^{\tilde{\tilde{H}}_2} x_1^{\tilde{\tilde{H}}_1} \prod_{j=1}^{N-3} V(-x_j (p_{j+1} a_j); -x_j b_j; \beta_i) e^{\sum_{\ell=1}^{N-3} \xi_\ell} | 0 \rangle . \qquad (3.13)$$

The vector $e^{a_k^+ p_k + b_k^+} | 0 \rangle$ is an eigenstate of the operators $a_{(k)\mu}$ $b_{(k)}$ with the eigenvalues $p_{(k)\mu}$ and I respectively. For this reason:

$$\prod_{j=1}^{N-3} V[-x_j (p_{j+1} a_j); -x_j b_j; \beta_j] e^{\sum_{\ell=1}^{N-3} \xi_\ell} | 0 \rangle =$$

$$= \prod_{j=1}^{N-3} V(-x_j (p_{j+1} p_j); -x_j; \beta_j) e^{\sum_{\ell=1}^{N-3} \xi_\ell} | 0 \rangle . \qquad (3.14)$$

From the definition of the operators $V((pa);b;\beta)$ (2.13) and (2.14) it foolows that

$$\prod_{k=1}^{N-3} V(-(p_{k+1}p_k)x_k;-x_k;\beta_k)=\prod_{k=1}^{N-3}(1-x_k)^{-a_{k+1,k}}, \tag{3.15}$$

where

$$a_{k+1,k}=a'(p_{k+1}+p_k)^2+a(0)\equiv 2a'(p_{k+1},p_k)-\beta_k+1.$$

So the integrand (3.13) is transformed to

$$\prod_{i=1}^{N-3} x_i^{-a(s_i)}(1-x_i)^{-a_{i+1,i}}<0\,|\prod_{k_1=1}^{N-4} V_{k_1}(p_{N-2})x_{N-3}^{\tilde{H}_{N-3}}\prod_{k_2=1}^{N-5} V_{k_2}(p_{N-3})x_{N-4}^{\tilde{\tilde{H}}_{N-4}}\cdots \tag{3.16}$$

$$\cdots x_2^{\tilde{\tilde{H}}_2}x_1^{\tilde{\tilde{H}}_1}e^{\sum_{\ell=1}^{N-3}\xi_\ell}\,|0>.$$

Using the method described above for the remaining operators. in (3.16) we see that (3.9) coincides identically with the Bardakci-Ruegg formula for the N -point dual amplitude. So the N-point dual diagram is represented by the expression (3.9) which is factorized in the momenta p_i $(i=1,2,\ldots N-2)$.

The essential feature of this factorization is that it is achieved by a finite number of creation and annihilation operators. These operators, as already clarified in the first section, give a finite-dimensional representation of the group U(3,2) according to which the residues of the resonances are expanded. It can be seen from (3.16) that the N -point diagram contains N-3 mutually commuting sets of operators $(a_\mu^{(i)}\ b^{(i)})$, $(i=1,2,\ldots,N\text{-}3)$. In this sense we can consider a given factorization to be determined by the group

$$\underbrace{U(3,2)\times U(3,2)\ldots\times U(3,2)}_{(N-3)\text{ times}} \tag{3.17}$$

The formula (3.9) can be expressed in a more symmetric form noting that on the right-hand side of the propagator $G_i(s_i)$ only the creation and annihilation operators carrying indeces $k<i$ appear.

Hence (3.9) does not change its form if one substitutes

$$G(s_i)=(\tilde{\tilde{H}}-\alpha(s_i)+1)^{-1}=\int_0^1 dx_i\, x_i^{-\alpha(s_i)+\tilde{\tilde{H}}}, \tag{3.18}$$

where

$$\tilde{\tilde{H}}=\sum_{k=1}^{\infty} H_k$$

and H_k is given by (3.8). This means that we extend the chain (3.17) to infinity. For the same reason the expression (3.9) remains unchanged if instead of Γ_i from (3.5) we put

$$\Gamma(p_i)=\prod_{k=1}^{\infty} V_k(p_i). \tag{3.19}$$

Consequently we obtain new expressions for the propagators and vertices which do not depend on their position in the formula (3.9). Finally we have

$$B_N=\langle 0|\, e^{\xi_{N-2}}\,\Gamma(p_{N-2})G(s_{N-3})\ldots G(s_2)e^{\xi_2}\,\Gamma(p_2)G(s_1)e^{\xi_1}|0\rangle. \tag{3.20}$$

Although an infinite set of operators occurs in the last formula, in fact, we deal with only some of them, as could be seen from (3.9) because all remaining operators will be annihilated by the vacuum.

Summarizing, we obtain a diagrammatic technique of building-up the N-point dual amplitude, according to the following rules:

a) The operator e^{ξ_i} (3.4) corresponds to the i-th external line of the diagram (Fig. 3).

b) The operator $\Gamma(p_i)$ (3.19) corresponds to the i-th vertex (see Fig. 3).

c) The propagator $G(s_i)$ (3.18) corresponds to the internal line indicated by s_i on Fig. 3.

d) Preserving the order stated in Fig. 3 we multiply the operators a), b), c) and taking the vacuum expectation value we obtain the exact expression for the N -point function.

The factorization of the N -particle dual diagram, we have obtained, probably is not the most economical one in the sense of the dimensionality of the group. Let us illustrate this statement by an example concerning the five-point diagram. It is seen that

$$<0|\, e^{p_3 a^+ + b^+} \frac{1}{H - \alpha(s_2)+1} V(p_2 a^+; b^+; \beta) D(\alpha', \beta') \times$$

$$\times V(p_2 a; b; \beta) \frac{1}{H - \alpha(s_1)+1} e^{p_1 a^+ + b^+} |0>$$

coincides with (2.11). Thus we have built up the five- and four-point function factorization using in both cases only the group $U(3,2)$.

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Appendix A

Let us rewrite in detail the definition (1.13) for coefficients $D^n_k(\alpha', \beta)$

$$\frac{\Gamma(-2\alpha' x + \beta)}{\Gamma(-2\alpha' x + \beta - n)} = (\beta - 2\alpha' x - 1)(\beta - 2\alpha' x - 2) \ldots (\beta - 2\alpha' x - n) =$$
$$= \Sigma\, D^n_k(\alpha', \beta) x^k \quad \text{and} \quad D^n_k(\alpha', \beta) \equiv 0 \quad , k > n . \qquad (A.1)$$

1) Translating the argument

$$x \to x - \frac{\gamma}{2\alpha'}$$

we get:

$$\frac{\Gamma(-2\alpha' x + \beta + \gamma)}{\Gamma(-2\alpha' x + \beta + \gamma - n)} = \sum_{k,k'} D^n_k(\alpha', \beta) \frac{k!}{k'!(k-k')!} \left(-\frac{\gamma}{2\alpha'}\right)^{k-k} x^{k'} .$$

On the other hand

$$\frac{\Gamma(-2\alpha' x+\beta+\gamma)}{\Gamma(-2\alpha' x+\beta+\gamma-n)} = \sum_k D^n_k(\alpha',\beta+\gamma)x^k .$$

Consequently we have

$$\sum_k D^n_k(\alpha',\beta)A_{kk'}(\gamma) = D^n_{k'}(\alpha',\beta+\gamma), \qquad (A.2)$$

where

$$A_{kk'}(\gamma) = \frac{k!}{k'!(k-k')!}\left[-\frac{\gamma}{2\alpha'}\right]^{k-k'}.$$

The matrices $A_{kk'}(\gamma)$ form an one-parameter commutative group. Indeed:

$$A_{kk'}(\beta)A_{k'k''}(\gamma) = A_{kk''}(\beta+\gamma).$$

With normalization condition

$$A_{kk'}(0) = \lim_{\gamma\to 0}\frac{k!}{k'!(k-k')!}\left[-\frac{\gamma}{2\alpha'}\right]^{k-k'} = \delta_{kk'};$$

2) Using the identity

$$\frac{\partial}{\partial x}\frac{\Gamma(-2\alpha' x+\beta)}{\Gamma(-2\alpha' x+\beta-n)} = (-2\alpha')\frac{\partial}{\partial\beta}\frac{\Gamma(-2\alpha' x+\beta)}{\Gamma(-2\alpha' x+\beta-n)} \qquad (A.4)$$

the following representation of the coefficients $D^n_k(\alpha',\beta)$ holds

$$D^n_k(\alpha',\beta) = \frac{(-2\alpha')^k}{k!}\frac{d^k}{d\beta^k}\frac{\Gamma(\beta)}{\Gamma(\beta-n)}. \qquad (A.5)$$

Indeed from (A.1) follows that

$$D^n_k(\alpha',\beta) = \frac{1}{k!}\left(\frac{\partial^k}{\partial x^k}\frac{\Gamma(-2\alpha' x+\beta)}{\Gamma(-2\alpha' x+\beta-n)}\right)_{x=0}.$$

If now we use the identity (A.4) to transform the right-hand side of the last relation we get (A.5).

3) If $\beta=0$ we have

$$\frac{\Gamma(-2\alpha' x)}{\Gamma(-2\alpha' x-n)} = \sum D^n_k(\alpha',0)x^k \qquad (A.6)$$

multiplying this equality by $(-2\alpha' x-n-1)$ we get

$$\frac{\Gamma(-2\alpha' x)}{\Gamma(-2\alpha' x-n-1)} = -2\alpha' \Sigma D^{n}_{k-1}(\alpha', 0)x^{k} - (n+1)\Sigma D^{n}_{k}(\alpha', 0)x^{k}.$$

Hence

$$D^{n+1}_{k}(\alpha', 0) = -2\alpha' D^{n}_{k-1}(\alpha', 0) - (n+1) D^{n}_{k}(\alpha', 0) \qquad (A.7)$$

the last gives the relation between the coefficients and the Stirling numbers $s(n,k)$.

4) The polynomials $\frac{\Gamma(\beta)}{\Gamma(\beta-n)}$ are related with the Laguerre polynomials $L^{\alpha}_{n}(0)$.

$$L^{-\beta}_{n}(0) = \frac{(-1)^{n}}{n!}\,\frac{\Gamma(\beta)}{\Gamma(\beta-n)}. \qquad (A.8)$$

Let us use one known property of the $L^{\alpha}_{n}(x)$

$$\sum_{m} L^{\alpha}_{m}(x) L^{\beta}_{n-m}(y) = L^{\alpha+\beta+1}_{n}(x+y)$$

in the form

$$\sum_{m} L^{-\beta}_{m}(0)\, L^{-\gamma}_{n-m}(0) = L^{-\beta-\gamma+1}_{n}(0).$$

Differentiating the last equality k times in β and s times in γ, and taking into account (A.8) and (A.5) we obtain

$$\sum_{m} \frac{D^{m}_{k}(\beta) D^{n-m}_{s}(\gamma)}{m!(n-m)!} = \frac{(k+s)!}{n!k!s!} D^{n}_{k+s}(\beta+\gamma-1). \qquad (A.9)$$

5) The binomial

$$(1-x)^{-2\alpha' y+\beta-1} \qquad (A.10)$$

has the following x-power expansion

$$(1-x)^{-2\alpha' y+\beta-1} = \Sigma \frac{\Gamma(-2\alpha' y+\beta)(-1)^{n}}{n!\,\Gamma(-2\alpha' y+\beta-n)} x^{n}$$

and using (A.1) we obtain

$$(1-x)^{-2\alpha' y+\beta-1} = \Sigma \frac{(-1)^n}{n!} D_k^n(\alpha',\beta) x^n y^k . \tag{A.11}$$

So the function (A.10) is a generator function for the coefficients $D_k^n(\alpha',\beta)$. Therefore directly the following integral representation holds

$$\frac{D_k^n(\alpha',\beta)}{n!} = -\frac{1}{4\pi} \int_{C_1}\int_{C_2} dx\,dx\,(1-x)^{-2\alpha' y+\beta-1} (-x)^{-n-1} y^{-k-1}, \tag{A.12}$$

where C_1 and C_2 are closed contours which contain the points $x=0$ and $y=0$. They must intersect the positive axis in the interval (0,1).

From (A.11) we get

$$\frac{1}{k!}(-2\alpha')^k \ell n^k (1-x)(1-x)^{\beta-1} = \Sigma \frac{x^n}{n!} D_k^n(\alpha',\beta). \tag{A.13}$$

The formulae (1.18) can be obtained from (A.10) and (A.13) if we put $x \to -x$ in (A.10) and $\beta=1$ in (A.13).

Appendix B

To prove the formula (2.16) let us multiply $V(pa,b,\beta)$ and $V(qa,b,\gamma)$ by each other:

$$V(pa;b;\beta)V(qa;b;\gamma) = \Sigma \frac{D_k^n(\alpha',\beta) D_{k'}^{n'}(\alpha',\gamma)}{n!\,n'!} (pa)^k (qa)^{k'} b^{n+n'-k-k'}$$

The substitutions $n \to m-n$ and $k \to s-k$ give

$$V(pa;b;\beta)V(qa;b;\gamma) = \Sigma \frac{D_k^n(\alpha',\beta) D_{s-k}^{m-n}(\alpha',\gamma)}{n!(m-n)!} (pa)^k (qa)^{s-k} b^{m-s}.$$

Taking into account the identity (A.9) we get

$$V(pa;b;\beta)V(qa;b;\gamma) = \sum \frac{s!D_s^m(a',\beta+\gamma-1)}{m!\,k!(s-k)!}(pa)^k (qa)^{s-k} b^{m-s} =$$

$$\sum \frac{1}{m!} D_s^m [(p+q)a]^s b^{m-s}$$

and consequently

$$V(pa;b;\beta)V(qa;b;\gamma) = V((p+q)a;b;\beta+\gamma-1).$$

Hence the formula (2.16) can be obtained by substituting $\beta \to \beta+1 \quad \gamma \to \gamma+1$.

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