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ON THE OFF-MASS-SHELL CONTINUATION OF THE VENEZIANO AMPLITUDE

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1. Introduction

Recently, Suura[1] has proposed a method to continue the Veneziano amplitude[2,3] off the mass shell. The method invokes the principle that chirality conjugate form factors should have similar structure. This principle has been originally applied to the continuation in one of the pion masses of the Veneziano amplitude for $\pi\pi$ scattering. However, in the course of manipulations it has been assumed that the amplitude extrapolates in a definite way[3] in another mass variable in the range $(0, m_\pi^2)$. Although this assumption is generally accepted as harmless, in our opinion it is worth to continue the amplitude without it, and treat all the continuations on an equal footing. Then one can verify, that the above assumption is really valid. Apart from this the continuation in 2 mass variables may prove to be interesting by itself.

In this paper we discuss the continuation in 2 mass variables of the $\pi\pi$ scattering amplitude. Suura's principle is enough to carry out this continuation. However, it turns out that factorization properties of the amplitude are important. In view of this the Veneziano model [2] and another model [4] (which is claimed to have simple factorization properties [5]) is investigated. Accepting a factorization similar to that of ref. [6], we see that the continuation does not introduce further problems (except for new ghosts) in the Veneziano model. The factorization in the model of ref. [4] requires modification.

2. The Off-Shell Amplitude

We study the amplitude for the process

$$\pi^+(p) + \pi^-(q) \to \pi^+(k) + \pi^-(q') \quad .$$

The amplitude as given by the Veneziano model [3] is as follows:

$$\langle_{out}\, \pi^+ k,\ \pi^- q' \mid \pi^+ p,\ \pi^- q \rangle^c_{in} = -i(2\pi)^4 \delta(k+q'-p-q)\, B(s,t)\ , \qquad (1)$$

where

$$B(s,t) = \beta_0 \frac{\Gamma(1-\alpha(s))\ \Gamma(1-\alpha(t))}{\Gamma(1-\alpha(s)-\alpha(t))} \quad . \qquad (2)$$

(The index c means the connected part of the amplitude).

The off-shell amplitude we define as

$$T(p,q,k,q') = \int d^4x\, e^{-ipx} \langle \pi\ q' \mid T(\partial^\lambda A^+_\lambda(x)\, \partial^\mu A^-_\mu(0)) \mid \pi^-\ q \rangle^c \ . \qquad (3)$$

Then we have on the mass shell

$$(k^2-m_\pi^2)(p^2-m_\pi^2)\, T(p,q,k,q') = i\,2 f_\pi^2 m_\pi^4\, 2(2\pi)^3\, B(s,t)\ . \qquad (4)$$

We shall use the Adler condition for T, which is given by the eqs.

$$\lim_{k\to 0} T(p,q,k,q') = -\int d^3x \langle \pi^- q' \mid [A_0^-(x),\ \partial^\mu A_\mu^+(0)]\, /_{x_0=0} \mid \pi^- q \rangle$$

$$= -2i\ \tilde{\Sigma}^1((q-q')^2)\ , \qquad (5)$$

$$\lim_{p\to 0} T(p,q,k,q') = -\int d^3x \langle \pi^- q' \mid [A_0^+(x),\ \partial^\mu A_\mu^-(0)]\, /_{x_0=0} \mid \pi^- q \rangle$$

$$= -2i\ \tilde{\Sigma}((q-q')^2)\ . \qquad (6)$$

If the $[A_0^i(x), \partial^\mu A_\mu^k(0)]\ \delta(x_0)$ commutator contains only isospi symmetric parts, then

$$\tilde{\Sigma}((q-q')^2) = \tilde{\Sigma}'((q-q')^2) \qquad (7)$$

as we always assume in the following.

We try to give the off-mass-shell continuation T by writtin

$$T(p,q,k,q') = \frac{i\,2 f_\pi^2 m_\pi^4\, 2(2\pi)^3}{(k^2-m_\pi^2)(p^2-m_\pi^2)\,\Phi(k^2,p^2)}\, B(s,t)\ , \qquad (8$$

where of course $\Phi(m_\pi^2, m_\pi^2) = 1$.

Taking the spin one part of the residue at the ρ pole, from factorization it is easy to see that we must have $\Phi(k^2, p^2) = f(k^2)\,\bar{f}(p^2)$. The Adler condition (eqs. (5) (6) (7)) then yields $f(k^2) = \bar{f}(k^2)$. Take now the spin zero part of the residue at the $s = m_\rho^2$ pole of eq.(8), then we get

$$\frac{i2 f_\pi^2 m_\pi^4 \tilde{\tilde{\pi}}(k^2)\,\tilde{\tilde{\pi}}(p^2)}{2(2\pi)^3} = \frac{i2 f_\pi^2 m_\pi^4 2(2\pi)^3}{f(k^2)(k^2-m_\pi^2) f(p^2)(p^2-m_\pi^2)}\,\beta_0\,\frac{\frac{1}{2} + b m_\pi^2 - 2q_0 q_0'}{b}, \qquad (9)$$

where we have used the definition

$$\langle \pi^- q' \,|\, \Phi_\pi(0) \,|\, \sigma q \rangle = \frac{1}{2(2\pi)^3}\,\tilde{\tilde{\pi}}((q-q')^2) \qquad (10)$$

σ is the $I = 0$ spin zero daughter of the ρ, Φ_{π^-} is the π^- field. The ρ trajectory $\alpha(s)$ is given by[3] $\alpha(s) = \frac{1}{2} + b(s - m_\pi^2)$.

It is now easy to see that eq. (9) cannot be valid as the left hand side factorizes, while the right hand side does not. Nevertheless take $p^2 = k^2$, then we get

$$\tilde{\tilde{\pi}}(p^2) = \frac{2(2\pi)^3}{f(p^2)(p^2 - m_\pi^2)} \sqrt{\frac{\beta_0}{b}\left(\frac{1}{2} + b\, m_\pi^2 - 2b\, q_0^2\right)} \;. \qquad (11)$$

(The square root factor of eq. (11) will of course be modified in Sec. 3, where we take into account factorization). Next we require $\tilde{\tilde{\pi}}(p^2)$ to have poles at $p^2 =$ (mass of the π and $\pi - A_1$ daughters)^2, therefore we choose

$$\frac{1}{f(p^2)(p^2 - m_\pi^2)} = -\,b\,\Gamma\left(\frac{1}{2} - \alpha(p^2)\right) P(p^2) \;. \qquad (12)$$

The factor $\Gamma(\frac{1}{2} - \alpha(p^2))$ gives the required poles (for the π trajectory we accept $\alpha_\pi(p^2) = \alpha(p^2) - \frac{1}{2}$), while the function $P(p^2)$ is an undetermined (good, probably smooth) function, which satisfies the normalization condition $P(m_\pi^2) = 1$.

From the Adler condition we also get the expression for $\tilde{\tilde{\Sigma}}(k^2)$

$$\tilde{\tilde{\Sigma}}(k^2) = -f_\pi^2\, m_\pi^4\, 2(2\pi)^3 \beta_0 b^2\, P(0)\,\Gamma\left(\frac{1}{2} - \alpha(0)\right) P(p^2) \times$$
$$\times\, \Gamma\left(\frac{1}{2}\right) \Gamma\left(1 - \alpha(p^2)\right) \qquad (13)$$

which has the right poles (at the $I = 0$ daughters of the $\rho - f_0$ trajectory). Choosing $P(p^2)$ in a special way

$$P(p^2) = \frac{\pi}{\Gamma(\frac{1}{4})} \frac{\Gamma\left(\frac{1}{2} - \frac{\alpha(p^2)}{2}\right)}{\Gamma\left(\frac{3}{4} - \frac{\alpha(p^2)}{2}\right) \Gamma\left(1 - \alpha(p^2)\right)}$$

we get back the form of Suura[1]. This function, however, kills half of the poles of both $\tilde{\tilde{\Sigma}}(k^2)$ and $\tilde{\tilde{\pi}}(p^2)$ therefore we do not insist on it.

In order to obtain some information on the high p^2 behaviour of $P(p^2)$ we may assume the validity of a Bjorken type procedure[8] for the scalar amplitude $T(p, q, k, q')$. As in the Veneziano model we have infinitely many resonances, with arbitrarily large masses it is clear that the assumption of the Bjorken limit is a strong one. Using the canonical commutation rules for the pion field the commutators determining the coefficients of the p_0^{-1}, p_0^{-2} terms of the expansion are known. We have

$$T(p,q,k,q') \approx -\frac{i}{p_0} \int d^3x\, e^{ipx} \langle \pi^- q' \,|\, [\Phi(x), \Phi_{\pi^+}'(0)]/_{x_0=0} \,|\, \pi^- q \rangle$$

$$-\frac{1}{p_0^2} \int d^3x\, e^{ipx} \langle \pi^- q' \,|\, [\Phi_{\pi^+}(x), \Phi_{\pi^-}(0)]/_{x_0=0} \,|\, \pi^- q \rangle \;.$$

The first commutator is zero, the second is given by $-i\delta(\vec{x})$. It can be easily shown that this second term arises from the disconnected part of T, thus we get $T = 0(p_0^{-3})$ in the Bjorken limit. Due to the $\Gamma(\frac{1}{2} - \alpha(p^2))^2$ factor $T(p,q,k,q')$ decreases very quicky, like $P(p_0^2)e^{2\alpha(p_0^2)}\ \alpha(p_0^2)^{-2\alpha(p_0^2)+\alpha(t)}$ so $P(p^2)$ may even have a wild increase for large positive values of p^2.

In our model the expression for $<\pi^- q' | \partial^\lambda A_\lambda^- | \pi^+ p\ \pi^- q>$ is as follows

$$<\pi^- q | \partial^\lambda A_\lambda^-(0) | \pi^+ p,\ \pi^- q> =$$

$$=\sqrt{2}\ f_\pi\ m_\pi^2 \sqrt{2(2\pi)^3}\, b^2\, \Gamma(\tfrac{1}{2} - \alpha(k^2)\, P(k^2)(p^2 - m_\pi^2)\Gamma(\tfrac{1}{2} - \alpha(p^2))P(p^2)B(s,t). \tag{14}$$

Now, the assumption of Suura[1] is that the variation of the factor $(p^2 - m_\pi^2)\Gamma(\frac{1}{2} - \alpha(p^2))\ P(p^2)$ in eq. (14) is small, when p^2 varies from 0 to m_π^2. As $(p^2 - m_\pi^2)\Gamma(\frac{1}{2} - \alpha(p^2))$ varies little in this region, we obtain the natural requirement that $P(p^2)$ is a slowly varying function. From this we also get the $\Sigma(0) \approx -\frac{m_\pi^2}{2(2\pi)^3}$ result of ref.[7].

We note that eq. (13) as well as the result for the off-shell amplitude $T(p,q,k,q')$ (eqs. (8) (12)) will not be modified when we take factorization properties of the amplitude into account. Thus $T(p,q,k,q')$ is given by eqs. (8),(12) and $\Phi(k^2, p^2) = f(k^2)\ f(p^2)$.

3. Factorization

The factorization properties of the N point Veneziano amplitude have been treated in ref.[6], in the case when all the external particles are spin zero and the (single) trajectory appearing in the model has negative intercept at $s = 0$. As the ρ trajectory has positive intercept, the above results are not straightforwardly applicable in our case.

We write the Veneziano amplitude in the following form

$$(1 - \alpha(s) - \alpha(t))\ B(1-\alpha(s), 1-\alpha(t)), \tag{15}$$

where $B(x,y)$ is the Euler beta function. The $(1-\alpha(s)-\alpha(t))$ factor, which ensures the correct polynomial behaviour of the residue in t , does not factorize in the simplest way, so this factor introduces further degeneracy x/. In the spirit of ref. /6/, we write

$$1-\alpha(s) - \alpha(t) = (\frac{1}{2} - \alpha(s) - b\, m_\pi^2) + 2 b\, q\, q' . \tag{16}$$

So the residue at $\alpha(s) = \ell$ is

$$-\frac{\beta_0}{b}\left(\frac{1}{2} - \ell - b\, m_\pi^2 + 2b\, q\, q'\right) \sum_{i+2j+\ldots+mk=\ell-1} \frac{(\ell-1)!}{i!\, j! \ldots k!} \left(2b\, q\, q' - \frac{1}{2} - b\, m_\pi^2\right)^i$$

$$\left(\frac{2bqq' - \frac{1}{2} - b m_\pi^2}{2}\right)^j \ldots \left(\frac{2b\, q\, q' - \frac{1}{2} - b\, m_\pi^2}{m}\right)^k . \tag{17}$$

We see that the residue factorizes even if p^2 and k^2 have arbitrary values. However, the elimination of the ghost states by the Ward identities (which is, in general, incomplete) spoils the off-shell factorization. Identifying the σ contribution with the good scalar at $\alpha(s) = 1$ we get for $\tilde{\tilde{\pi}}(p^2)$:

x/ Of course, this statement is rather heuristic, as we do not know how to include the analogous factors into a general N point function.

$$\tilde{\pi}(p^2) = -2(2\pi)^3 \, b \, \Gamma\left(\frac{1}{2} - \alpha(p^2)\right) P(p^2) \left(\frac{\beta_0}{b}\left(\frac{1}{2} + b\, m_\pi^2\right)\right)^{1/2} . \quad (18)$$

We now turn to the model of ref.[3] which is claimed to have the same degeneracy in the general N point function case, as for the 4 point function[4]. In our case the function replacing $B(s,t)$ of eq. (2) is

$$B'(s,t) = \beta' \left\{ \gamma(s) \frac{\Gamma(1-\alpha(s))}{\Gamma(\frac{1}{2}-\alpha(s))} (\alpha(t))^{\alpha(s)} + \gamma(t) \frac{\Gamma(1-\alpha(t))}{\Gamma(\frac{1}{2}-\alpha(t))} (\alpha(s))^{\alpha(t)} \right\} . \quad (19)$$

This function has all the good features of the Veneziano amplitude[3].

The spin zero part of the residue at $\alpha(s) = 1$ is

$$-\frac{1}{b} \gamma(m_\rho^2) \frac{1}{\Gamma(-\frac{1}{2})} \left[\frac{1}{2} + b\,(m_\pi^2 - 2 q_0 q_0') \right] \quad (20)$$

which is not a product. The way out of this difficulty is the introduction of a more complicated factorization as in the Veneziano model. The $\frac{1}{2} + b\, m_\pi^2$ part of the residue corresponds to a good scalar, while $-2b\, q_0 q_0'$ is a factorizable scalar ghost. (The $\alpha(s) = \ell$ residue can be factorized similarly). The degeneracy of the model is thus higher if we claim off-shell factorization.

Nevertheless the N point function still has the same degree of degeneracy as the 4 point function.

Continuation in all the four masses does not introduce further problems concerning factorization. It is clear that the difficulty of new ghosts persists in all models where the residue is a polynomial of t.

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