ОБЪЕДИНЕННЫЙ
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Дубна

E2 - 4838

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# A REMARK ON THE CONNECTION OF SCATTERING MATRIX WITH AUTOMORPHIC FUNCTIONS

ЛАБОРАТОРИЯ ВЫЧИСЛИТЕЛЬНОЙ ТЕХНИКИ И АВТОМАТИЗАЦИИ

1969

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# A REMARK ON THE CONNECTION OF SCATTERING MATRIX WITH AUTOMORPHIC FUNCTIONS

Submitted to "Годишник на Софийския университет"

We consider the one-meson Low equations of the static baryon model /1/

$$f_\alpha(z) = \frac{\lambda_\alpha}{z} + \frac{1}{\pi}\int_1^\infty \frac{d\omega\,\rho(\omega)\,|f_\alpha(\omega+i0)|^2}{\omega - z} + \frac{1}{\pi}\sum_{\beta=1}^{n} C_{\alpha\beta}\,\frac{1}{\pi}\int_1^\infty \frac{d\omega\,\rho(\omega)\,|f_\beta(\omega+i0)|^2}{\omega + z} \quad (\alpha = 1,\dots,n). \tag{1}$$

The meson energy is $\omega$ in a system of units, where the meson mass is unity. The amplitude $f_\alpha(\omega + i0)$ describes scattering of pseudoscalar mesons in $p$-waves. In terms of the cut-off function $v(k)$ the function $\rho(\omega)$ is defined by

$$\rho(\omega) = \frac{k^3 v^2(\omega)}{12\pi}, \qquad k = (\omega^2 - 1)^{1/2}.$$

We suppose that $\rho(\omega)$ is analytic on the complex plane $\omega$ with cuts $(-\infty, -1), (1, \infty)$, for the exeption of possible poles and possesses the properties

$$\rho(z^*) = -\rho^*(z), \quad \rho(-z) = \rho(z).$$

The real constants $\lambda_\alpha$ and $C_{\alpha\beta}$ satisfy the conditions

$$C = C^{+}\ , \ C^{2} = 1\ , \ C\lambda = -\lambda\ ,$$

where $C$ is the matrix of the numbers $C_{\alpha\beta}$ and $\lambda$ is a vector with components $\lambda_{\alpha}$.

It may be then shown /2-3/ that the solving of eqs. (1) reduces to the problem of finding the functions $S_{\alpha}(w)$ $(\alpha = 1,..., n)$ with the following properties:

1. $S_{\alpha}(w)$ are meromorphic functions
2. $S_{\alpha}(w^{*}) = S_{\alpha}(w)^{*}$
3. $S_{\alpha}(w)\, S_{\alpha}(\pi - w) = 1$
4. $S_{\alpha}(-w) = d_{\alpha} + \sum_{\beta=1}^{n} C_{\alpha\beta} S_{\beta}(w), \quad d_{\alpha} = 1 - \sum_{\beta=1}^{n} C_{\alpha\beta}$
5. $S_{\alpha}(w) = 1 + 0(\dot{\rho}(w))$ when $w \to \infty$, $-\frac{\pi}{2} < \mathrm{Re}\, w < \frac{\pi}{2}$.

We give here some considerations concerning the explicit solving of the problem.

Let $E_n^c$ be an $n$-dimensional complex vector space with elements $s = (s_1, ..., s_n) = \{s_{\alpha}\}$; $s_{\alpha}$ are complex number. We introduce the following transformations in $E_n^c$:

$$\Lambda_1 : E_n^c \ni s \to \left(\frac{1}{s_1}, ..., \frac{1}{s_n}\right) \in E_n^c$$

$$\Lambda_2 : E_n^c \ni s \to \{ d_{\alpha} + \sum_{\beta=1}^{n} C_{\alpha\beta} s_{\beta} \} \in E_n^c .$$

Denote by $\mathcal{G}$ the group generated by $\Lambda_1$ and $\Lambda_2$. The function $\Phi(s_1, ..., s_n) = \Phi(s)$ is called <u>automorphic</u> with respect to the group $\mathcal{G}$ if it has the following properties:

A) $\Phi(s)$ is a single-values analytic function.

B) When the point $s$ lies in the domain of definition of the function $\Phi(s)$, then the point $gs$ lies in the same domain for any $g \in \mathcal{G}$

C) $\Phi(gs) = \Phi(s)$ for any $g \in \mathcal{G}$.

We shall prove the following theorem.

<u>Theorem. Let $S_a(w)$ $(a = 1, \dots, n)$ be some functions possessing properties 1,3,4. We assume that</u> $\lim S_a(w) = p_a$ as $w \to \infty$, $-\frac{\pi}{2} < \operatorname{Re} w < \frac{\pi}{2}$.

<u>Let $\Phi(s)$ be a function automorphic with respect to the group $\mathcal{G}$ and let it tend to a certain limit (finite or infinite) when $s_a \to p_a$</u> $(a = 1, \dots, n)$. <u>Then the equality</u>

$$\Phi(S_1(w), \dots, S_n(w)) = R(\cos 2w), \tag{2}$$

<u>where $R$ is a certain rational function, is valid.</u>

<u>Proof.</u> For the function $h(w) = \Phi(S_1(w), \dots, S_n(w))$ we have

$$h(\pi - w) = \Phi(S(\pi - w)) = \Phi\left(\frac{1}{S_1(w)}, \dots, \frac{1}{S_n(w)}\right) =$$

$$= \Phi(\Lambda_1 S(w)) = \Phi(S(w)) = h(w)$$

$$h(-w) = \Phi(S(-w)) = \Phi(\Lambda_2 S(w)) = \Phi(S(w)) = h(w).$$

This means that $h(w)$ is automorphic with respect to the group generated by the transformations $w' = \pi - w$ and $w' = -w$. The half-band $-\frac{\pi}{2} < \mathrm{Re}\, w < \frac{\pi}{2}$, $\mathrm{Im}\, w > 0$ may serve as a fundamental domain of this group. There is a parabolic point at infinity. According to the conditions of the theorem the function $h(w)$ has a limit at this point. Hence, $h(w)$ is a simple automorphic function /4/. On the other hand, $\cos 2w$ is also a simple automorphic function and has the only simple pole in the fundamental domain (at the point $\infty$). According to the well-known theorem of automorphic functions /4/, the function $h(w)$ is a rational function of $\cos 2w$, i.e. eq.(2) holds. Thus, the theorem has been proved.

Let $\Phi_1(s), \dots, \Phi_n(s)$ be functionally independent functions, automorphic with respect to $\mathcal{G}$ and satisfying the conditions of the theorem.

Consider the following equations

$$
\begin{aligned}
&\Phi_1(s_1, \dots, s_n) = R_1(\cos 2w), \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
&\Phi_n(s_1, \dots, s_n) = R_n(\cos 2w),
\end{aligned}
\tag{3}
$$

where $R_1, \dots, R_n$ are arbitrary rational functions. These equations can, generally speaking, be solved with respect to $s_1, \dots, s_n$. It follows from the theorem that the solutions $S(w)$ of our problem for which there exists a limit for $w \to \infty$, $-\frac{\pi}{2} < \mathrm{Re}\, w < \frac{\pi}{2}$ are among the solutions of the system (3).

<u>Example</u>:

Let $n = 2$ and

$$
C = \begin{pmatrix} -\frac{1}{3} & \frac{4}{3} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}.
$$

The example was studied by G. Wanders [5]. It is not difficult to check that the functions

$$\Phi_1(s_1, s_2) = \cos(2\pi \frac{s_1 + 2s_2}{s_1 - s_2})$$

$$\Phi_2(s_1, s_2) = \frac{2s_1 + s_2}{s_1 + 2s_2} s_2 + \frac{s_1 + 2s_2}{(2s_1 + s_2)s_2}$$

are automorphic with respect to the group $\mathcal{G}$ in our case. Solving (3) we obtain

$$S_1(w) = \frac{B(w)(B(w)-2)}{B^2(w)-1} D(w),$$

$$S_2(w) = \frac{B(w)}{B(w)-1} D(w), \tag{4}$$

where

$$B(w) = \frac{1}{2\pi} \arccos R_1(\cos 2w),$$

$$D(w) = \frac{1}{2}(R_2(\cos 2w) \pm \sqrt{R_2^2(\cos 2w) - 4}). \tag{5}$$

If the rational functions $R_1$ and $R_2$ are such that the functions $S_1(w)$, $S_2(w)$ obtained by (4) and (5) are meromorphic and real then they are a solution of our problem.

<u>Remark:</u> The theorem we proved shows that there exists a certain connection between the scattering matrix and the functions

automorphic with respect to the group $\mathcal{G}$. Roughly speaking, the functions $S_a(w)$ are the inverse functions of the automorphic $\{\Phi_\beta(s)\}$. In the case of one variable a theorem /4/ is known according to which the inverse function of a certain automorphic function is represented in the form of the quotien of two solutions of a linear differential equation. It is interesting to know whether there exists an analog of this theorem for the functions of many variables and what it can yield for the scattering matrix.

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Receieved by Publishing Department
on December 3, 1969.