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REPRESENTATIONS OF THE LORENTZ GROUP ON THE TWO-DIMENSIONAL COMPLEX SPHERE AND TWO-PARTICLE STATES

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Introduction

The definition of the proper Lorentz group as transformations leaving invariant the upper sheet of the hyperboloid makes it natural to associate the spherical functions with the one-particle states labelled by the four-momentum p^μ. These functions have been derived by J. Smorodinsky and N. Vilenkin by introducing several coordinate systems on the hyperboloid. There is however a number of homogeneous spaces according to various subgroups constituting the little group of a fixed point of the space in consideration. In the case of the subgroup $SO(2)\times SO(1,1)=SO(2,C)$ we arrive at the two-dimensional complex sphere $S_1^2+S_2^2+S_3^2=S^2$. (The surface of this sphere is a complex two-dimensional or real-four dimensional manifold). The Lorentz group can be regarded as a group of motion of the complex sphere. The reason for considering the Lorentz group in this manner is that with the parameters inherent to the complex sphere unitary representations take probably the simplest possible form.

In Section 1 it is proved that the connected part of the three-dimensional complex rotation group is isomorphic to the proper Lorentz group.

In Section 2 we review some results of ref./2/ where the explicit form of matrix elements of the unitary representations of the

Lorentz group has been obtained. Similarly to the real rotation group the matrix elements contain two of the Euler-angles ϕ and ψ (and in the present case their complex conjugate ϕ^*, ψ^* as well) in exponential factors. The dependence on θ (and θ^*) is contained in the $R^{jj^*}_{mm^*,nn^*}$ function which is the complex analogue of the familiar d^j_{mn} function. In addition to the dependence of the R-function on the complex conjugate angle θ^* the R-function differs from the d-function by the appearance of the d-function of second kind, which is the other linearly independent solution of the differential equation satisfied by d^j_{mn}.

In Section 3 we consider the spin-zero two-particle momentum eigenstates $|m_1, m_2; p_{(1)}, p_{(2)}\rangle$ and we produce an arbitrary $p_{(1)}, p_{(2)}$, by a Lorentz transformation from a standard system. (Since $p_{(1)}$ and $p_{(2)}$ are transformed simultaneously, $s = (p_{(1)} + p_{(2)})^2$ is fixed). By definition the standard frame is one where three-momenta are directed along the z-axis and their absolute values have a given ratio. This frame may be the C.M. system or the equal-velocity system etc. Afterwards we perform a Lorentz transformation characterized by the complex Euler-angles ϕ, θ, ψ. It is clear that one of the 6 parameters, namely $\mathrm{Re}\,\psi$ (rotation about z-axis) is irrelevant, while the remaining 5 parameters together with the total C.M. energy s, determine unambiguously the two momenta.

It is expedient to give the 5 parameters of the Lorentz transformation in the following manner. We form an antisymmetric tensor from the two momenta by putting $S^{\mu\nu} = \epsilon^{\mu\nu\kappa\lambda} p^{(1)}_{\kappa} p^{(2)}_{\lambda}$. The self-dual part of this tensor defines the semi-bivector $\vec{S} = p^0_{(1)}\vec{p}_{(2)} - p^0_{(2)}\vec{p}_{(1)} - i\vec{p}_{(1)} \times \vec{p}_{(2)}$ which forms a complex sphere $\vec{S}^2 = \mathrm{const}$. It is easy to see that a boost along z-axis by an arbitrary angle $\mathrm{Im}\,\psi = \chi$ leaves unaltered the north pole and the subsequent 4 transformations translate it to an other position which is characterized just by the above complex polar-angles θ, ϕ, while it remains, of course, on the surface of a complex sphere. Since the radius of the sphere considered is in a straightforward connection with the centre-of-mass energy s we get finally that momenta of the two particles can be parametrized by giving the radius of the complex sphere s, the complex polar angles of the unit vector $\vec{n} = \vec{s}/s$ and the (real) boost angle χ i.e. $| m_1, m_2; p_{(1)}, p_{(2)} \rangle = | m_1, m_2; s, \vec{n}, \chi \rangle$.

Construction of the two-particle spherical functions – treated in Section 4 – with the above parametrization is straightforward. We separate the motion due to the boost χ and form the functions $\langle \vec{n} | j, j^*; m, m^* \rangle$ transforming according to the unitary irreducible representation labelled by j, j^*. In a certain sense the above construction is a generalization of the Jacob-Wick formalism which separates the centre-of-mass motion and builds up the spherical functions (spin-zero case) and the representations (non-zero spin case) of the three-dimensional rotation group on the two particle states. It is worth mentioning that though we have treated here spinless particles, the generalization to the non-zero spin case is straightforward.

1. Lorentz Group and the Three-Dimensional Complex Rotation Group

The group of three-dimensional complex rotations is one leaving invariant the quadratic form $s_1^2 + s_2^2 + s_3^2 = s^2$ i.e. it consists of matrices satisfying $0^T 0 = 1$. The Lie-algebra of the $0(3, C)$ group coincides with that of the Lorentz group. In order to make clear the global properties it will be shown that the connected part of the $0(3, C)$ group is isomorphic to the proper Lorentz group $L_+^\uparrow$.

On multiplying by the Pauli matrices we associate with $\vec{s}$ a 2x2 matrix:

$$\hat{s} = s_i \sigma_i \qquad (i = 1, 2, 3) . \tag{1.1}$$

The reverse relation

$$s_i = \frac{1}{2} \mathrm{Tr} (\sigma_i \hat{s}) . \tag{1.2}$$

Let us perform a transformation of (1.1) by an arbitrary non-singular 2x2 matrix A':

$$\hat{S}' = A' \hat{S} A'^{-1} .$$

Clearly $\mathrm{Det}\,\hat{S}' = \mathrm{Det}\,\hat{S}$, which implies $\vec{S}'^2 = \vec{S}^2$. Consider now an element of $SL(2,C)$:

$$A = \frac{1}{\sqrt{\mathrm{Det}\,A'}} A'$$

Since

$$\hat{S}' = A' \hat{S} A'^{-1} = A \hat{S} A^{-1} \tag{1.3}$$

the values of $\mathrm{Det}\,A'$ is irrelevant, so it is sufficient to restrict ourselves to the elements of $SL(2,C)$.

Transformation (1.3) induces by (1.2) and $SO(3,C)$ transformation : $S_i' = O_{ik} S_k$ where

$$O_{ik} = \frac{1}{2} \mathrm{Tr}\,(\sigma_i A \sigma_k A^{-1}) . \tag{1.4}$$

Since, as is well known , the $SL(2,C)$ group is connected [4] and O_{ik} depends continuously on A , eq. (1.4) maps the $SL(2,C)$ group onto the connected part of the $O(3,C)$ group. Conversely, it is easy to see that A and B correpond to the same rotation if and only if $A = \pm B$. The inverse mapping reads:

$$A = \frac{\sigma_i (O_{ik} + \frac{1}{3} \delta_{ik}) \sigma_k}{\pm \sqrt{\mathrm{Det}\,\sigma_i (O_{ik} + \frac{1}{3} \delta_{ik}) \sigma_k}}$$

and thus there is a two-to-one homomorphy between $SL(2,C)$ and the connected part of $O(3,C)$. The same connection holds between the $SL(2,C)$ and $L_+^\uparrow$ groups and thus the chain $O_{ik} \begin{smallmatrix} \nearrow +A \searrow \\ \searrow -A \nearrow \end{smallmatrix} L_+$

involves an isomorphy between the proper Lorentz group and the connected part of the three-dimensional rotation group. In what follows we shall need some simple relations transforming Lorentz covariant quantities to the three-dimensional complex ones. To an antisymmetric tensor $S^{\mu\nu}$ we can associate a complex vector and its complex conjugate by

$$S_i = -\frac{1}{4i}\, \sigma_{i\alpha}{}^{\dot\alpha}\, \sigma_{0\dot\alpha}{}^{\beta}\, \sigma_{\mu\beta}{}^{\dot\gamma}\, \sigma_{\nu\dot\gamma}{}^{\alpha}\, S^{\mu\nu} \tag{1.5}$$

$$S_i = -\frac{1}{4i}\, \sigma_{i\alpha}{}^{\dot\alpha}\, \sigma_{\mu\dot\alpha}{}^{\beta}\, \sigma_{\nu\beta}{}^{\dot\gamma}\, \sigma_{0\dot\gamma}{}^{\alpha}\, S^{\mu\nu} \tag{1.6}$$

$(i = 1,2,3;\ \mu,\nu = 0,\ldots,3,\ \alpha,\beta,\gamma = 1,2)$,

where $(\sigma_0)_{\alpha\dot\alpha}$, $(\sigma_i)_{\alpha\dot\alpha}$ are the unit and Pauli-matrices. The raising and lowering of the indices is accomplished with the aid of the metric tensor $\epsilon_{\alpha\beta} = \epsilon^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. (2.5) and (2.6) project out the self-dual and anti-self-dual parts of the antisymmetric tensor $S^{\mu\nu}$. Inverse formulas read:

$$S_{\mu\nu} = \frac{1}{4i}\left(S_i\, \sigma_{i\alpha}{}^{\dot\alpha}\, \sigma_{0\dot\alpha}{}^{\beta}\, \sigma_{\mu\beta}{}^{\dot\gamma}\, \sigma_{\nu\dot\gamma}{}^{\alpha} + S_i^*\, \sigma_{i\alpha}{}^{\dot\alpha}\, \sigma_{\mu\dot\alpha}{}^{\beta}\, \sigma_{\nu\beta}{}^{\dot\gamma}\, \sigma_{0\dot\gamma}{}^{\alpha} \right) \tag{1.7}$$

$$S^D_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\kappa\lambda} S^{\kappa\lambda} = -\frac{1}{4i}\left(S_i\, \sigma_{i\alpha}{}^{\dot\alpha}\, \sigma_{0\dot\alpha}{}^{\beta}\, \sigma_{\mu\beta}{}^{\dot\gamma}\, \sigma_{\nu\dot\gamma}{}^{\alpha} - S_i^*\, \sigma_{i\alpha}{}^{\dot\alpha}\, \sigma_{\mu\dot\alpha}{}^{\beta}\, \sigma_{\nu\beta}{}^{\dot\gamma}\, \sigma_{0\dot\gamma}{}^{\alpha} \right) \tag{1.8}$$

($\epsilon_{\mu\nu\kappa\lambda}$ is totally antisymmetric, $\epsilon_{0123} = -1$).

The complex vector S_i under Lorentz transformations transforms by the familiar three-dimensional representation of the rotation group but instead of real Euler angles we have to put complex ones.

Be $S_{\mu\nu}$, $T_{\mu\nu}$ two antisymmetric tensors then the invariants can be expressed simply as

$$\frac{1}{2} S^{\mu\nu} T_{\mu\nu} = \mathrm{Re}\,(\vec S\, \vec T\,) \tag{1.9}$$

$$\frac{1}{4}\epsilon_{\mu\nu\kappa\lambda} S^{\mu\nu} T^{\kappa\lambda} = \mathrm{Im}\,(\vec S\, \vec T\,)\ . \tag{1.10}$$

2. Unitary Representations of the Lorentz Group

Denoting the generators of spatial and hyperbolic rotations by M_k, N_k, respectively ($k = 1,2,3$) and introducing the combinations

$$J_k = \frac{1}{2}(M_k + iN_k), \quad K_k = \frac{1}{2}(M_k - iN_k) \tag{2.1}$$

we define a complex rotation about the k -axis by

$$C_k(\alpha) = e^{-i\alpha J_k} e^{-i\alpha^* K_k} = e^{-i(\alpha_1 M_k - \alpha_2 N_k)} \tag{2.2}$$

with $\alpha = \alpha_1 + i\alpha_2$, $\alpha^* = \alpha_1 - i\alpha_2$. Then an element of the proper Lorentz group can be decomposed as

$$C(\phi, \theta, \psi) = C_3(\phi)\ C_2(\theta)\ C_3(\psi), \tag{2.3}$$

where

$$\phi = \phi_1 + i\phi_2, \quad \theta = \theta_1 + i\theta_2, \quad \psi = \psi_1 + i\psi_2 \tag{2.4}$$

with the range of parameters

$$0 \leq \phi_1, \psi_1 < 2\pi, \quad 0 \leq \theta_1 < \pi, \quad -\infty < \phi_2, \theta_2, \psi_2 < \infty . \tag{2.5}$$

The product (2.1) decomposes the Lorentz transformations into a complex rotation $C_3(\psi)$, which constitutes the little group $SO(2) \times \times SO(1,1) = SO(2,C)$ of the north pole $\overset{0}{S} = (0,0,S)$ of the complex sphere, and subsequent rotations $C_3(\phi)\ C_2(\theta)$ which translate $\overset{0}{S}$ to an arbitrary position on the surface of the sphere.

Matrix elements of the unitary representations will be computed in a basis where M_3 and N_3 are diagonal. They are the generators of $SO(2,C)$ subgroup. The eigenvalue of M_3 take the values $\mu = 0, \pm 1, \pm 2, ..$and $\mu = \pm\frac{1}{2}, \pm\frac{3}{2}, ...$ for single-valued and double-valued representations respectively, while the eigenvalues of N_3, denoted by ν, are continuous. In view of (2.1) states can be labelled by the complex eigenvalues of the generators J_3 and K_3 i.e. by $m = \frac{1}{2}(\mu + i\nu)$ and $m^* = \frac{1}{2}(\mu - i\nu)$.

The matrix elements of the unitary representations are simultaneous eigenfunctions of the Casimir operators $\vec{J}^2 = \frac{1}{4}(\vec{M}^2 - \vec{N}^2 + 2i\vec{M}\vec{N})$ and $\vec{K}^2 = \frac{1}{4}(\vec{M}^2 - \vec{N}^2 - 2i\vec{M}\vec{N})$:

$$\vec{J}^2 T^{jj^*}_{mm^*;nn^*}(\phi,\theta,\psi) = j(j+1) T^{jj^*}_{mm^*;nn^*}$$

$$K^2 T^{jj^*}_{mm^*,nn^*}(\phi,\theta,\psi) = j^*(j^*+1) T^{jj^*}_{mm^*;nn^*} . \tag{2.6}$$

By virtue of (2.2) the ϕ and ψ dependence of T separates:

$$T^{jj^*}_{mm^*;nn^*} = e^{-i(m\phi + m^*\phi^* + n\psi + n^*\psi^*)} R^{jj^*}_{mm^*;nn^*}(\cos\theta, \cos\theta^*).$$

Write j in the form $j = \frac{1}{2}(j_0 - 1 + i\sigma)$ where j_0 and σ are real for the principal series. Then a detailed investigation shows [2] that Eqs. (2.6) have a regular solution if and only if j_0 takes integer (or half-integer) values. It has the form

$$R^{jj^*}_{mm^*;nn^*}(z, z^*) = \frac{N^j_{mn}}{4i\sqrt{\sin\pi(m-n)\sin\pi(m^*-n^*)}} (P^*Q - PQ^*) \tag{2.7}$$

with

$$N^j_{mn} = \left[\frac{\sin\pi(j-n)\sin\pi(j^*-n^*)}{\sin\pi(j-m)\sin\pi(j^*-m^*)}\right]^{1/2}.$$

Here P is the familiar d-function:

$$P = P^j_{mn}(z) = n^j_{mn}\left(\frac{1-z}{2}\right)^{\frac{m-n}{2}} \left(\frac{1+z}{2}\right)^{\frac{m+n}{2}} F\left(-j+m, j+m+1, m-n+1; \frac{1-z}{2}\right) \tag{2.8}$$

$$n^j_{mn} = \frac{1}{\Gamma(m-n+1)} \left[\frac{\Gamma(j+m+1)\Gamma(j-n+1)}{\Gamma(j+n+1)\Gamma(j-m+1)}\right]^{1/2}, \quad z = \cos\theta .$$

Q is the d-function of the second kind, namely

$$Q = Q^{j}_{mn} = \frac{\sin\pi(j-m)}{\sin\pi(j-n)}\, n^{j}_{nm} \left(\frac{1-z}{2}\right)^{\frac{n-m}{2}} \left(\frac{1+z}{2}\right)^{\frac{n+m}{2}} F\left(-j+n, j+n+1; n-m+1; \frac{1-z}{2}\right). \quad (2.9)$$

Eqs. (2.7), (2.8), (2.9) are valid for $\mathrm{Re}(m \pm n) \geq 0$. The generalization for arbitrary m, n is trivial/2/.

$T^{jj^*}_{mm^*;nn^*}$ functions satisfy orthogonality and completeness relations (cf./2/).

The normalized spherical functions are

$$f^{jj^*}_{mm^*}(\theta,\phi) = \left[\frac{(2j+1)(2j^*+1)}{8\pi^2}\right]^{1/2} e^{i(m\phi+m^*\phi^*)} R^{jj^*}_{mm^*;00}(\cos\theta, \cos\theta^*). \quad (2.10)$$

3. Two-Particle States and the Complex Sphere

Consider the momentum eigenstate of two spin-zero particles: $|m_1, m_2; p_{(1)}, p_{(2)}>$. It can be produced by a Lorentz transformation from a standard system, where the three-momenta directed along z-axis and their absolute values have a given ratio δ, i.e. where $p_{(1)}$ and $p_{(2)}$ have the components:

$$\overset{0}{p}_{(1)} \equiv (E_1, 0, 0, P_1), \; \overset{0}{p}_{(2)} \equiv (E_2, 0, 0, P_2) \qquad \text{with } \frac{P_1}{P_2} = -\delta. \quad (3.1)$$

E.g. for the equal velocity frame $\delta = m_1/m_2$ for the centre-of-mass frame $\delta = 1$ etc.

E_A and P_A ($A = 1,2$) can be expressed in terms of invariants as:

$$E_A = \frac{1}{\sqrt{D}}(m_A^2 + p_{(1)} p_{(2)} \delta), \; P_1 = \frac{S\delta}{\sqrt{D}}, \; P_2 = -\frac{S}{\sqrt{D}}, \quad (3.2)$$

with

$$D = m_1^2 + m_2^2 \delta^2 + 2 p_{(1)} p_{(2)} \delta .$$

S is given by eq. (3.8).

The standard frame has been fixed by the value of δ. Its relation to the boost angle χ_0 which connects the standard and centre-of-mass frames is:

$$\delta = -\frac{P_1}{P_2} = \frac{S + \mathrm{sh}\,\overset{0}{\chi}\ (m_1^2 + p_{(1)} p_{(2)})}{S - \mathrm{sh}\,\overset{0}{\chi}\ (m_2^2 + p_{(1)} p_{(2)})}. \tag{3.3}$$

In view of (2.3) an arbitrary state $|p_{(1)}, p_{(2)}>$ can be produced from the standard frame as follows. At first we boost along z-axis by an angle $\chi \equiv \mathrm{Im}\,\psi$. (Spatial rotation about z-axis by an angle is unnecessary since it leaves unaltered $\overset{0}{p}_{(1)}$ and $\overset{0}{p}_{(2)}$). Afterwards we perform two subsequent complex rotations by angles θ and ϕ. Thus we have

$$|p_{(1)}, p_{(2)}> = C_3(\phi) C_2(\theta) C_3(\chi) | \overset{0}{p}_{(1)}, \overset{0}{p}_{(2)} >. \tag{3.4}$$

Using the matrix elements of the 4x4 representation listed in Appendix we obtain the parametrization of the two momenta defined by (3.4):

$$p_A^0 = (\mathrm{ch}\phi_2 \mathrm{ch}\,\theta_2 \mathrm{ch}\chi + \mathrm{sh}\,\phi_2 \cos\theta_1 \mathrm{sh}\chi) E_A - (\mathrm{ch}\,\phi_2 \mathrm{ch}\theta_2 \mathrm{sh}\chi + \mathrm{sh}\phi_2 \cos\theta_1 \mathrm{ch}\chi) P_A$$

$$p_A^1 = (\sin\phi_1 \mathrm{sh}\theta_2 \mathrm{ch}\chi - \cos\phi_1 \sin\theta_1 \mathrm{sh}\chi) E_A + (\cos\phi_1 \sin\theta_1 \mathrm{ch}\chi - \sin\phi_1 \mathrm{sh}\theta_2 \mathrm{sh}\chi) P_A$$

$$(3.5)$$

$$p_A^2 = -(\cos\phi_1 \mathrm{sh}\theta_2 \mathrm{ch}\chi + \sin\phi_1 \sin\theta_1 \mathrm{sh}\chi) E_A + (\cos\phi_1 \mathrm{sh}\theta_2 \mathrm{sh}\chi + \sin\phi_1 \sin\theta_1 \mathrm{ch}\chi) P_A$$

$$p_A^3 = -(\mathrm{sh}\phi_2 \mathrm{ch}\,\theta_2 \mathrm{ch}\,\chi + \mathrm{ch}\phi_2 \cos\theta_1 \mathrm{sh}\chi) E_A + (\mathrm{sh}\phi_2 \mathrm{ch}\theta_2 \mathrm{sh}\chi + \mathrm{ch}\phi_2 \cos\theta_1 \mathrm{ch}\chi) P_A$$

$(A = 1,2)$.

Let us form now an atisymmetric tensor from the two momenta

$$S^{\mu\nu} = \epsilon^{\mu\nu\kappa\lambda} p^{(1)}_{\kappa} p^{(2)}_{\lambda} \tag{3.6}$$

by eq. (1.5) the corresponding complex vector is:

$$\vec{S} = \vec{p}_{(1)} p^{0}_{(2)} - \vec{p}_{(2)} p^{0}_{(1)} - i\, \vec{p}_{(1)} \times \vec{p}_{(2)} . \tag{3.7}$$

It has only one (real) invariant, the radius of the complex sphere

$$S^2 = \vec{S}^2 = p_{(1)} p_{(2)} - m_1^2 m_2^2 = \frac{1}{4}[s-(m_1+m_2)^2][s-(m_1-m_2)^2], \tag{3.8}$$

where

$$s = (p_{(1)} + p_{(2)})^2 .$$

(In what follows, it is supposed that $S^2 \neq 0$. Though the singular $S^2=0$ case is simple, it requires a particular treatment therefore it will be discussed elsewhere).

Under simultaneous Lorentz transformations (cf. Appendix) of the two momenta, $\vec{S}$ transforms simply by the familiar rotation matrix

$$\begin{bmatrix} -\sin\phi\sin\psi + \cos\theta\cos\phi\cos\psi & -\sin\phi\cos\psi - \cos\theta\cos\phi\sin\psi & \sin\theta\cos\phi \\ \cos\phi\sin\psi + \cos\theta\sin\phi\cos\psi & \cos\phi\cos\psi - \cos\theta\sin\phi\sin\psi & \sin\theta\sin\phi \\ -\sin\theta\cos\psi & \sin\theta\sin\psi & \cos\theta \end{bmatrix} \tag{3.9}$$

It is obvious that in the standard frame $\vec{S}'$ lies at the north pole of the complex sphere $S_1^2+S_2^2+S_3^2=S^2$, i.e. here $\vec{S} = \overset{0}{\vec{S}} = (0,0,\overset{0}{S})$. Under a rotation about z-axis and a boost along z-axis $\overset{0}{S}$ remaines unaltered so it has the little group $SO(2,C) = SO(2)\times SO(1,1)$. As it was mentioned a rotation about z-axis is irrelevent since the momenta are unaffected by this transformation as well. The boost by an angle χ along z-axis leaves $\overset{0}{S}$ unaltered and the subsequent Lorentz transformations $C_2(\theta)$, $C_3(\phi)$ as indicated in eq. (3.4) translate it to a position $\vec{S}$ where

$$S_1 = S\sin\theta\cos\phi\,,\quad S_2 = S\sin\theta\sin\phi\,,\quad S_3 = S\cos\theta\,. \tag{3.10}$$

Since by the Eq. (3.8) the components of $\overset{0}{p}_1$ and $\overset{0}{p}_2$ can be simply expressed in terms of the invariant s , and alternative label of the state $|p_{(1)}, p_{(2)}> = C_3(\phi)C_2(\theta)C_3(\chi)|\overset{0}{p}_1 \overset{0}{p}_2>$ is given by

$$|\, p_{(1)}\; p_{(2)} > = |\vec{S}\,,\chi > = |\, S\,,\vec{n}\,,\chi\; >, \tag{3.11}$$

where $\vec{n}$ is a complex unit vector in the direction of $\vec{S}$.

By means of eqs. (3.2),(3.5) we get the invariant two-particle volume element in the form

$$\frac{d^3p_{(1)}}{p^0_{(1)}}\,\frac{d^3p_{(2)}}{p^0_{(2)}} = \frac{S^2}{\sqrt{m_1^2 m_2^2 + S^2}}\, dS\, d\Omega\, d\chi = \frac{1}{2}\, S\, ds\, d\Omega\, d\chi \tag{3.12}$$

with

$$d\Omega = \sin\theta\,\sin\theta^*\, d\theta_1\, d\theta_2\, d\phi_1\, d\phi_2\,. \tag{3.13}$$

Thus the integration over the momentum space can be decomposed as integrations over the boost angle χ over the surface of the complex sphere and finally over the radius. It has been shown in/2/ that $d\Omega$ is just the invariant measure on the complex sphere. Using (3.12) we obtain the normalization of states in the form:

$$<S';\vec{n}',\chi'|S,\vec{n},\chi> = 4(2\pi)^6\,\frac{\sqrt{m_1^2 m_2^2 + S^2}}{S^2}\,\delta(S'-S)\,\delta(n'-n)\,\delta(\chi'-\chi)\,. \tag{3.14}$$

Normalization on the s scale yields:

$$<s',\vec{n}',\chi'|s,\vec{n},\chi> = 8(2\pi)^6\,\frac{1}{S}\,\delta(s'-s)\,\delta(\vec{n}'-\vec{n})\,\delta(\chi'-\chi)\,, \tag{3.15}$$

where $\delta(\vec{n}'-\vec{n})$ is the delta-function on the complex sphere, namely

$$\delta(\vec{n}'-\vec{n}) = \delta(\phi_1'-\phi_1)\delta(\phi_2'-\phi_2)\delta(\cos\theta_1'\cos\theta_2' - \cos\theta_1\cos\theta_2)\delta(\sin\theta_1'\mathrm{sh}\,\theta_2' - \sin\theta_1\mathrm{sh}\,\theta_2). \tag{3.16}$$

In terms of momenta, normalizations (3.14), (3.15) correspond to the following convention of normalization of two-particle states

$$\langle p_1' p_2' | p_1 p_2 \rangle = (2\pi)^6\, 2p^0_{(1)}\, 2p^0_{(2)}\, \delta(\vec{p}_1'-\vec{p}_1)\delta(\vec{p}_2'-\vec{p}_2). \tag{3.17}$$

4. Two Particle Spherical Functions

In what follows we fix the total centre-of-mass energy and so the label S is supressed. If in addition one separates the boost single χ and uses the standard system (3.1) the function $\langle\vec{n}|j,m\rangle$ can be identified with the spherical function (2.10) that is

$$\langle\vec{n}|jm\rangle \equiv f^{jj^*}_{mm^*}(\theta,\theta^*) = \left[\frac{(2j+1)(2j^*+1)}{8\pi^2}\right]^{1/2} e^{i(m\phi+m^*\phi^*)} R^{jj^*}_{mm^*;00}(\cos\theta,\cos\theta^*). \tag{4.1}$$

It transforms according to irreducible unitary representations of the Lorentz group and satisfies the orthogonality and completeness relations

$$\int d\Omega \langle j'm'|\vec{n}\rangle\langle\vec{n}|jm\rangle = \delta_{j_0'j_0}\,\delta(\sigma'-\sigma)\delta_{\mu'\mu}\,\delta(\nu'-\nu) \tag{4.2}$$

$$(\sigma,\sigma' \geq 0)$$

$$\sum_{\mu=-\infty}^{\infty}\sum_{j_0=-\infty}^{\infty}\int_0^{\infty} d\sigma \int_{-\infty}^{\infty} d\nu \langle\vec{n}'|jm\rangle\langle jm|\vec{n}\rangle = \delta(\vec{n}'-\vec{n}), \tag{4.3}$$

where

$$j = \frac{1}{2}(j_0 - 1 + i\sigma), \quad m = \frac{1}{2}(\mu + i\nu),$$

j_0 and μ take integer values, while σ and ν continuous. $d\Omega$ is given by Eq. (3.13).

APPENDIX

A contravariant four-vector transforms as $x'^{\mu} = L^{\mu}{}_{\nu} x^{\nu}$. Matrix elements $L^{\mu}{}_{\nu}$ can be expressed in terms of complex Euler angles (2.3) as follows: x/

$$L^0{}_0 = \mathrm{ch}\,\phi_2\,\mathrm{ch}\,\theta_2\,\mathrm{ch}\,\psi_2 + \mathrm{sh}\,\phi_2 \cos\theta_1\,\mathrm{sh}\,\psi_2$$

$$L^0{}_1 = \mathrm{sh}\,\phi_2 \sin\theta_1 \cos\psi_1 - \mathrm{ch}\,\phi_2\,\mathrm{sh}\,\theta_2 \sin\psi_1$$

$$L^0{}_2 = -\mathrm{sh}\,\phi_2 \sin\theta_1 \sin\psi_1 - \mathrm{ch}\,\phi_2\,\mathrm{sh}\,\theta_2 \cos\psi_1$$

$$L^0{}_3 = -\mathrm{ch}\,\phi_2\,\mathrm{ch}\,\theta_2\,\mathrm{sh}\,\psi_2 - \mathrm{sh}\,\phi_2 \cos\theta_1\,\mathrm{ch}\,\psi_2$$

$$L^1{}_0 = \sin\phi_1\,\mathrm{sh}\,\theta_2\,\mathrm{ch}\,\psi_2 - \cos\phi_1 \sin\theta_1\,\mathrm{sh}\,\psi_2$$

$$L^1{}_1 = \cos\phi_1 \cos\theta_1 \cos\psi_1 - \sin\phi_1\,\mathrm{ch}\,\theta_2 \sin\psi_1$$

$$L^1{}_2 = -\cos\phi_1 \cos\theta_1 \sin\psi_1 - \sin\phi_1\,\mathrm{ch}\,\theta_2 \cos\psi_1$$

$$L^1{}_3 = -\sin\phi_1\,\mathrm{sh}\,\theta_2\,\mathrm{sh}\,\psi_2 + \cos\phi_1 \sin\theta_1\,\mathrm{ch}\,\psi_2$$

$$L^2{}_0 = -\cos\phi_1\,\mathrm{sh}\,\theta_2\,\mathrm{ch}\,\psi_2 - \sin\phi_1 \sin\theta_1\,\mathrm{sh}\,\psi_2$$

$$L^2{}_1 = \sin\phi_1 \cos\theta_1 \cos\psi_1 + \cos\phi_1\,\mathrm{ch}\,\theta_2 \sin\psi_1$$

x/ Matrix elements listed in ref./2/ correspond to another definition of complex Euler angles.

$$L^2_2 = -\sin\phi_1 \cos\theta_1 \sin\psi_1 + \cos\phi_1 \,\mathrm{ch}\,\theta_2 \cos\psi_1$$

$$L^2_3 = \cos\phi_1 \,\mathrm{sh}\,\theta_2 \,\mathrm{sh}\,\psi_2 + \sin\phi_1 \sin\theta_1 \,\mathrm{ch}\,\psi_2$$

$$L^3_0 = -\mathrm{sh}\,\phi_2 \,\mathrm{ch}\,\theta_2 \,\mathrm{ch}\,\psi_2 - \mathrm{ch}\,\phi_2 \cos\theta_1 \,\mathrm{sh}\,\psi_2$$

$$L^3_1 = -\mathrm{ch}\,\phi_2 \sin\theta_1 \cos\psi_1 + \mathrm{sh}\,\phi_2 \,\mathrm{sh}\,\theta_2 \sin\psi_1$$

$$L^3_2 = \mathrm{ch}\,\phi_2 \sin\theta_1 \sin\psi_1 + \mathrm{sh}\,\phi_2 \,\mathrm{sh}\,\theta_2 \cos\psi_1$$

$$L^3_3 = \mathrm{sh}\,\phi_2 \,\mathrm{ch}\,\theta_2 \,\mathrm{sh}\,\psi_2 + \mathrm{ch}\,\phi_2 \cos\theta_1 \,\mathrm{ch}\,\psi_2 .$$

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