2 8/XII

О - 40

ОБЪЕДИНЕННЫЙ
ИНСТИТУТ
ЯДЕРНЫХ
ИССЛЕДОВАНИЙ

Дубна

E2-5160

A.I. Oksak

ЛАБОРАТОРИЯ ТЕОРЕТИЧЕСКОЙ ФИЗИКИ

CPT AND INFINITE-COMPONENT FIELDS

1970

E2-5160

84443/2 pr

A.I. Oksak

CPT AND INFINITE-COMPONENT FIELDS

C O N T E N T S

Introduction

The contemporary interest in infinite-component fields is due to the fact that they allow to describe in the unified manner the second quantization and interactions of infinite multiplets. Such multiplets appear in attempts to relativize internal symmetries of elementary particles/1/. Nevertheless, at the preliminary stage of developing infinite-component quantum fields it is useful (as it has been proposed by Feldman and Matthews/2/) to digress from inner structure of particles and to study the "pure infinite-componentness", treating infinite-component fields as fields transforming under infinite dimensional (mainly, irreducible) representations of the connected Lorentz group $L_+^\uparrow$ x/.

In accordance with the Wightman scheme a quantized field transforming under irreducible representation $\chi=[k,c]$ of the group $L_+^\uparrow$ (where k is a (half-)integer and c is an arbitrary complex number) is defined/4/ as a continuous bilinear operator-valued functio-

---

x/ Proceeding from analogy with the construction of local algebras on basis of the Wightman finite-component fields one can (under usual assumptions) define the Haag-Araki algebra/3/ starting from infinite-component quantum fields. The properties of infinite-component fields differ considerably from those of finite-component fields (e.g., CPT-theorem and the connection of spin with statistics fail), hence the study of infinite-component fields can give an insight into problems of the theory of local algebras).

nal $\varphi(u;f)$ over $(u,f)\in S(M_4)\times\mathfrak{D}_{-\chi}$ with common dense domain D_0 in the Hilbert space $\mathcal{H}$ of state vectors; here $S(M_4)$ is the Schwartz space of rapidly decreasing C^{∞}-functions in Minkowsky space M_4; $\mathfrak{D}_{-\chi}$ is the nuclear space/5/ of representation $T_{-\chi}$ of the group $SL(2,C)$. Therefore, for fixed $f\in\mathfrak{D}_{-\chi}$, $\varphi(u;f)$ is an operator-valued distribution in M_4.

All the Wightman axioms/6/ are imposed (except the finite-componentness of fields). In particular, the condition of covariance under the connected Poincaré group $\mathcal{P}_+^{\uparrow}$ (the connected inhomogeneous Lorentz group) is

$$U(a,A)\varphi(x;f)U(a,A)^{-1}=\varphi(\Lambda(A)x+a;\,T_{-\chi}(A)f), \tag{0.1}$$

where $a\in T_4$ (the group of translations), $A\in SL(2,C)$, $U(a,A)$ is the unitary representation in $\mathcal{H}$ of the universal covering group $\tilde{\mathcal{P}}_+^{\uparrow}$ of the group $\mathcal{P}_+^{\uparrow}$. Note that if the field φ transforms under the representation χ then the hermitian conjugate field φ^*, defined by

$$\varphi^*(u;f)=\left(\varphi(\bar{u};\bar{f})\right)^*\Big|_{D_0}, \quad \left(u\in S(M_4),\, f\in\mathfrak{D}_{-\chi^+}\right) \tag{0.2}$$

transforms under the complex conjugate representation $\chi^+=[-k,\bar{c}]$.

The field φ transforming under the representation χ can be realized in another equivalent way/7-8/ x/ as an operator-valued $SL(2,C)$-covariant distribution $\varphi(x;z)$ in the domain $M_4\times\mathring{\mathbb{C}}_2$ (where $\mathring{\mathbb{C}}_2=\mathbb{C}_2\setminus\{0\}$ is the two-dimensional complex space without the origine), the following homogeneity condition being fulfilled:

---

x/ The equivalence of this realization and the one presented above is based on an isomorphism of the space $\mathfrak{D}'_{-\chi}$, topological dual to $\mathfrak{D}_{-\chi}$, and the subspace d_χ of distributions in $\mathring{\mathbb{C}}_2=\mathbb{C}_2\setminus\{0\}$ homogeneous of index χ (cf./4/, Appendix A).

$$\varphi(x;\sqrt{\rho}\, e^{i\frac{\alpha}{2}} z) = \rho^{c-1} e^{ik\alpha} \varphi(x;z) \quad \text{for any} \quad \rho > 0,\ \alpha = \bar{\alpha}.$$

A finite-component field with m undotted and n dotted indices transforming under the finite-dimensional representation (m /2 , n /2) of $L_+^\uparrow$ is a special case of infinite-component fields when $\varphi(x;z)$ is a homogeneous polynomial in $(z,\bar{z})$ of the bi-degree (m , n):

$$\varphi(x;z) = \sum_{\alpha_1,\dots,\dot{\beta}_n = 1,2} \varphi_{\alpha_1\dots\alpha_m\dot{\beta}_1\dots\dot{\beta}_n}(x) \cdot z^{\alpha_1} \dots z^{\alpha_m} (\bar{z})^{\dot{\beta}_1} \dots (\bar{z})^{\dot{\beta}_n} . \tag{0.3}$$

There is the famous CPT -theorem for the Wightman finite-component fields: an antiunitary operator, Θ , exists such that for any field φ transforming under an irreducible representation of $L_+^\uparrow$

$$\Theta\, \varphi(x;f)\, \Theta^{-1} = \eta(\varphi) \cdot \varphi(-x;f)^*, \qquad \left(|\eta(\varphi)| = 1,\ f \in \mathcal{D}_{-\chi}\right) \tag{0.4}$$

holds. Feldman and Matthews(/2/, 1967) have posed the problem whether the CPT-theorem is valid for infinite-component fields x/. The positive answer has been of doubt since all proofs of the CPT-theorem are based essentially upon the finite-componentness of fields (see the discussion by Stoyanov and Todorov/10/). Indeed, there are examples/11/ of CPT-noncovariant free fields transforming under irreducible representations of the group $L_+^\uparrow$ and satisfying all the Wightman axioms, except the finite-componentness of fields; (the connection of spin with statistics may be chosen normal as well as abnormal).

In view of absence of CPT-theorem for infinite-component fields it is desirable to investigate other possible forms of covariance condition (compatible with the locality) for infinite-component fields

x/ A similar problem has been posed by Epstein in the local algebra approach/9/.

under space-time reflection. Of course, such covariance is to have an obvious group theoretical meaning and to be based on an analysis of symmetries with respect to the special Poincaré group $\mathcal{P}_+ = \mathcal{P}_+^\uparrow \cup \mathcal{P}_+^\downarrow$ generated by the connected Poincaré group, $\mathcal{P}_+^\uparrow$, and the space-time reflection, I.

§1 of the present paper is devoted to the analysis of symmetries with respect to the group $\mathcal{P}_+$ in quantum field theory. An analysis of this kind would be incomplete without taking into account superselection rules: it is well known that the neglect of these does not permit a consistent treatment of symmetries even for the Wightman ordinary fields. To clarify our point of view we expose in section 1.1. what we mean by symmetry in the presence of superselection rules. Respectively, in section 1.2 we state the notion (adopted further) of local covariance of quantized fields with respect to a symmetry group[x]. In section 1.4 the characterization of symmetries with respect to the group $\mathcal{P}_+$ in the presence of superselection rules is given. Using the Wigner analysis/12/ of projective unitary representations of $\mathcal{P}_+^\uparrow$ we show that every $\mathcal{P}_+$-symmetry can be described by a unitary-antiunitary representation of the group $\tilde{\mathcal{P}}_+$ (the description of the group $\tilde{\mathcal{P}}_+$ and other groups associated with $\mathcal{P}_+$ is presented in section 1.3). In turn, any such representation $U(g)$ of $\tilde{\mathcal{P}}_+$ is uniquelly defined by a unitary representation $U(a,A)$ of $\tilde{\mathcal{P}}_+^\uparrow$ and an antiunitary operator $\mathcal{J}$ satisfying the conditions:

$$\mathcal{J}\, U(a,A)\, \mathcal{J}^{-1} = U(-a,A)\,, \tag{0.5a}$$

$\mathcal{J}^2$ is a unitary superselection operator, $(\mathcal{J}^2)^2 = 1.$ (0.5b)

[x] Most of concepts of §1 are far from being original and are drawn to make the treatment self-contained.

Sections 1.5–1.6 deal with the statement of $\mathcal{P}_+$- and CPT-covariance conditions. Let $\mathfrak{A}=\{\varphi_1,\dots,\varphi_N\}$ be a complete finite set (which involves with every fields its hermitian conjugate) of infinite-component fields transforming under irreducible representations of $SL(2,C)$ and satisfying $\mathcal{P}_+^\uparrow$-covariance condition (0.1). Let further the $\mathcal{P}_+$-symmetry in the Hilbert space $\mathcal{H}$ be defined by the unitary representation $U(a,A)$ of $\tilde{\mathcal{P}}_+^\uparrow$ and the antiunitary operator $\mathfrak{J}$ satisfying (0.5). Then the $\mathcal{P}_+$-covariance condition of fields $\varphi_1,\dots,\varphi_N$ means that the field $\mathfrak{J}\varphi_i\mathfrak{J}^{-1}$ $(i=1,\dots,N)$ at an arbitrary point $x\in M_4$ is a linear combination of the fields $\varphi_1,\dots,\varphi_N$ at the point $-x$ x/. The CPT-covariance condition (0.4) is the special case of the $\mathcal{P}_+$-covariance condition.

To show that the $\mathcal{P}_+$-covariance condition for infinite-component quantized fields is independent of the basic principles, examples of free $\mathcal{P}_+$-noncovariant infinite-component fields are given in section 2.1xx/. Since free fields are characterized completely by two-point functions, it is a straightforward task to verify all the conditions of the Wightman reconstruction theorem for these fields. An example of section 2.2 illustrates the fact that the $\mathcal{P}_+$-covariance condition is more flexible than the CPT-covariance. Namely, we consider fields φ and φ^* (introduced earlier in/11/) transforming under the Majorana representation $\chi=[0,1/2]$; there is no operator Θ satisfying (0.4) but an operator $\mathfrak{J}$ exists such that

$$\mathfrak{J}\varphi(x;f)\mathfrak{J}^{-1}=\varphi(-x;\bar{f}),\quad \mathfrak{J}\varphi^*(x;f)\mathfrak{J}^{-1}=\varphi^*(-x;\bar{f}) \tag{0.6}$$

---

x/ Though a quantized field $\varphi(x)$ is an operator-valued distribution one may say about the value of the field at a point x bearing in mind a sesquilinear form defined on a dense domain in $\mathcal{H}$.

xx/ It is likely that using fields similar to those of section 2.1 one can obtain CPT-noninvariant local algebras.

holds. The interpretation of the operator $\mathcal{J}$ depends on superselection rules. E.g., if the fields φ and φ^* describe respectively "particles" and "antiparticles" (separated by superselection rules), then the operator $\mathcal{J}$ in (0.6) has the meaning of the PT-operator. Thus, infinite-component field theory allow PT-covariant but CPT-non-covariant local fields.

§1. Representations of the Special Poincaré Group in Quantum Field Theory

1.1. Symmetry and Superselection Rules

In the sequel we follow the Wigner notion of symmetry of a quantum system with respect to a group G .

Let $\mathcal{H}$ be a Hilbert space of a system and let $\mathfrak{M}$ be the set of all unit rays (in $\mathcal{H}$) representing pure states[x]. We shall suppose that the linear hull of the set $\mathfrak{M}$ is dense in $\mathcal{H}$. This condition is sufficient (/15/, ch. II, section 1.3) to represent (uniquelly) $\mathcal{H}$ as a direct sum $\mathcal{H}=\bigoplus_{\alpha}\mathcal{H}_{\alpha}$ of a family of subspaces called coherent subspaces so that i) $\mathfrak{M}=\bigcup_{\alpha}\mathfrak{M}_{\alpha}$ where $\mathfrak{M}_{\alpha}$ is the part of $\mathfrak{M}$ contained in $\mathcal{H}_{\alpha}$, ii) for every α the set of projections on rays of the set $\mathfrak{M}_{\alpha}$ is an irreducible set of operators in $\mathcal{H}_{\alpha}$.

---

[x] Nowadays the concept of pure state is the one derived from the concept of the observable considered as the primary object/3/. We shall restrict ourselves to the special scheme which corresponds to the traditional approach to the concept of pure state in quantum field theory/13,14/ and which seems to be sufficient for up-to-date needs. This approach takes for granted pure state vectors (otherwise, physically realizable vectors/6/, which may be "prepared experimentally"). The closed linear hull of the pure state vectors can be uniquelly decomposed into direct sum of so-called coherent subspaces (see below), and observables are defined as hermitian operators with vanishing matrix elements between different coherent subspaces.

A group G is called a symmetry group of the quantum system if every $g \in G$ is in line with a transformation, τ_g, of the set $\mathcal{M}$ of pure state vectors onto itself which preserves transition probabilities $|(\Phi_1, \Phi_2)|^2$ (where $\Phi_1, \Phi_2 \in \mathcal{M}$), the group law being fulfilled:

$$\tau_{g_1} \circ \tau_{g_2} = \tau_{g_1 g_2} . \tag{1.1}$$

Due to the ramification of the Wigner well-known theorem (/6/, Theorem 1-1), every transformation τ_g can be realized by an operator $T(g)$, a direct sum of a unitary operator and of an antiunitary operator, such that

any $\mathcal{H}_\alpha$ is mapped by $T(g)$ onto some $\mathcal{H}_\beta$ $\left(\beta = \beta(g,\alpha)\right)$, (1.2)

the restriction of $T(g)$ to a coherent subspace $\mathcal{H}_\alpha$ being unitary or antiunitary operator defined uniquelly up-to a phase. Due to (1.1) operators $T(g_1)\cdot T(g_2)$ and $T(g_1 \cdot g_2)$ realize the same trsansformation $\tau_{g_1 g_2}$ of $\mathcal{M}$, hence (in view of irreducibility in $\mathcal{H}_\alpha$ of the set of projections on rays of $\mathcal{M}_\alpha$) they may differ at most by a phase factor on any coherent subspace. Let us introduce the set $\mathcal{U}$ of unitary superselection operators in $\mathcal{H}$, i.e. operators of the form

$$\Omega = \sum_\alpha \omega_\alpha \cdot E_\alpha ,$$

where E_α is the projection on $\mathcal{H}_\alpha$ and ω_α is an arbitrary complex number of modulus 1. Then the multiplication law of operators $T(g)$ can be written in the form

$$T(g_1)\cdot T(g_2) = \Omega(g_1, g_2)\cdot T(g_1 \cdot g_2), \quad \Omega(g_1,g_2) \in \mathcal{U} . \tag{1.3}$$

In all cases of practical importance every $T(g)$ must be either unitary or antiunitary; (in particular, this is the only possibility for the special Poincaré group $\mathcal{P}_+$ as a symmetry group in a theory with the spectrum condition). Therefore we adopt the following definition.

Definition 1. We call the family $(T(g) : g \in G)$ of (anti) unitary operators satisfying (1.2), (1.3) the Unitary-Antiunitary Representation up-to a Superselection Factor of the symmetry group G, briefly, UARSF of the group G.

There is an important class of UARSF.

Definition 2. We say that UARSF, T, of a group G is generated by a unitary-antiunitary representation, U, of a group $\tilde{G}$ if i) there is a homomorphism, f, of $\tilde{G}$ onto G, ii) for any $\tilde{g} \in \tilde{G}$ operators $U(\tilde{g})$ and $T(f(g))$ differ at most by a superselection factor, $\lambda(g)$, i.e. $U(\tilde{g}) = \lambda(g) \cdot T(f(g))$ where $\lambda(g) \in \mathcal{U}$.

The following proposition is evident : if f is a homomorphism of a group $\tilde{G}$ onto G and U is a unitary-antiuritary representation of $\tilde{G}$, for any $\tilde{g}$ of the kernel of f $U(g)$ being a unitary superselection operator, then U generates an UARSF of G x/.

1.2. Covariance of Quantized Fields with Respect to a Symmetry Group

To formulate the notion of local covariance of fields it is useful to have a unique "big" field Φ instead of a complete set of fields $\varphi, \varphi^*, \psi, \psi^*, \ldots$ This is a bilinear continuous hermitian functional over $u \in S(M_4), F \in \mathfrak{D}$, where $\mathfrak{D}$ is some topological vector space with an involution $\mathfrak{L}$, $\Phi(u;F)$ being operators defined on a domain D_0 dense in $\mathcal{H}$. Hermiticity of the functional Φ implies

$$\Phi(\bar{u}; \mathfrak{L} F) = \left(\Phi(u;F) \right)^* \Big|_{D_0} . \tag{1.4}$$

x/ As to projective unitary-antiunitary representations (which correspond to the case when the set $\mathcal{U}$ consists only of multiples of the identity operator, i.e. when there is no superselection rules) of topological groups see /16,17/.

By $\Phi(x;F)$ we shall denote the operator-valued distribution in $x \in M_4$.

For example, if $\mathfrak{A} = \{\varphi_1, \dots, \varphi_N\}$ is a set of (in)finite-component fields, transforming under representations $\chi_1, \dots, \chi_N$ of $SL(2,C)$, any φ_i belonging to $\mathfrak{A}$ which its hermitian conjugate $\varphi_i^* = \varphi_{i'}$, then one may introduce a "big" field by

$$\Phi(x;F) = \sum_{i=1}^{N} \varphi_i(x; f_i), \tag{1.5}$$

where $F = (f_1, \dots, f_N)$ is an element of the space $\mathfrak{D} = \mathfrak{D}_{-\chi_1} \oplus \dots \oplus \mathfrak{D}_{-\chi_N}$, the involution $\mathfrak{I}$ being defined by the formula $\mathfrak{I}(f_1, \dots, f_N) = (\bar{f}_{1'}, \dots, \bar{f}_{N'})$.

Let G be a symmetry group which acts on the Minkowsky space M_4 so that $(gx-gy)^2 = (x-y)^2$ (i.e. there is a homomorphism of G into the full Poincaré group $\mathcal{P}$). Given a UARSF, T, of the group G, the most straightforward way to formulate the notion of G-covariance of a "big" field Φ is to group the UARSF T into a unitary antiunitary representation, U, of a group $\tilde{G}$, homomorphic to G, (hence, $\tilde{G}$ also acts on M_4). Then the local G-covariance condition means, roughly, that, for any $\tilde{g} \in \tilde{G}$, the field $U(\tilde{g})\Phi\, U(\tilde{g})^{-1}$ at any point x is a (anti)linear functional of the field Φ at the point $\tilde{g}x$.

<u>Definition 3 (Local G-covariance of fields).</u> Let Φ be a "big" quantized field in Hilbert space $\mathcal{H}$. Let G be a symmetry group which acts moreover in the Minkowsky space M_4, and let T be its UARSF in $\mathcal{H}$. We call the field Φ G-covariant if there are a group $\tilde{G}$, homomorphic to G, and a pair of unitary-antiunitary representation, U, and linear-antilinear representation, $\mathcal{T}$, of the group $\tilde{G}$ in $\mathcal{H}$ and $\mathfrak{D}$ respectively such that i) UARSF T of G is generated by the representation U of $\tilde{G}$ (see Definition 2), ii)

$$U(\tilde{g})\, D_0 = D_0, \tag{1.6a}$$

$$U(\tilde{g}) \cdot \Phi(x;F) \cdot U(\tilde{g})^{-1} = \Phi(\tilde{g}x; \mathcal{T}(\tilde{g})F). \tag{1.6b}$$

1.3. The Group $\mathcal{P}_+$ and its Covering Groups $\underset{\sim}{\mathcal{P}}_+$, $\widetilde{\mathcal{P}}_+$, $\bar{\mathcal{P}}_+$.

Let us define a few groups which are of use in the analysis of symmetries with respect to the Poincaré group involving the space-time reflection.

a) $\mathcal{P}_+^\uparrow$ is the connected Poincaré group; this is the semi-direct product[x] $T_4 \odot L_+^\uparrow$ of the group T_4 (of translations of the Minkowsky space M_4) and of the connected Lorentz group $L_+^\uparrow$.

b) $\widetilde{\mathcal{P}}_+^\uparrow = T_4 \odot \widetilde{L}_+^\uparrow$ (where $\widetilde{L}_+^\uparrow = SL(2,C)$ is the universal covering group of $L_+^\uparrow$), the action of $SL(2,C)$ on T_4 therewith is defined by the formula $a^A = \Lambda(A)a$ where $A \to \Lambda(A)$ is the standard homomorphism of $SL(2,C)$ onto $L_+^\uparrow$ /6/.

c) $\mathcal{P}_+$, the special Poincaré group, is the group of transformations $x \to (a,\Lambda)x \equiv \Lambda x + a$ of the Minkowsky space, where $a \in T_4$ and Λ is Lorentz transformation with determinant equal to 1. It is essential that $\mathcal{P}_+$ is isomorphic to the semi-direct product of two subgroups, $\mathcal{P}_+^\uparrow$ and $\mathcal{R}$; the latter consists of two elements— unity, $e=(0,1)$, and space-time reflection, I, with respect to the origine. The action of an $\alpha \in \mathcal{R}$ on $\mathcal{P}_+^\uparrow$ is defined by $g^\alpha = \alpha g \alpha^{-1}$, i.e.

$$g^e = g, \quad g^I \equiv (a,\Lambda)^I = (-a,\Lambda) \quad \text{for any} \quad g \in \mathcal{P}_+^\uparrow . \tag{1.7}$$

By the isomorphism $[g,\alpha] \leftrightarrow g \cdot \alpha$ (where $g \in \mathcal{P}_+^\uparrow$, $\alpha \in \mathcal{R}$) we shall identify the groups $\mathcal{P}_+^\uparrow \odot \mathcal{R}$ and $\mathcal{P}_+$.

d) $\underset{\sim}{\mathcal{P}}_+$ is the semi-direct product of $\widetilde{\mathcal{P}}_+^\uparrow$ and the free group $\underset{\sim}{\mathcal{R}} = \{\underset{\sim}{I}^n\}_{n=0,\pm1,\pm2,\dots}$ with one generator $\underset{\sim}{I}$. The action of the $\underset{\sim}{I}$ on $g \in \widetilde{\mathcal{P}}_+^\uparrow$ is defined by

---

[x] Let a group K act on a group G , for any $k \in K$ the mapping $g \to g^k$ being an automorphism of G . The semi-direct product $G \odot K$ of groups G and K is the set of all pairs $[g,k]$ $(g \in G, k \in K)$ with the composition law $[g_1,k_1]\cdot[g_2,k_2] = [g_1 \cdot g_2^{k_1}, k_1 k_2]$.

$$\tilde{g}^{\underset{\sim}{I}} = (a, A)^{\underset{\sim}{I}} = (-a, A). \tag{1.8}$$

The mapping $\underset{\sim}{I}^n \to I^n$ is a homomorphism of $\underset{\sim}{\mathcal{R}}$ onto $\mathcal{R}$; analogously, the mapping $[(a, A), \underset{\sim}{I}^n] \to [(a, \tilde{\Lambda}(A)), I^n]$ is a homomorphism of $\underset{\sim}{\mathcal{P}}_+$ onto $\mathcal{P}_+$.

e) $\tilde{\mathcal{P}}_+$ is the semi-direct product of $\tilde{\mathcal{P}}_+^{\uparrow}$ onto the cyclic group $\tilde{\mathcal{R}} = \{e, \tilde{I}, \tilde{I}^2, \tilde{I}^3\}$ of order 4, the action of $\tilde{I} \in \tilde{\mathcal{R}}$ on $\tilde{\mathcal{P}}_+^{\uparrow}$ being defined by a formula of the type (1.8) (with $\tilde{I}$ instead of $\underset{\sim}{I}$).

f) $\bar{\mathcal{P}}_+$ is the quotient group $\tilde{\mathcal{P}}_+/Q$, where Q is a central subgroup in $\tilde{\mathcal{P}}_+$ which consists of the identity and $[(0,-1), \tilde{I}^2]$.

There are obvious homomorphisms of every of the group a) - f) into $\mathcal{P}_+$, and actions of these group on the Minkowsky space is defined thereby. E.g., for $g' = [(a, A), \tilde{I}^n] \in \tilde{\mathcal{P}}_+$,

$$g'x = \Lambda(A)\, I^n x + a. \tag{1.9}$$

In every of the groups $\mathcal{P}_+^{\uparrow}, \tilde{\mathcal{P}}_+^{\uparrow}, \mathcal{P}_+, \underset{\sim}{\mathcal{P}}_+, \tilde{\mathcal{P}}_+, \bar{\mathcal{P}}_+$ we may choose elements with the translation vector $a=0$, we obtain thus subgroups $L_+^{\uparrow}, \tilde{L}_+^{\uparrow} = SL(2,C), L_+, \underset{\sim}{L}_+, \tilde{L}_+, \bar{L}_+$, respectively (i.e. the quotients of the above groups by the translation subgroup).

1.4. $\mathcal{P}_+$ -symmetry

We are interested in the group $\mathcal{P}_+$ as a symmetry group. Our aim is to reduce $\mathcal{P}_+$-symmetries to unitary-antiunitary representations of $\tilde{\mathcal{P}}_+$, the covering group of $\mathcal{P}_+$.

We start from the description of symmetries with respect to the connected subgroup $\mathcal{P}_+^{\uparrow} \subset \mathcal{P}_+$ (/6/, section 1-4). Of course, it is reasonable to restrict ourselves to continuous in g transformations $\tau(g): \mathcal{M} \to \mathcal{M}$ of pure state rays; in this case the equality of $\tau(e)$ to the identity transformation entails that every coherent subspace is invariant under $(T(g): g \in \mathcal{P}_+^{\uparrow})$. Since any $g \in \mathcal{P}_+^{\uparrow}$ can be represented in the form $g = (g_1)^2 \ldots (g_n)^2$, all $T(g)$ are unitary operators (due to (1.3)). Hence there is a unitary representation up-to a phase of the group $\mathcal{P}_+^\uparrow$ in every coherent subspace $\mathcal{H}_\alpha$. The Wigner analysis /12,16/ of projective unitary representations of $\mathcal{P}_+^\uparrow$ gives now that every UARSF of $\mathcal{P}_+^\uparrow$ is generated by a unitary representation U of $\tilde{\mathcal{P}}_+^\uparrow$ (see section 1.1, Definitions 1,2).

The representation of the abelian subgroup of translations, $(U(a,1) : a \in T_4)$ has the form $U(a,1) = \exp(iPa)$ where P is the 4-momentum operator (whose components are self-adjoint commuting operators). We suppose that the spectrum condition is fulfilled: the spectrum of the operator P lies in $\overline{V}_+ = \{p \in R_4 : p^0 \geq |\vec{p}|\}$, the point $0 \in \overline{V}_+$ corresponding to unique (vacuum) ray in $\mathcal{H}$.

Now we can characterize UARSF of the group $\mathcal{P}_+$.

Proposition. Let T be a UASRF of the group $\mathcal{P}_+^\uparrow$ in $\mathcal{H}$, hence there is a unitary representation U of the group $\tilde{\mathcal{P}}_+$ generating T. Let the spectrum condition be fulfilled. The family $(T(g) : g \in \mathcal{P}_+^\uparrow)$ can be continued to a UARSF $(T(g') : g' \in \mathcal{P}_+)$ of the group $\mathcal{P}_+$ if and only if there is an antiunitary operator, J, satisfying (1.2) such that

$$\text{i)} \quad J \cdot U(\tilde{g}) \cdot J^{-1} = U(\tilde{g}^{\underset{\sim}{I}}), \qquad \tilde{g} \in \tilde{\mathcal{P}}_+^\uparrow \tag{1.10}$$

where $\tilde{g}^{\underset{\sim}{I}} \in \tilde{\mathcal{P}}_+^\uparrow$ is defined by (1.8);

ii) J^2 is a unitary superselection operator x/. (1.11)

As this takes place, a UARSF T of $\mathcal{P}_+$ may be defined by

$$T([g, I^n]) = T(g) \cdot J^n \quad \text{where} \quad g \in \mathcal{P}_+^\uparrow, \; n = 0,1 \; ; \tag{1.12}$$

this may be considered as a UARSF of $\mathcal{P}_+$ generated by the unitary-antiunitary representation $V(\underset{\sim}{g})$ of the group $\underset{\sim}{\mathcal{P}_+}$ (see section 1.3, d) defined by

$$V([\tilde{g}, \underset{\sim}{I}^n]) = U(\tilde{g}) \cdot J^n, \qquad \tilde{g} \in \tilde{\mathcal{P}}_+^\uparrow, \quad n = 0, \pm 1, \pm 2, \ldots \tag{1.13}$$

---

x/ Condition (1.11) means that J^2 is a unitary operator with discrete spectrum and every pure state vector of $\mathcal{H}$ is an eigenvector of J^2. We call the operator J^2 the superselection operator with respect to the space-time reflection.

Moreover, multiplying the operator $\mathcal{J}$ by a superselection factor one is able to fulfill the equality

iii) $(\mathcal{J}^2)^2 = 1$; (1.14)

then the UARSF T of the group $\mathcal{P}_+$ may be considered as being generated by the unitary-antiunitary representation U of the group $\tilde{\mathcal{P}}_+$ (see section 1.3, e) defined by

$$U([\tilde{g}, \tilde{I}^n]) = U(\tilde{g}) \cdot \mathcal{J}^n , \qquad \tilde{g} \in \tilde{\mathcal{P}}_+^{\uparrow}, \qquad n = 0,1,2,3. \tag{1.15}$$

Proof. The sufficiency of (1.10), (1.11) follows from the formulae (1.12), (1.13). Let us prove the necessity of (1.10), (1.11)x/.

Let $\left(T([g,\alpha]) : g \in \mathcal{P}_+^{\uparrow}, \alpha \in \mathcal{R}\right)$ be the UARSF of $\mathcal{P}_+$:

$$T([g_1,\alpha_1]) \cdot T([g_2,\alpha_2]) = \Omega([g_1,\alpha_1],[g_2,\alpha_2]) \cdot T([g_1 g_2^{\alpha_1}, \alpha_1\alpha_2]). \tag{1.16}$$

It has been pointed out that, under suitable choice of superselection factors $\lambda(a,A) \in \mathcal{U}$, operators

$$U(a,A) = \lambda(a,A) \cdot T([(a,\Lambda(A)), e])$$

form a unitary representation of $\tilde{\mathcal{P}}_+^{\uparrow}$.

Denoting $\mathcal{J} = T(I)$ we want to prove that $\mathcal{J}$ satisfies (1.10). Indeed, (1.16) implies that the left and the right sides of (1.10) may differ at most by a superselection operator:

$$\mathcal{J}\, U(\tilde{g}) \mathcal{J}^{-1} = \Omega'(\tilde{g}) \cdot U(\tilde{g}^{\tilde{I}}) \equiv U(\tilde{g}^{\tilde{I}}) \cdot \Omega'(\tilde{g}) , \tag{1.17}$$

where

$$\Omega'(\tilde{g}) = \sum_{\alpha} \omega'_{\alpha}(\tilde{g}) \cdot E_{\alpha} , \qquad |\omega'_{\alpha}(\tilde{g})| = 1. \tag{1.18}$$

---

x/ Our argument is close to the Wigner analysis /18/ of invariance under the full Poincaré group, the only difference is that we incorporate superselection rules.

But $\tilde{g} \to \tilde{g}^{\mathrm{I}}$ is an automorphism of $\tilde{\mathcal{P}}_+^\uparrow$, hence $\mathcal{J}\, U(\tilde{g})\, \mathcal{J}^{-1}$ and $U(\tilde{g}^{\mathrm{I}})$ are two representations of $\tilde{\mathcal{P}}_+^\uparrow$. Eq. (1.17) implies then that $\Omega'(\tilde{g})$ is also a representation of $\tilde{\mathcal{P}}_+^\uparrow$ which is reduced to a one-dimensional representation, $\omega'_\alpha(\tilde{g})$, of $\tilde{\mathcal{P}}_+^\uparrow$ on every coherent subspaces $\mathcal{H}_\alpha$. The only possible such representation $\omega'_\alpha(\tilde{g})$ is trivial, hence $\Omega'(\tilde{g}) = 1$ and (1.10) is proved.

In particular, $\mathcal{J}\, U(a,1)\, \mathcal{J}^{-1} = U(-a,1)$ for any $a \in T_4$. Recall that, with our definition of a UARSF, the operator $\mathcal{J}$ is either unitary (hence $\mathcal{J} P \mathcal{J}^{-1} = -P$) or antiunitary (hence $\mathcal{J} P \mathcal{J}^{-1} = P$). But the only possibility (except for trivial one-dimensional $\mathcal{H}$) compatible with the spectrum condition is the antiunitarity of $\mathcal{J}$.

We have: $\mathcal{J}$ is the antiunitary operator, and $\mathcal{J}^2$ is (due to (1.16)) a unitary superselection operator:

$$\mathcal{J}^2 = \sum_\alpha \xi_\alpha \cdot E_\alpha , \qquad |\xi_\alpha| = 1. \tag{1.19}$$

For a coherent subspace $\mathcal{H}_\alpha$, there is one of two possibilities:

a) $\mathcal{H}_\alpha$ is invariant under $\mathcal{J}$, hence the equality $\mathcal{J} E_\alpha = E_\alpha \mathcal{J}$ holds; then the identity $\mathcal{J}(\mathcal{J}^2) = (\mathcal{J}^2)\mathcal{J}$ implies $\bar{\xi}_\alpha = \xi_\alpha$, i.e. $\xi_\alpha = \pm 1$;

b) There is another coherent subspace $\mathcal{H}_{\alpha'}$ such that $\mathcal{J}\mathcal{H}_\alpha = \mathcal{H}_{\alpha'}$ and $\mathcal{J}\mathcal{H}_{\alpha'} = \mathcal{H}_\alpha$, hence $\mathcal{J} E_\alpha = E_{\alpha'} \mathcal{J}$ and $\mathcal{J} E_{\alpha'} = E_\alpha \mathcal{J}$; then the identity $\mathcal{J}(\mathcal{J}^2)\mathcal{J}^{-1} = \mathcal{J}^2$ implies $\bar{\xi}_{\alpha'} = \xi_\alpha$.

Let us intriduce instead of $\mathcal{J}$ a new operator, $\mathcal{J}_1$, which differs from $\mathcal{J}$ by a superselection factor Ω:

$$\mathcal{J}_1 = \Omega\, \mathcal{J}, \qquad \text{where } \Omega = \sum_\alpha \eta_\alpha \cdot E_\alpha \quad \text{and} \quad \eta_\alpha = \pm\sqrt{\xi_\alpha} . \tag{1.20}$$

Using the equalities $\xi_\alpha = \pm 1$ in the case a) and $\bar{\xi}_{\alpha'} = \xi_\alpha$ in the case b), one easily verifies that $\mathcal{J}_1$ so defined satisfies (1.14). This completes the proof.

Example. Let us describe possible $\mathcal{P}_+$-symmetries assuming that the space $\mathcal{H}$ does not contain zero-mass states. A UARSF of $\mathcal{P}_+$ in $\mathcal{H}$ commutes with operators of mass and spin. Hence the subspace, $\mathcal{H}'$, orthogonal to the vacuum can be represented as a direct integral (or sum) of spaces $\mathcal{H}^{ms}$ with definite values of mass $(m>0)$ and spin $(s=0,\frac{1}{2},1,\ldots)$ and the representation of $\tilde{\mathcal{P}}_+^\uparrow$ in $\mathcal{H}'$ is reduced to representations U^{ms} of $\tilde{\mathcal{P}}_+^\uparrow$ in $\mathcal{H}^{ms}$. By $\mathcal{H}_\alpha^{ms}$ we shall denote a contribution of a coherent subspace $\mathcal{H}_\alpha\subset\mathcal{H}$ to $\mathcal{H}^{ms}$, $\mathcal{H}^{ms}=\bigoplus_\alpha \mathcal{H}_\alpha^{ms}$. Let H^{ms} be the space of the irreducible representation of $\tilde{\mathcal{P}}_+^\uparrow$ with mass m and spin s, which may be realized by functions $f(p;\zeta)$ of p, a vector on the upper hyperboloid of mass m in R_4, and of ζ, a two-dimensional complex vector, $f(p;\zeta)$ being a homogeneous polynomial of degree $2s$ in ζ. The scalar product in H^{ms} is given by

$$\langle f',f\rangle=\int\left(\frac{\partial}{\partial\zeta}\cdot\frac{\tilde{p}}{m}\cdot\frac{\partial}{\partial\bar{\zeta}}\right)^{2s}\overline{f'(p;\zeta)}\cdot f(p;\zeta)\,\frac{d^3p}{\sqrt{\vec{p}^{\,2}+m^2}}\,,$$

where

$$\tilde{p}=\begin{pmatrix} p^0-p^3 & -p^1+ip^2\\ -p^1-ip^2 & p^0+p^3\end{pmatrix}$$

(cf.[6], section 1-4). Every $\mathcal{H}_\alpha^{ms}$ as a space of representation of $\tilde{\mathcal{P}}_+^\uparrow$ with mass m and spin s, can be represented in the form $\mathcal{H}_\alpha^{ms}=H^{ms}\otimes\mathfrak{H}_\alpha$. Then elements of the $\mathcal{H}_\alpha^{ms}$ are vector-valued measurable functions $F_\alpha^{ms}(p;\zeta)$ with values in $\mathfrak{H}_\alpha$ such that

$$\|F_\alpha^{ms}\|^2\equiv\int\left(\frac{\partial}{\partial\zeta}\cdot\frac{\tilde{p}}{m}\cdot\frac{\partial}{\partial\bar{\zeta}}\right)^{2s}\left(F_\alpha^{ms}(p;\zeta),F_\alpha^{ms}(p;\zeta)\right)\frac{d^3p}{\sqrt{\vec{p}^{\,2}+m^2}}<\infty,$$

and the representation of $\tilde{\mathcal{P}}_+^\uparrow$ has the form

$$\left(U_\alpha^{ms}(a,A)\,F_\alpha^{ms}\right)(p;\zeta)=e^{ipa}\,F_\alpha^{ms}(\Lambda^{-1}(A);\zeta A). \tag{1.21}$$

Due to Proposition, for the definition of a UARSF of $\mathcal{P}_+$ it is sufficient to define in every $\mathcal{H}^{ms}$ an antiunitary operator $\mathfrak{J}^{ms}$ which maps every $\mathcal{H}_\alpha^{ms}$ onto some $\mathcal{H}_{\alpha'}^{ms}$ ($(\alpha')'$ being equal to α) and satisfies the equation

$$\mathfrak{J}^{ms}\cdot U^{ms}(a,A)\cdot(\mathfrak{J}^{ms})^{-1}=U^{ms}(-a,A). \tag{1.22}$$

As it has been mentioned above, there are two possibilities:
a) $\alpha' = \alpha$, then in view of (1.21) and (1.22) $\mathcal{J}_\alpha^{ms}$, the restriction of $\mathcal{J}^{ms}$ to $\mathcal{H}_\alpha^{ms}$ has the form

$$(\mathcal{J}_\alpha^{ms} F_\alpha^{ms})(p;\zeta) = \left(\zeta\epsilon\frac{\tilde{p}}{m}\cdot\frac{\partial}{\partial\bar{\zeta}}\right)^{2s} j_\alpha F_\alpha^{ms}(p;\zeta),$$

where $\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and j_α is an antiunitary operator in $\mathfrak{H}_\alpha$ such that $(j_\alpha)^2$ equals up-to a phase factor to the identity operator;
b) $\alpha' \neq \alpha$, then $\mathcal{J}^{ms}$ maps $\mathcal{H}_\alpha^{ms}$ onto $\mathcal{H}_{\alpha'}^{ms}$ and, vice versa, $\mathcal{H}_{\alpha'}^{ms}$ onto $\mathcal{H}_\alpha^{ms}$; $\tilde{\mathcal{J}}_\alpha^{ms}$, the restriction of $\mathcal{J}^{ms}$ to $\mathcal{H}_\alpha^{ms} \oplus \mathcal{H}_{\alpha'}^{ms}$, has the form

$$\left[\tilde{\mathcal{J}}_\alpha^{ms}\begin{pmatrix} F_\alpha^{ms} \\ F_{\alpha'}^{ms}\end{pmatrix}\right](p;\zeta) = \left(\zeta\epsilon\frac{\tilde{p}}{m}\cdot\frac{\partial}{\partial\bar{\zeta}}\right)^{2s}\cdot\begin{pmatrix} 0 & \eta_\alpha \cdot j_{\alpha'\alpha}^{-1} \\ j_{\alpha'\alpha} & 0 \end{pmatrix}\cdot\begin{pmatrix} F_\alpha^{ms}(p;\zeta) \\ F_{\alpha'}^{ms}(p;\zeta)\end{pmatrix},$$

where $j_{\alpha'\alpha}$ is an antilinear isometry of $\mathfrak{H}_\alpha$ onto $\mathfrak{H}_{\alpha'}$, η_α is an arbitrary phase factor.

1.5. $\mathcal{P}_+$-Covariance Condition for Infinite-Component Fields

We have proved in section 1.4 that any $\mathcal{P}_+$-symmetry is defined by a unitary-antiunitary representation, U, of the group $\tilde{\mathcal{P}}_+$. In line with Definition 3, we may take $G = \mathcal{P}_+$ and $\tilde{G} = \tilde{\mathcal{P}}_+$ to formulate a $\mathcal{P}_+$-covariance condition for a "big" field Φ. As the group $\tilde{\mathcal{P}}_+$ being generated by the subgroup $\tilde{\mathcal{P}}_+^\uparrow$ and the element $\tilde{I}$, it is sufficient to fulfil (1.6b) for $\tilde{g} \in \tilde{\mathcal{P}}_+^\uparrow$ and $\tilde{g} = \tilde{I}$ separately. It is natural therefore to take the following

<u>$\mathcal{P}_+$-covariance condition:</u>

i). There is a UARSF of $\mathcal{P}_+$ in $\mathcal{H}$ which due to Proposition of section 1.4 is defined by a unitary representation, U, of $\mathcal{P}_+^\uparrow$ and an antiunitary operator, $\mathcal{J}$, satisfying (1.2), (1.10), (1.11), (1.14). The $\mathcal{P}_+^\uparrow$-covariance condition for a "big" field $\Phi(x;F)$ is fulfilled in the form

$$U(a,A)\cdot\Phi(x;F)\cdot U(a,A)^{-1} = \Phi(\Lambda(A)x+a;\ \mathcal{T}(A)\,F), \tag{1.23}$$

where $\mathcal{T}(A)$ is a representation of $SL(2,C)$ in $\mathfrak{D}$ (cf. (0.1)).
ii). There is an antilinear operator $\mathcal{J}$ in $\mathfrak{D}$ such that

$$\mathcal{J}\,\mathcal{T}(A) = \mathcal{T}(A)\,\mathcal{J}, \tag{1.24.a}$$

$$(\mathcal{J})^4 = 1, \tag{1.24.b}$$

and the field Φ satisfies the condition

$$\mathfrak{J}\ \Phi(x;F)\ \mathfrak{J}^{-1} = \Phi(-x;\mathcal{J}\,F). \tag{1.25}$$

Taking hermitian conjugation of (1.25) and using antiunitary of we obtain

$$\mathfrak{J}\ \Phi(x;F)\,\mathfrak{J}^{-1} = \Phi(-x;\mathfrak{I}\,\mathcal{J}\,\mathfrak{I}^{-1}\,F), \tag{1.26}$$

where $\mathfrak{I}$ is the involution in $\mathfrak{D}$ (cf. (1.4)). For compatibility of (1.26) with (1.25) we shall assume that operators $\mathcal{J}$ and $\mathfrak{I}$ commute:

$$\mathfrak{I}\,\mathcal{J} = \mathcal{J}\,\mathfrak{I}. \tag{1.27}$$

We discuss in more detail <u>the case of infinite-component fields</u> (see Introduction). Let $\mathfrak{U} = \{\varphi, \varphi^*, \psi, \psi^*, \ldots\} = \{\varphi_1, \ldots, \varphi_N\}$ be a complete finite set of infinite-component fields transforming under irreducible representations $\chi_1, \ldots, \chi_N$ of $SL(2,C)$. All the Wightman axioms are assumed. To get rid of non-significant over-loading we make additional assumption: for any two fields $\varphi_i, \varphi_j \in \mathfrak{U}$

either $\chi_i = \chi_j$ or $\chi_i \neq \pm\chi_j$ (the latter means that χ_i and χ_j are non-equivalent). By $\varphi_{i'} \in \mathfrak{A}$ we denote the field hermitian conjugate to φ_i (see (0.2)):

$$\varphi_{i'} = (\varphi_i)^* . \qquad (1.28)$$

Let $\Phi(x;F)$ be the $\mathcal{P}_+$-covariant big field corresponding to the set $\mathfrak{A}$ (see section 1.2) and $\mathcal{J}$ be corresponding anti-linear $SL(2,C)$-invariant (due to (1.24)) operator in $\mathfrak{D} = \mathfrak{D}_{-\chi_1} \oplus \dots \oplus \mathfrak{D}_{-\chi_N}$. According to /5/, the most general form of $\mathcal{J}$ is

$$\mathcal{J}(f_1, \dots, f_N) = \Big(\sum_{j=1}^{N} a_{1j} \bar{f}_{j'}, \dots, \sum_{j=1}^{N} a_{Nj} \bar{f}_{j'} \Big), \qquad (1.29)$$

where $f_j \in \mathfrak{D}_{-\chi_j}$ and (a_{ij}) is $N \times N$ complex matrix such that $a_{ij} = 0$ if $\chi_i \neq \chi_j$. Now (1.25) takes the form

$$J\, \varphi_i(x;f)\, J^{-1} = \sum_{j=1}^{N} a_{ij} \big(\varphi_j(-x;f)\big)^*, \qquad f \in \mathfrak{D}_{-\chi_j}. \qquad (1.30)$$

Equations (1.24) and (1.26) impose additional restrictions on the matrix $\mathcal{A} = (a_{ij})$:

$$\mathcal{A}^4 = 1, \qquad (1.31)$$

$$\overline{a_{i'j'}} = a_{ij} . \qquad (1.32)$$

Note that by a linear nonsingular transformation,

$$\psi_i(x;f) = \sum_j B_{ij} \cdot \varphi_j(x;f), \quad \text{where } B_{ij} = 0 \text{ if } \chi_i \neq \chi_j, \qquad (1.33)$$

identity (1.30) allows to come from the set $\mathfrak{A} = \{\varphi_1, \dots, \varphi_N\}$ of fields to a new set $\mathfrak{A}_1 = \{\psi_1, \dots, \psi_N\}$ of fields (also transforming under irreducible representations of $SL(2,C)$) so that the new matrix $\mathcal{A}_1$ in (1.30) will be diagonal, the diagonal elements being equal to i^k $(k = 0,1,2,3)$:

$$J\, \psi_j(x;f)\, J^{-1} = \eta(\psi_j) \cdot \big(\psi_j(-x;f)\big)^* ; \quad \eta(\psi_j) = 1, -1, i, -i. \qquad (1.34)$$

Indeed, the new matrix $\mathcal{A}_1$ is obtained from $\mathcal{A}$ by a transformation $\mathcal{A}_1 = \bar{B}\mathcal{A}\bar{B}^{-1}$, and equation (1.31) guarantees the existence of a $\bar{B}$ which transforms $\mathcal{A}$ to diagonal form.

It is easy to translate the $\mathcal{P}_+$-covariance condition into the language of Wightman distributions. Let

$$\mathcal{W}_{i_1 \dots i_n}(x_1,\dots,x_n; f_1,\dots,f_n) = \langle 0 | \varphi_{i_1}(x_1; f_1) \cdots \varphi_{i_n}(x_n; f_n) | 0 \rangle \tag{1.35}$$

be vacuum expectation values of infinite-component fields $\varphi_{i_1},\dots,\varphi_{i_n} \in \mathfrak{A}$ (here $n=0,1,\dots$; $i_1,\dots,i_n = 1,\dots,N$; $f_j \in \mathcal{D}_{-\chi_j}$). Let $\mathcal{A} = (a_{ij})$ be $N \times N$ matrix satisfying (1.31), (1.32). One verifies immediately that the $\mathcal{P}_+$-covariance condition implies the following relations between the Wightman distributions

$$\mathcal{W}_{i_1 \dots i_n}(x_1,\dots,x_n; f_1,\dots,f_n) = \sum_{j_1,\dots,j_n = 1,\dots,N} \bar{a}_{i_1 j_1} \cdots a_{i_n j_n} \times \tag{1.36}$$

$$\times \; \mathcal{W}_{j_n \dots j_1}(-x_n,\dots,-x_1; f_n,\dots,f_1),$$

for any $n=0,1,\dots$; $i_1,\dots,i_n = 1,\dots,N$. On the other hand, due to the reconstruction theorem/6/ the distributions (1.35) define completely the set $\mathfrak{A}$ of quantized fields, and the fulfillment of (1.36) is the sufficient condition for the existence of an antiunitary operator $\mathfrak{J}$ satisfying (1.10), (1.14), (1.30) (/6/, theorem 3-9). If in addition $\mathfrak{J}$ satisfies the condition: $\mathfrak{J}$ maps every coherent subspace $\mathcal{H}_\alpha$ onto some coherent subspace $\mathcal{H}_{\alpha'}$ and $(\alpha')' = \alpha$ (cf.(1.2), (1.11)), then the set $\mathfrak{A}$ of fields defined by distributions (1.35) satisfies the $\mathcal{P}_+$-covariance condition.

In particular, a set $\mathfrak{A}$ of infinite-component free fields (for which (anti)commutators are multiples of the identity operator) are uniquelly defined by two-point functions, and the covariance condition (1.36) is equivalent to the following restrictions on two-point functions

$$W_{ij}(x;f,g) = \sum_{l,m=1,\dots,N} \bar{a}_{il}\bar{a}_{jm}\, W_{ml}(x;g,f), \tag{1.37}$$

where

$$i,j=1,\dots,N; \qquad W_{ij}(x-y;f,g) = \mathcal{W}_{ij}(x,y;f,g). \tag{1.38}$$

Remark. The G-covariance condition (section 1.2) involves a freedom in the choice of a group $\tilde{G}$, homomorphic to G. Given a UARSF of a group G, the larger the group $\tilde{G}$, the easier is to satisfy the resulting G-covariance condition. In the case of $G=\mathcal{P}_+$ one could take the group $\underset{\sim}{\mathcal{P}}_+$ (see section 1.3,d) as $\tilde{G}$. This results in abandoning the restriction (1.14), (1.24,b), (1.31) of the above treatment. There is a case when the choices of the groups $\underset{\sim}{\mathcal{P}}_+$ and $\tilde{\mathcal{P}}_+$ are equivalent. Namely, if, for any unitary superselection operator $\Omega \in \mathcal{U}$, there is an operator $\mathcal{O}$ in $\mathcal{D}$ such that $\Omega\Phi(x;F)\Omega^{-1}=\Phi(x;\mathcal{O}F)$, then the operator $\mathcal{J}$ satisfying (1.25) but not (1.14) may be substituted by an equivalent operator $\mathcal{J}_1=\Omega\mathcal{J}$ such that $(\mathcal{J}_1)^4=1$ (cf. Proposition of section 1.4).

1.6. CPT-covariance

It is complicated to formulate in field theory general CPT-covariance condition with reasonable physical meaning; this would demand a detailed description of superselection rules (in particular, the separation of "particle" and "antiparticle" pure states). We shall content ourselves with the standard version of CPT-covariance which requires that, for a field φ transforming under an irreducible representation of $SL(2,C)$, the field $\mathcal{J}\varphi\mathcal{J}^{-1}$ is equal up-to a factor to φ^* ($\mathcal{J}$ being space-time reflection operator).

CPT-covariance. Let $\mathcal{U}=\{\varphi_1,\dots,\varphi_N\}$ be a complete set of infinite-component $\mathcal{P}_+$-covariant fields (see section 1.5) transforming under irreducible representations of $SL(2,C)$. We call the fields CPT-covariant and write Θ instead of $\mathcal{J}$ if

$$\Theta\,\varphi(x;f)\,\Theta^{-1}=\eta(\varphi)\cdot\left(\varphi(-x;f)\right)^{*}, \qquad \eta(\varphi)=1,-1,i,-i, \tag{1.39}$$

for any field φ, the linear combination of the fields $\varphi_1,\dots,\varphi_N$, which transforms under an irreducible representation, χ, of $SL(2,C)$ (i.e. $\varphi(x;f)=\sum_j c_j\,\varphi_j(x;f)$ and $c_j=0$ if $\chi_j\neq\chi$).

The standard interpretation of Θ as the CPT-operator is as follows. First, Θ transforms fields at a point x to fields at $-x$. Second, let, for a field φ, one-particle states obtained by applying φ and φ^* to the vacuum be called - in line with some superselection rules - respectively "particle" and "antiparticle" states; then, due to (1.39), Θ converts particles into antiparticles and vice versa[x].

We have seen in section 1.5 that, starting from a set $\mathfrak{A}=\{\varphi_1,\dots,\varphi_N\}$ of $\mathcal{P}_+$-covariant fields transforming under irreducible representations of $SL(2,C)$, by a (local) linear transformation (1.38) one may obtain a new set, $\mathfrak{A}_1$, of fields so that the transformation low under space-time reflection takes the "diagonal" form (1.34). It is evident that this diagonal form of $\mathcal{P}_+$-covariance is basic independent, hence means CPT-covariance, if $\eta(\psi_i)$ are the same for fields ψ_i transforming under the same representations of $SL(2,C)$.

---

[x] In the theory of Wightman finite-component fields the condition $\Theta^2=(-1)^{2S}$ is fulfilled moreover, S being the spin operator, and $(-1)^{2S}$ being a superselection operator called the univalence operator. Using notations of sections 1.3 and 1.4 one can write this condition in the form $U([(0,1),\tilde{I}^2])\cdot U([(0,-1),e])\equiv U([(0,-1),\tilde{I}^2])=1$ which implies that $(U(g'):g'\in\tilde{\mathcal{P}}_+)$ is actually a representation of the group $\tilde{\mathcal{P}}_+$ (section 1.3,f).

§2. The Condition of Covariance Under Space-Time Reflection for Infinite-Component Fields is Independent of the Basic Principles (the Wightman Axioms)

2.1. Examples of $\mathcal{P}_+$-Noncovariant (Bose and Fermi) Infinite-Component Fields

We shall construct free infinite-component field, φ, transforming (with its hermitian conjugate, φ^*) under the self-adjoint Majorana representation $\chi = \chi^+ = [0, 1/2]$ of the group $SL(2,C)$. The fields φ and φ^* satisfy the Klein-Gordon equation and their (anti)commutators are multiples of the identity operator. All the Wightman axioms are fulfilled, the fields φ and φ^* are $\mathcal{P}_+$-noncovariant nonetheless.

Let $\mathfrak{D}_{-\chi}$ be the space of the irreducible representation $\chi = [0, 1/2]$ of $SL(2,C)$ realized by homogeneous C^∞-functions in the complex domain $\dot{\mathbb{C}}_2 = \mathbb{C}_2 \setminus \{0\}$ /5/. There is an invariant hermitian positive-definite form $(f|g)$ on $\mathfrak{D}_{-\chi}$, and we denote by X the completion of the pre-Hilbert space $\mathfrak{D}_{-\chi}$ with respect to this scalar product. Closures, $V(A)$, of the representation operators, $T_{-\chi}(A)$, are unitary operators in X. Moreover (see/19/, section 2.2.7), there is a covariant 4-vector of essentially self-adjoint operators Γ^μ defined (with all polynomials in Γ^μ) on $\mathfrak{D}_{-\chi}$ which commutes with the involution operator in X (the involution being the complex conjugation). Covariance properties of Γ^μ are expressed by $V(A)\Gamma^\mu V(A)^{-1} = \Lambda(A)^\mu{}_\nu \Gamma^\nu$. The last property we need is the following: the spectrum of the operator Γ^0 is discrete and bounded from below:

$$\Gamma^0 \geqq a \quad , \qquad a > 0. \tag{2.1}$$

The covariance of Γ^μ implies then

$$p_\mu \Gamma^\mu \geqq a\sqrt{(p)^2} \quad \text{for} \quad p \in V_+ = \{ p \in R_4 : p^0 > |\vec{p}| \}. \tag{2.2}$$

We define a free field $\varphi_1(x;f)=\varphi(x;f)$ and its hermitian conjugate $\varphi_2(x;f)=(\varphi(x;\bar{f}))^*$ (where $f\in \mathfrak{D}_{-\chi}$) by the following two-point functions

$$\langle 0|\,\varphi_\alpha(x;f)^*\cdot\varphi_\beta(y;g)|0\rangle=\Big(f\Big|\sum_{K=0}^{n} M^{(K)}_{\alpha\beta}\Big(i\Gamma^\mu\frac{\partial}{\partial x^\mu}\Big)^K\Big|g\Big)\cdot\frac{1}{i}D^-_m(x-y), \tag{2.3}$$

where $M^{(K)}\equiv\big(M^{(K)}_{\alpha\beta}\big)$ are 2 x 2 complex matrices, and $\frac{1}{i}D^-_m(x-y)=$ $=\frac{1}{(2\pi)^3}\int\theta(p^0)\delta(p^2-m^2)e^{-ipx}d^4p$ is the two-point function of the free hermitian scalar field with mass $m>0$. The φ_1 and φ_2 are Bose (respectively, Fermi) fields if n is even (respectively, odd). The trancated vacuum expectation values of order $n>2$ are assumed to be zero.

Proposition. Let $M^{(K)}$ be 2 x 2 hermitian matrices $(K=0,1,\dots,n)$ of the form

$$M^{(K)}=\begin{pmatrix}\gamma_K & \zeta_K\\ \bar{\zeta}_K & \gamma_K\end{pmatrix}\qquad \text{for } n+K \text{ even,} \tag{2.4.a}$$

$$M^{(K)}=\rho_K\cdot\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\qquad \text{for } n+K \text{ odd,} \tag{2.4.b}$$

where γ_K and ρ_K are real numbers, and ζ_K are complex numbers. Let the following conditions be fulfilled (where >0 stands for positive definiteness of a matrix):

$$M^{(n)}>0,\quad \sum_{K=0}^{n} M^{(K)}\lambda^K\geq 0\qquad \text{for any } \lambda\in[ma,\infty) \tag{2.5}$$

(a being defined by (2.1)). Then the two-point functions (2.3) define infinite-component free fields $\varphi_1=\varphi$ and $\varphi_2=\varphi^*$ transforming under representation $\chi=[0,1/2]$ of the Lorentz group and satisfying all the Wightman axioms, the statistics being of the Bose (respectively, of the Fermi) type if n is even (respectively, odd).

Proof. According to the Wightman reconstruction theorem/6/ it is sufficient to verify the following condition, 1) through 6), for the two-point functions (2.3).

1) The condition for hermitian conjugation of fields,

$$\langle 0|\varphi_\alpha(x;f)^* \cdot \varphi_\beta(y;g)|0\rangle = \overline{\langle 0|\varphi_\beta(y;g)^* \cdot \varphi_\alpha(x;f)|0\rangle}, \tag{2.6}$$

is implied by the hermiticity of the matrices $M^{(\kappa)}$.

2) The positive definiteness condition implies

$$\sum_{j,\ell=1}^{N} \sum_{\alpha,\beta=1}^{2} \int \langle 0|\varphi_\alpha(x;f_\alpha^{(j)})^* \cdot \varphi_\beta(y;f_\beta^{(\ell)})|0\rangle \cdot \overline{u^{(j)}(x)}\, u^{(\ell)}(y)\, d^4x\, d^4y \geq 0, \tag{2.7}$$

for any $f_\alpha^{(j)} \in \mathfrak{D}_{-\chi}, u^{(j)} \in S(M_4), N$. Vector functions $H(x;z) = \begin{pmatrix} h_1(x;z) \\ h_2(x;z) \end{pmatrix}$ with $h_\alpha(x;z) = \sum_{j=1}^{N} u^{(j)}(x) \cdot f_\alpha^{(j)}(z)$ form an algebraic tensor product $S(M_4) \otimes \mathfrak{D}$ where $\mathfrak{D} = \mathfrak{D}_{-\chi} \oplus \mathfrak{D}_{-\chi}$, and (2.7) is the condition of positive definiteness of a hermitian functional defined on this space.

The necessary condition for (2.7) is the inequality

$$\sum_{\alpha,\beta} (f_\alpha| \sum_{\kappa=0}^{n} M_{\alpha\beta}^{(\kappa)} (p_\mu \Gamma^\mu)^\kappa | f_\beta) \geq 0 \tag{2.8}$$

for any $f_1, f_2 \in \mathfrak{D}_{-\chi}$ and $p \in \mathcal{V}_+^m = \{p:\ p^0 > 0, (p)^2 = m^2\}$. Indeed, due to (2.8) for $N=1$, the distribution $F(x)$ defined by

$$F(x) = \frac{1}{\sqrt{2\pi}} \sum_{\alpha,\beta} \langle 0|\varphi_\alpha(x;f_\alpha)^* \cdot \varphi_\beta(y;f_\beta)|0\rangle \tag{2.9}$$

is positive definite. This implies (/20/, ch. II) that the Fourier transform of $F(x)$,

$$\tilde{F}(p) = \sum_{\alpha,\beta} (f_\alpha| \sum_{\kappa=0}^{n} M_{\alpha\beta}^{(\kappa)} (p_\mu \Gamma^\mu)^\kappa | f_\beta) \cdot \theta(p^0)\delta(p^2 - m^2), \tag{2.10}$$

is a non-negative measure, hence (2.8) is implied by (2.7).

On the other hand, (2.8) is sufficient for (2.7). Substituting $\sum_{j=1}^{N} \tilde{u}^{(j)}(p) f_\alpha^{(j)}$ instead of f_α into (2.8), where $\tilde{u}_\alpha^{(j)}(p)$ are arbitrary functions of $S(R_4)$, we obtain, for $p \in \mathcal{V}_+^m$,

$$\sum_{j,l=1}^{N} \sum_{\alpha,\beta=1}^{2} \overline{\tilde{u}^{(j)}(p)} \cdot \tilde{u}^{(l)}(p) \left(f_\alpha^{(j)} \Big| \sum_{\kappa=0}^{n} M_{\alpha\beta}^{(\kappa)} (p_\mu \Gamma^\mu)^\kappa \Big| f_\beta^{(l)} \right) \geq 0 ,$$

and this is equivalent to (2.7).

It remains to see that conditions (2.5) on the matrices $M^{(\kappa)}$ are sufficient for inequality (2.8) to be satisfied. We have pointed out that the operator Γ^0, hence $p_\mu \cdot \Gamma^\mu$ for $p \in V_+$, is self-adjoint and possesses a discrete spectrum. Let $\{e_\nu\}_1^\infty$ be the basis in X of eigen-vectors of the operator $p_\mu \Gamma^\mu$, e_ν corresponding to eigen-value λ_ν, and let $\{\xi^\nu\}$ be components of a vector $f \in \mathfrak{D}_{-\chi}$. Then the left hand side of (2.8) takes the form

$$\sum_\nu \sum_{\alpha,\beta} \sum_\kappa \left(M_{\alpha\beta}^{(\kappa)} \ (m\lambda_\nu)^\kappa \right) \cdot \overline{\xi_\alpha^\nu} \cdot \xi_\beta^\nu$$

and is non-negative due to (2.5).

3) Relativistic covariance,

$$\langle 0 | \varphi_\alpha(\Lambda(A)x + a; V(A)f)^* \cdot \varphi_\beta(\Lambda(A)y + a; V(A)g) | 0 \rangle = \tag{2.11}$$

$$= \langle 0 | \varphi_\alpha(x; f)^* \cdot \varphi_\beta(y; g) | 0 \rangle \qquad \text{for any } a \in T_4 ,\ A \in SL(2,C),$$

is evident.

4) Spectrum condition implies that the Fourier transform, (2.10), of the distribution $F(x)$ defined by (2.9) has a support in $\overline{V}_+$.

5) Locality implies

$$\langle 0 | \varphi_\alpha(x;f)^* \cdot \varphi_\beta(y;g) | 0 \rangle = (-1)^n \langle 0 | \varphi_\beta(y;g) \cdot \varphi_\alpha(x;f)^* | 0 \rangle \tag{2.12}$$

for $(x-y)^2 < 0$, or, equivalently,

$$\langle 0 | \varphi_\alpha(x;f)^* \varphi_\beta(y;g) | 0 \rangle = (-1)^n \sum_{\gamma,\delta=1}^{2} \sigma_{\alpha\gamma} \cdot \sigma_{\beta\delta} \cdot \langle 0 | \varphi_\delta(y;\overline{g})^* \cdot \varphi_\gamma(x;\overline{f}) | 0 \rangle ,$$

where $\sigma_{\alpha\beta} = \delta_{\alpha,3-\beta}$, $(x-y)^2 < 0$. We substitute (2.3) here and remember that the operators Γ^μ commute with the involution in X [hence, $(\overline{g} | \Gamma^{\mu_1} \ldots \Gamma^{\mu_k} | \overline{f}) = (f | \Gamma^{\mu_k} \ldots \Gamma^{\mu_1} | g)$]. Since $\frac{1}{i} D_m^-(x-y) = \frac{1}{i} D_m^-(y-x)$ for $(x-y)^2 < 0$, we obtain

$$M^{(\kappa)}_{\alpha\beta}=(-1)^{n+\kappa}\cdot\sum_{\gamma,\delta}\sigma_{\alpha\gamma}\cdot\sigma_{\beta\delta}\cdot M^{(\kappa)}_{\delta\gamma}\text{, i.e. } M^{(\kappa)}=(-1)^{n+\kappa}\sigma\cdot M^{(\kappa)T}\cdot\sigma\,, \tag{2.13}$$

where A^T denotes the transposed of a matrix A. This is the way we come to (2.4).

6) Cluster decomposition property[x/],

$$F(x+\lambda\cdot a)\to 0 \quad \text{in } S'(R_4) \quad \text{for } \lambda\to\infty\,, \tag{2.14}$$

where $F(x)$ is defined by (2.9) and a is an arbitrary space-like vector, is implied by the fast decrease of $D^-_m(x)$ for $x^2\to-\infty$.

Proposition is proved.

Let us investigate when the fields $\varphi_1=\varphi$ and $\varphi_2=\varphi^*$ defined by Proposition are $\mathcal{P}_+$ -covariant. According to section 1.5, $\mathcal{P}_+$ -covariance implies the existence of an antiunitary operator. $\mathcal{J}$, which leaves the vacuum invariant such that

$$\mathcal{J}\ \varphi_\alpha(x;f)\ \mathcal{J}^{-1}=\sum_\beta a_{\alpha\beta}\cdot\varphi_\beta(-x;f)^*\,, \tag{2.15}$$

where $\mathcal{A}=(a_{\alpha\beta})$ is a 2 x 2 matrix such that

$$\mathcal{A}^4=1\,, \tag{2.16}$$

$$a_{22}=\bar a_{11}\,,\quad a_{21}=\bar a_{12} \tag{2.17}$$

(cf. (1.31), (1.32)). In terms of the two-point functions, (2.15) takes the form

$$\langle 0|\varphi_\alpha(x;f)\cdot\varphi_\beta(y;g)|0\rangle=\sum_{\gamma,\delta=1}^{2}\bar a_{\alpha\gamma}\bar a_{\beta\delta}\cdot\langle 0|\varphi_\delta(x;g)\cdot\varphi_\gamma(y;f)|0\rangle \tag{2.18}$$

---

x/ The cluster decomposition property guarantees the uniqueness of the vacuum ray.

(cf. (1.37)). These results in the following restriction on matrices in (2.3):

$$\Pi^{(\kappa)} = \bar{\mathcal{A}} \cdot \Pi^{(\kappa)T} \cdot \bar{\mathcal{A}}^{T}, \qquad \kappa = 0, 1, \ldots, n, \tag{2.19}$$

where $\Pi^{(\kappa)} = \sigma \cdot M^{(\kappa)}$ are matrices with elements

$$\Pi^{(\kappa)}_{\alpha\beta} = M^{(\kappa)}_{3-\alpha,\beta} . \tag{2.20}$$

We assume that the following condition is fulfilled: at least one of $M^{(\kappa)}$ with odd $\kappa + n$ is non-zero. Then (2.4.b) and (2.19) give

$$\det \mathcal{A} = -1 . \tag{2.21}$$

Using (2.4.a) we reduce the equation (2.19) for even $\kappa + n$ to

$$a_{11} \cdot r_{\kappa} = - a_{22} \cdot r_{\kappa} = - \frac{1}{2} \left(a_{12} \zeta_{\kappa} - a_{21} \bar{\zeta}_{\kappa} \right) . \tag{2.22}$$

Equations (2.17), (2.21) and the inequality $r_n \neq 0$ give

$$a_{11} = 0, \ |a_{12}| = 1, \ \mathrm{Im}\left(a_{12} \cdot \zeta_{\kappa} \right) = 0 \quad \text{for even } \kappa + n . \tag{2.23}$$

Our final result is as follows. Let, for at least one odd $\kappa + n$, $M^{(\kappa)} \neq 0$. There is an antiunitary operator, $\mathcal{J}$, such that (2.15) is fulfilled (the matrix $\mathcal{A}$ in (2.15) being equal to

$$\mathcal{A} = \begin{pmatrix} 0 & \bar{\vartheta} \\ \vartheta & 0 \end{pmatrix} , \qquad |\vartheta| = 1 , \tag{2.24}$$

and $\mathcal{J}^2$ being equal to 1) if and only if

$$\zeta_{\kappa} = \pm |\zeta_{\kappa}| \cdot \vartheta \qquad \text{for all even } \kappa + n \text{ and some } \vartheta, \ |\vartheta| = 1 \tag{2.25}$$

(ζ_{κ} being defined by (2.4.a)).

It is clear that the condition (2.25) is not fulfilled in general. Let us consider two examples. We define Bose fields φ and φ^* putting in (2.3) $n=2$ and

$$M^{(0)}=\varepsilon\cdot\begin{pmatrix}0 & i\\ -i & 0\end{pmatrix},\quad M^{(1)}=\varepsilon\cdot\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\quad M^{(2)}=\begin{pmatrix}1 & \varepsilon\\ \varepsilon & 1\end{pmatrix}. \tag{2.26a}$$

Analogously, Fermi fields may be defined if we put $n=3$ and choose

$$M^{(0)}=\varepsilon\cdot\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\ M^{(1)}=\varepsilon\cdot\begin{pmatrix}0 & i\\ -i & 0\end{pmatrix},\ M^{(2)}=0,\ M^{(3)}=\begin{pmatrix}1 & \varepsilon\\ \varepsilon & 1\end{pmatrix}, \tag{2.26b}$$

ε in (2.26) being a sufficiently small positive number which guarantees the fulfillment of (2.6). The fields so defined satisfy all the Whightman axioms but are $\mathcal{P}_+$ -noncovariant.

2.2. An Example of a PT-Covariant but CPT-Noncovariant Infinite-Component Field

At last we consider an example which illustrates that the $\mathcal{P}_+$ -covariance condition in the form of the CPT-covariance (automatically fulfilled in the Wightman theory of finite-component fields) is too restrictive in the infinite-component field theory.

In the notations of section 2.1, we define Fermi fields φ and φ^* (first introduced in/11/) by the following two-point functions:

$$\left.\begin{aligned}
\langle 0|\varphi(x;f)^*\cdot\varphi(y;g)|0\rangle &= (f|i\Gamma^\mu\frac{\partial}{\partial x^\mu}+\varepsilon|g)\cdot\frac{1}{i}D_m^-(x-y),\\
\langle 0|\varphi(x;g)\cdot\varphi(y;f)^*|0\rangle &= (f|i\Gamma^\mu\frac{\partial}{\partial x^\mu}-\varepsilon|g)\cdot\frac{1}{i}D_m^-(x-y),\\
\langle 0|\varphi(x;f)\cdot\varphi(y;g)|0\rangle &= \langle 0|\varphi(x;f)^*\cdot\varphi(y;g)^*|0\rangle = 0,
\end{aligned}\right\} \tag{2.27}$$

where ε is a real number, $0<|\varepsilon|<m\cdot a$. (2.27) is a special case of (2.3) for $n=1$ and

$$M^{(0)}=\varepsilon\cdot\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\quad M^{(1)}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

The criterion (2.25) gives that, for any ϑ of modulus 1, there is antiunitary operator $J=J(\vartheta)$ (with $J^2=1$) such that

$$\mathcal{J}\varphi(x;f)\mathcal{J}^{-1}=\bar{\vartheta}\cdot\varphi(-x;\bar{f}),\ \mathcal{J}\varphi^*(x;f)\mathcal{J}^{-1}=\vartheta\cdot\varphi^*(-x;\bar{f}). \qquad (2.28)$$

For definiteness, we take $\vartheta=1$. In agreement with the section 1.5, the matrix $\mathcal{A}$ in (2.15) can be diagonalized if we pass to "hermitian" fields $\psi_1(x;f)=\frac{1}{2}\left(\varphi(x;f)+\varphi^*(x;f)\right)$, $\psi_2(x;f)=$ $=\frac{1}{2i}\left(\varphi(x;f)-\varphi^*(x;f)\right)$:

$$\left.\begin{aligned}\mathcal{J}\psi_1(x;f)\mathcal{J}^{-1}&=\psi_1(-x;f)^*[\equiv\psi_1(-x;\bar{f})],\\ \mathcal{J}\psi_2(x;f)\mathcal{J}^{-1}&=-\psi_2(-x;f)^*[\equiv-\psi_2(-x;\bar{f})].\end{aligned}\right\} \qquad (2.29)$$

But there is no antiunitary operator Θ satisfying the condition (1.39):

$$\Theta\varphi(x;f)\Theta^{-1}=\eta\cdot\varphi(-x;f)^*,\ \Theta\varphi(x;f)^*\Theta^{-1}=\bar{\eta}\cdot\varphi(-x;f)\ , \qquad (2.30)$$

since the necessary condition of CPT-covariance, namely,
$\langle 0|\varphi(x;f)^*\cdot\varphi(y;g)|0\rangle=\langle 0|\varphi(x;g)\cdot\varphi(y;f)^*|0\rangle$,
is not fulfilled (cf. (2.27)).

The structure of the two-point functions (2.27) implies that one-particle subspaces $\mathcal{H}_\varphi$ and $\mathcal{H}_{\varphi^*}$ obtained by applying fields φ and φ^*, respectively, to the vacuum are orthogonal. Hence, one may imagine superselection rules which separate $\mathcal{H}_\varphi$ from $\mathcal{H}_{\varphi^*}$ Let us agree to call the vectors of $\mathcal{H}_\varphi$ and $\mathcal{H}_{\varphi^*}$ "particle" and "antiparticle" states, respectively. Then the operator $\mathcal{J}$ in (2.28) realizes space-time reflection and leaves particle and antiparticle subspaces invariant. It is naturally to call $\mathcal{J}$ the PT -operator. Thus, the fields φ and φ^* are PT -covariant but not CPT-covariant[x].

The author would like to thank M.K. Polivanov, I.T. Todorov and V.N. Sushko for valuable discussions.

---

[x] If we put $\varepsilon=0$ in (2.27) then there are both the operator $\mathcal{J}$ satisfying (2.28) and the operator Θ satisfying (2.30), i.e. the resulting fields are both PT- and CPT-covariant.

References

1. H. Ruegg, W. Rühl, T.S. Santhanam. Helv. Phys. Acta, 40, 9 (1967).
2. G. Feldman, P.T. Matthews. Ann. Phys., 40, 19 (1966); Phys. Rev., 151, 1176 (1966); Phys.Rev., 154, 1241 (1967).
3. R. Haag, D. Kastler. J.Math.Phys., 5, 848 (1964).
4. A.I. Oksak, I.T. Todorov. Commun. Math.Phys., 14, 271 (1969).
5. I.M. Gel'fand, M.I. Graev, N.Ya. Vilenkin. Generalized Functions, vol. 5, ch.III. Integral Geometry and Representation Theory. New York, Academic Press, 1966.
6. R.F. Streater, A.S. Wightman. PCT, Spin and Statistics, and All That. New York, Benjamin, 1964.
7. Cao Chi, Dao vong Dyc, Nguyen van Hieu. Ann.Phys., 49 173 (1968).
8. I.T. Todorov, R.P. Zaikov. J.Math.Phys., 10, 2014 (1969).
9. H. Epstein. J.Math.Phys., 8, 750 (1967).
10. D. Tz. Stoyanov, I.T.Todorov. J.Math.Phys., 9, 2146 (1968).
11. A.I. Oksak, I.T. Todorov. Commun.Math.Phys., 11, 125 (1968).
12. E.P. Wigner. Ann.Math., 40, 149 (1939).
13. G.C. Wick, A.S. Wightman, E.P. Wigner. Phys.Rev., 88, 101 (1952).
14. A.S. Wightman. Ann.Inst. H.Poincaré, sér. A, 1, 403 (1964).
15. N.N. Bogoliubov, A.A. Logunov, I.T. Todorov. Foundations of the Axiomatic Approach to Quantum Field Theory. Moscow, Nauka, 1969.
16. V. Bargmann. Ann.Math., 59, 1 (1954).
17. K.R. Parthasarathy. Commun.Math.Phys., 15, 305 (1969).
18. E.P. Wigner. In "Group Theoretical Concepts and Methods in Elementary Particle Physics", ed. F. Gürsey, New York, Gordon and Breach, 1964.
19. I.M. Gel'fand, R.A. Minlos, Z. Ya. Shapiro. Representations of the Rotation and Lorentz Group and Their Applications. London Pergamon Press, 1963.
20. I.M. Gel'fand, N.Ya. Vilenkin. Generalized Functions, vol. 4. Applications of Harmonic Analysis. New York, Academic Press, 1964.

Received by Publishing Department
on June, 5, 1970.