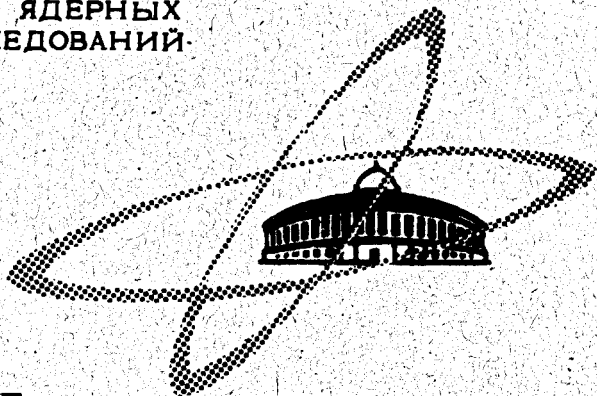


20/70

Д-40
ОБЪЕДИНЕННЫЙ
ИНСТИТУТ
ЯДЕРНЫХ
ИССЛЕДОВАНИЙ

Дубна:

E2 - 4838



ЛАБОРАТОРИЯ ВЫЧИСЛИТЕЛЬНОЙ ТЕХНИКИ
И АВТОМАТИЗАЦИИ

R.T. Denchev

A REMARK ON THE CONNECTION
OF SCATTERING MATRIX
WITH AUTOMORPHIC FUNCTIONS

1969

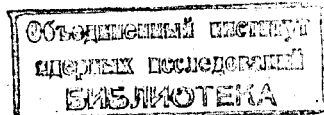
E2 - 4838

R.T. Denchev

8157/2 pr

**A REMARK ON THE CONNECTION
OF SCATTERING MATRIX
WITH AUTOMORPHIC FUNCTIONS**

Submitted to "Годишник на Софийския университет"



We consider the one-meson Low equations of the static baryon model ^{/1/}

$$f_a(z) = \frac{\lambda_a}{z} + \frac{1}{\pi} \int_1^\infty \frac{d\omega \rho(\omega) |f_a(\omega + i0)|^2}{\omega - z} +$$

$$+ \frac{1}{\pi} \sum_{\beta=1}^n C_{a\beta} \frac{1}{\pi} \int_1^\infty \frac{d\omega \rho(\omega) |f_\beta(\omega + i0)|^2}{\omega + z} \quad (a=1, \dots, n). \quad (1)$$

The meson energy is ω in a system of units, where the meson mass is unity. The amplitude $f_a(\omega + i0)$ describes scattering of pseudoscalar mesons in p -waves. In terms of the cut-off function $v(k)$ the function $\rho(\omega)$ is defined by

$$\rho(\omega) = \frac{k^3 v^2(\omega)}{12\pi}, \quad k = (\omega^2 - 1)^{1/2}.$$

We suppose that $\rho(\omega)$ is analytic on the complex plane ω with cuts $(-\infty, -1), (1, \infty)$, for the exception of possible poles and possesses the properties

$$\rho(z^*) = -\rho^*(z), \quad \rho(-z) = \rho(z).$$

The real constants λ_a and $C_{a\beta}$ satisfy the conditions

$$C = C^+, C^2 = 1, C\lambda = -\lambda,$$

where C is the matrix of the numbers $C_{\alpha\beta}$ and λ is a vector with components λ_α .

It may be then shown^[2-3] that the solving of eqs. (1) reduces to the problem of finding the functions $S_\alpha(w)$ ($\alpha = 1, \dots, n$) with the following properties:

1. $S_\alpha(w)$ are meromorphic functions
2. $S_\alpha(w^*) = S_\alpha(w)^*$
3. $S_\alpha(w) S_\alpha(\pi - w) = 1$
4. $S_\alpha(-w) = d_\alpha + \sum_{\beta=1}^n C_{\alpha\beta} S_\beta(w)$, $d_\alpha = 1 - \sum_{\beta=1}^n C_{\alpha\beta}$
5. $S_\alpha(w) = 1 + o(\rho(w))$ when $w \rightarrow \infty$, $-\frac{\pi}{2} < \operatorname{Re} w < \frac{\pi}{2}$.

We give here some considerations concerning the explicit solving of the problem.

Let E_n° be an n -dimensional complex vector space with elements $s = (s_1, \dots, s_n) = \{s_\alpha\}$, s_α are complex number. We introduce the following transformations in E_n° :

$$\Lambda_1 : E_n^{\circ} \ni s \rightarrow \left(\frac{1}{s_1}, \dots, \frac{1}{s_n} \right) \in E_n^{\circ}$$

$$\Lambda_2 : E_n^{\circ} \ni s \rightarrow \left\{ d_\alpha + \sum_{\beta=1}^n C_{\alpha\beta} s_\beta \right\} \in E_n^{\circ}$$

Denote by \mathcal{G} the group generated by Λ_1 and Λ_2 . The function $\Phi(s_1, \dots, s_n) = \Phi(s)$ is called automorphic with respect to the group \mathcal{G} if it has the following properties:

A) $\Phi(s)$ is a single-valued analytic function.

B) When the point s lies in the domain of definition of the function $\Phi(s)$, then the point gs lies in the same domain for any $g \in \mathcal{G}$.

C) $\Phi(gs) = \Phi(s)$ for any $g \in \mathcal{G}$.

We shall prove the following theorem.

Theorem. Let $S_a(w) (a=1, \dots, n)$ be some functions possessing properties 1,3,4. We assume that $\lim_{w \rightarrow \infty} S_a(w) = p_a$ as $w \rightarrow \infty$, $-\frac{\pi}{2} < \operatorname{Re} w < \frac{\pi}{2}$.

Let $\Phi(s)$ be a function automorphic with respect to the group \mathcal{G} and let it tend to a certain limit (finite or infinite) when $s_a \rightarrow p_a (a=1, \dots, n)$. Then the equality

$$\Phi(S_1(w), \dots, S_n(w)) = R(\cos 2w), \quad (2)$$

where R is a certain rational function, is valid.

Proof. For the function $h(w) = \Phi(S_1(w), \dots, S_n(w))$ we have

$$h(\pi - w) = \Phi(S(\pi - w)) = \Phi\left(\frac{1}{S_1(w)}, \dots, \frac{1}{S_n(w)}\right) =$$

$$= \Phi(\Lambda_1 S(w)) = \Phi(S(w)) = h(w)$$

$$h(-w) = \Phi(S(-w)) = \Phi(\Lambda_2 S(w)) = \Phi(S(w)) = h(w).$$

This means that $h(w)$ is automorphic with respect to the group generated by the transformations $w' = \pi - w$ and $w' = -w$. The

half-band - $\frac{\pi}{2} < \operatorname{Re} w < \frac{\pi}{2}$, $\operatorname{Im} w > 0$ may serve as a fundamental domain of this group. There is a parabolic point at infinity. According to the conditions of the theorem the function $h(w)$ has a limit at this point. Hence, $h(w)$ is a simple automorphic function^[4]. On the other hand, $\cos 2w$ is also a simple automorphic function and has the only simple pole in the fundamental domain (at the point ∞). According to the well-known theorem of automorphic functions^[4], the function $h(w)$ is a rational function of $\cos 2w$, i.e. eq.(2) holds. Thus, the theorem has been proved.

Let $\Phi_1(s), \dots, \Phi_n(s)$ be functionally independent functions, automorphic with respect to \mathcal{G} and satisfying the conditions of the theorem.

Consider the following equations

$$\Phi_1(s_1, \dots, s_n) = R_1(\cos 2w),$$

.....

$$\Phi_n(s_1, \dots, s_n) = R_n(\cos 2w),$$

(3)

where R_1, \dots, R_n are arbitrary rational functions. These equations can, generally speaking, be solved with respect to s_1, \dots, s_n .

It follows from the theorem that the solutions $S(w)$ of our problem for which there exists a limit for $w \rightarrow \infty$, $-\frac{\pi}{2} < \operatorname{Re} w < \frac{\pi}{2}$ are among the solutions of the system (3).

Example:

Let $n=2$ and

$$C = \begin{pmatrix} -\frac{1}{3} & \frac{4}{3} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

The example was studied by G. Wanders^{/5/}. It is not difficult to check that the functions

$$\Phi_1(s_1, s_2) = \cos\left(2\pi \frac{s_1 + 2s_2}{s_1 - s_2}\right)$$

$$\Phi_2(s_1, s_2) = \frac{2s_1 + s_2}{s_1 + 2s_2} s_2 + \frac{s_1 + 2s_2}{(2s_1 + s_2)s_2}$$

are automorphic with respect to the group \mathcal{G} in our case. Solving (3) we obtain

$$S_1(w) = \frac{B(w)(B(w) - 2)}{B^2(w) - 1} D(w),$$

$$S_2(w) = \frac{B(w)}{B(w) - 1} D(w), \quad (4)$$

where

$$B(w) = \frac{1}{2\pi} \arccos R_1(\cos 2w),$$

$$D(w) = \frac{1}{2} (R_2(\cos 2w) \pm \sqrt{R_2^2(\cos 2w) - 4}). \quad (5)$$

If the rational functions R_1 and R_2 are such that the functions $S_1(w)$, $S_2(w)$ obtained by (4) and (5) are meromorphic and real then they are a solution of our problem.

Remark: The theorem we proved shows that there exists a certain connection between the scattering matrix and the functions

automorphic with respect to the group \mathcal{G} . Roughly speaking, the functions $S_a(w)$ are the inverse functions of the automorphic $\{\Phi_\beta(s)\}$. In the case of one variable a theorem^[4] is known according to which the inverse function of a certain automorphic function is represented in the form of the quotient of two solutions of a linear differential equation. It is interesting to know whether there exists an analog of this theorem for the functions of many variables and what it can yield for the scattering matrix.

References

1. R.L. Warnock. Preprint ANL/HED 6821, Argonne Illinois (1968).
2. J. Rothleitner. Zeitschrift für Physik, 177, 287-299 (1964).
3. В.А. Мещеряков. Препринт ОИЯИ Р-2369, Дубна (1965).
4. L.R. Ford. Automorphic functions, McGraw-Hill, New York (1929).
5. G. Wanders. Nuovo Cimento, 23, 817-837 (1962).

Received by Publishing Department
on December 3, 1969.